Lesson 4

4.1

<u>Note</u>: If E is φ-measurable then E^c is φ-measurable

<u>Proof</u>: Let P, Q such that

$$P \subset E^c \ , \quad Q \subset (E^c)^c = E$$

Since $Q \subset E$ and $P \subset E^c$, the characterization proved in class for φ-measurable sets yields; since E is φ-measurable:

$$\varphi(Q \cup P) = \varphi(Q) + \varphi(P)$$

$$\therefore \quad \varphi(P \cup Q) = \varphi(P) + \varphi(Q)$$

$\therefore$ E^c is φ-measurable

<u>Another way</u>: Since both E and X are φ-measurable, then $X \setminus E$ is φ-measurable; that is, E^c is φ-measurable.

Lemma. Let $\{E_i\}$ be a sequence of arbitrary sets. (4.2)

Then, there exists a sequence of disjoints sets $\{A_i\}$ such that $A_i \subset E_i$ and:

$$\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} A_i$$

In case each E_i is φ-measurable, so is A_i.

Proof: Consider

$$A_1 := E_1$$

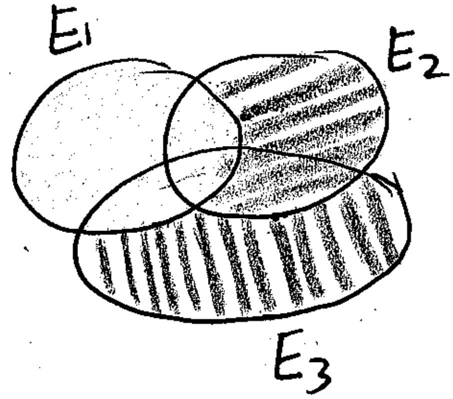
$$A_2 := (E_1 \cup E_2) \setminus E_1$$

$$A_3 := (E_1 \cup E_2 \cup E_3) \setminus (E_1 \cup E_2)$$

$$\vdots$$

$$A_{i+1} := (E_1 \cup E_2 \cup \cdots \cup E_{i+1}) \setminus (E_1 \cup \cdots \cup E_i)$$

$$\vdots$$



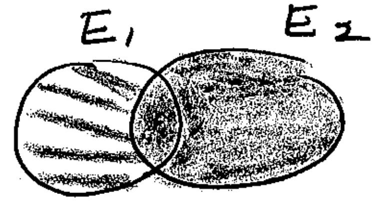
Clearly:

$$\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} A_i$$

If the $\{E_i\}$ are φ-measurable, we want to show that each A_i is measurable.

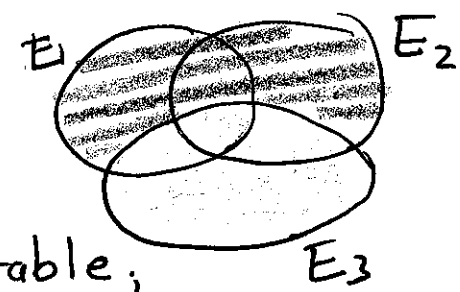
Notice that:

$$E_1 \cup E_2 = E_2 \cup (E_1 \setminus E_2)$$



- Thus, from last class, $E_1 \setminus E_2$ is measurable, and since E_2 and $E_1 \setminus E_2$ are disjoint then the union is measurable. Hence $E_1 \cup E_2$ is measurable.

$$E_1 \cup E_2 \cup E_3 = E_3 \cup [(E_1 \cup E_2) \setminus E_3]$$



- Since $E_1 \cup E_2$ is measurable, then the substraction of sets $(E_1 \cup E_2) \setminus E_3$ is φ-measurable; and, since E_3 and $(E_1 \cup E_2) \setminus E_3$ are disjoint, then the sum is measurable

In general, if we write: (4.4)

$$E_1 \cup \cdots \cup E_j = E_j \cup [(E_1 \cup \cdots \cup E_{j-1}) \setminus E_j]$$

we conclude as before that

$$\boxed{E_1 \cup \cdots \cup E_j \text{ is measurable}} \ \forall j$$

and, since, for all i:

$$A_{i+1} = (E_1 \cup \cdots \cup E_{i+1}) \setminus (E_1 \cup \cdots \cup E_i)$$

we conclude:

$$\boxed{A_i \text{ is } \varphi\text{-measurable}, \ i = 1, 2, 3, \ldots}$$

Note that we have proven that any <u>finite union</u> of φ-measurable sets is φ-measurable, and the decomposition $\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} A_i$ implies that the same is true for any <u>countable union</u> of φ-measurable sets.

<u>Thm</u>: If $\{E_i\}$ is a sequence of φ-measurable sets in X, then:

$$\bigcup_{i=1}^{\infty} E_i \quad \text{and} \quad \bigcap_{i=1}^{\infty} E_i \quad \text{are } \varphi\text{-measurable.}$$

<u>Proof</u>: From previous theorem:

$$\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} A_i$$

where A_i is φ-measurable, $i=1,2,3,\dots$

Since the $\{A_i\}$ are disjoints then

$$\bigcup_{i=1}^{\infty} A_i \text{ is } \varphi\text{-measurable},$$

and hence:

$$\bigcup_{i=1}^{\infty} E_i \text{ is } \varphi\text{-measurable}.$$

Now:

$$\bigcap_{i=1}^{\infty} E_i = \left(\bigcup_{i=1}^{\infty} E_i^c\right)^c,$$

and clearly, the countable intersection is also φ-measurable.

<u>Def</u>: A σ-algebra on X is a nonempty collection Σ of sets $E \subset X$ such that:

(i) $E \in \Sigma \Rightarrow E^c \in \Sigma$

(ii) $E_i \in \Sigma$, $i = 1, 2, \ldots \Rightarrow \bigcup_{i=1}^{\infty} E_i \in \Sigma$

<u>Remark</u>: From previous definition, if φ is an outer measure on an arbitrary set X, then the class of φ-measurable sets forms a σ-algebra.

<u>Corollary</u>: Let φ be an outer measure on X and $\{E_i\}$ a countable collection of φ-measurable sets. Then

(i) If $E_1 \subset E_2$, $\varphi(E_1) < \infty$ then:

$$\varphi(E_2 \setminus E_1) = \varphi(E_2) - \varphi(E_1)$$

(ii) If $E_i \subset E_{i+1}$, $i = 1, 2, 3, \ldots$, then

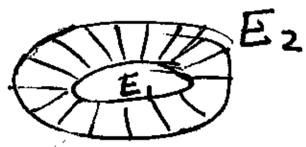
$$\varphi\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim_{i \to \infty} \varphi(E_i)$$

(iii) If $E_i \supset E_{i+1}$, $i = 1, 2, \ldots$, and $\varphi(E_{i_0}) < \infty$, for some i_0, then: $\varphi\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{i \to \infty} \varphi(E_i)$.

Proof: (4.7)

(i) We write:

$$E_2 = (E_2 \setminus E_1) \cup E_1$$



$$\therefore \ \varphi(E_2) = \varphi(E_2 \setminus E_1) + \varphi(E_1)$$

$$\Rightarrow \ \varphi(E_2) - \varphi(E_1) = \varphi(E_2 \setminus E_1);$$ which

is possible since $\varphi(E_1) < \infty$

We conclude:

$$\varphi(E_2 \setminus E_1) = \varphi(E_2) - \varphi(E_1)$$

(ii) We have:

$$E_1 \subset E_2 \subset E_3 \subset E_4 \subset \cdots$$

If $\varphi(E_{i_0}) = \infty$ for some i_0 then by monotonity:

$$\varphi(E_i) = \infty, \ \forall i \geq i_0$$

and $\displaystyle\lim_{i \to \infty} \varphi(E_i) = \infty.$

Again by monotonicity:

$$E_{i_0} \subset \bigcup_{i=1}^{\infty} E_i$$

Hence

$$\infty = \varphi(E_{i_0}) \leq \varphi\left(\bigcup_{i=1}^{\infty} E_i\right)$$

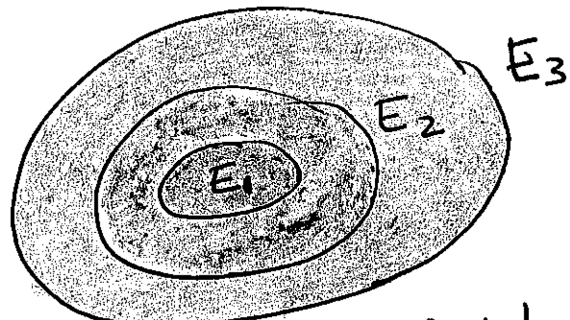
$$\therefore \quad \varphi\left(\bigcup_{i=1}^{\infty} E_i\right) = \infty = \lim_{i\to\infty} \varphi(E_i).$$

Thus, we can assume that:

$$\boxed{\varphi(E_i) < \infty, \quad i = 1, 2, \ldots}$$

We note that:

$$\bigcup_{i=1}^{\infty} E_i = E_1 \cup \left[\bigcup_{i=1}^{\infty} (E_{i+1} \setminus E_i)\right] \qquad (1)$$



The union in the right of (1) is disjoint. Thus.

$$\varphi\left(\bigcup_{i=1}^{\infty} E_i\right) = \varphi(E_1) + \sum_{i=1}^{\infty} \varphi(E_{i+1} \setminus E_i)$$

Define $E_0 := \phi$ and:

$$S_n = \sum_{i=0}^{n} \varphi(E_{i+1} \setminus E_i)$$

$$\therefore \quad \varphi\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=0}^{\infty} \varphi(E_{i+1} \setminus E_i)$$

$$= \lim_{n\to\infty} S_n \qquad (2)$$

The partial sums are:

$$
\begin{aligned}
S_n &= \sum_{i=0}^{n} \varphi(E_{i+1} \setminus E_i) \\
&= \cancel{\varphi(E_1)} + \cancel{\varphi(E_2)} - \cancel{\varphi(E_1)} \\
&\quad + \varphi(E_3) - \cancel{\varphi(E_2)} \\
&\quad + \dots + \varphi(E_{n+1}) - \varphi(E_n) \\
&= \varphi(E_{n+1}) \; ; \quad \text{since } \varphi(E_{i+1} \setminus E_i) = \varphi(E_{i+1}) - \varphi(E_i)
\end{aligned}
$$

From (2) we conclude:

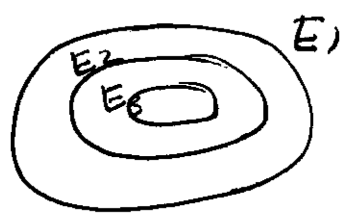
$$
\begin{aligned}
\varphi\Big(\bigcup_{i=1}^{\infty} E_i\Big) &= \lim_{n \to \infty} \varphi(E_{n+1}) \\
&= \lim_{i \to \infty} \varphi(E_i) \, . \qquad \blacksquare
\end{aligned}
$$

(iii) We have

$E_1 \supset E_2 \supset E_3 \supset \dots\dots$

$\varphi(E_{i_0}) < \infty$, for some i_0.

By replacing E_i with $E_i \cap E_{i_0}$, if necessary, we can assume $\varphi(E_1) < \infty$.



From (ii):

$$\varphi\left(\bigcup_{i=1}^{\infty}(E_1\setminus E_i)\right)=\lim_{i\to\infty}\varphi(E_1\setminus E_i); \quad E_1\setminus E_i\subset E_1\setminus E_{i+1}$$

$$=\lim_{i\to\infty}\varphi(E_1)-\varphi(E_i); \quad \text{since } \varphi(E_i)<\infty$$

$$=\varphi(E_1)-\lim_{i\to\infty}\varphi(E_i)$$

$$\therefore \boxed{\varphi\left(\bigcup_{i=1}^{\infty}E_1\setminus E_i\right)=\varphi(E_1)-\lim_{i\to\infty}\varphi(E_i)} \quad (3)$$

Note that:

$$\bigcup_{i=1}^{\infty}E_1\setminus E_i = E_1\setminus\bigcap_{i=1}^{\infty}E_i$$

Indeed:

$$x\in\bigcup_{i=1}^{\infty}E_1\setminus E_i \iff x\in E_1\setminus E_i, \text{ for some } i$$

$$\iff x\in E_1 \text{ and } x\notin E_i$$

$$\iff x\in E_1 \text{ and } x\notin\bigcap_{i=1}^{\infty}E_i$$

$$\iff x\in E_1\setminus\bigcap_{i=1}^{\infty}E_i$$

From (3): $\boxed{4.11}$

$$\varphi(E_1) - \lim_{i\to\infty} \varphi(E_i) = \varphi\left(\bigcup_{i=1}^{\infty} (E_1 \setminus E_i)\right)$$
$$= \varphi\left(E_1 \setminus \bigcap_{i=1}^{\infty} E_i\right)$$
$$= \varphi(E_1) - \varphi\left(\bigcap_{i=1}^{\infty} E_i\right); \text{ since } \varphi\left(\bigcap_{i=1}^{\infty} E_i\right) < \infty$$

We conclude:

$$\varphi\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} \varphi(E_i) \quad \blacksquare$$