Lesson 5

Recall the definition of σ-algebra.

<u>Def</u>: A nonempty collection Σ of sets $E \subset X$ satisfying the following two conditions is called a σ-algebra:

(i) $E \in \Sigma \Rightarrow \quad \in \Sigma$

(ii) $\bigcup_{i=1}^{\infty} E_i \in \Sigma$ provided each $E_i \in \Sigma$

We now introduce the concept of Borel-sets:

<u>Def</u>: In a topological space, the elements of the smallest σ-algebra that contains all open sets are called <u>Borel sets</u>. The term "smallest" is taken in the sense of inclusion. In exercise 4.4, you will show that such a smallest σ-algebra exists.

We will use the following definitions:

<u>Regular outer measure</u>: This is an outer measure φ on a set X that satisfies: for each $A \subset X$, there exists a φ-measurable set B, $A \subset B$ such that $\varphi(A) = \varphi(B)$.

<u>Borel regular outer measure</u>: This a regular outer measure with the additional property that B in the definition above can be taken as a Borel set (assuming that X is a topological space).

<u>Borel outer measure</u>: This is an outer measure φ on a topological space X where all Borel sets are φ-measurable

<u>Finite Borel outer measure</u>: This is a Borel outer measure φ such that $\varphi(X) < \infty$.

Ex., Let X be an arbitrary set. Define, for every $E \subset X$,

$$\varphi(E) = \begin{cases} 1 & E \neq \phi \\ 0 & E = \phi \end{cases}$$

φ is an outer measure

(i) $\varphi(\phi) = 0$ ✓

(ii) $0 \leq \varphi(E) \leq \infty$ ✓

(iii) $E_1 \subset E_2 \Rightarrow \varphi(E_1) \leq \varphi(E_2)$ ✓

(iv) If $\{E_i\}$ is any countable collection of sets in X

If one of the sets E_i is not empty then:

$$\varphi\left(\bigcup_{i=1}^{\infty} E_i\right) = 1 \leq \sum_{i=1}^{\infty} \varphi(E_i)$$

If $E_i = \phi \ \forall i$, then

$$\varphi\left(\bigcup_{i=1}^{\infty} E_i\right) = 0 = \sum_{i=1}^{\infty} \varphi(E_i)$$

From theorem 80.1 we obtain that ϕ and X are measurable.

Let $E \subset X$, $E \neq \emptyset, X$.

Let $P = E$, $Q = E^c$

$\Rightarrow$ $E \neq \emptyset$ and $E^c \neq \emptyset$

$\Rightarrow$ $\varphi(E \cup E^c) = 1 \neq \varphi(E) + \varphi(E^c) = 2$

$\Rightarrow$ Only $\emptyset$ and X are measurables.

We would like to have outer measures with a rich suply of measurable sets.

Section 4.2
Carathéodory Outer measure

Def: An outer measure φ defined on a metric space (X, ρ) is called a Carathéodory outer measure if:

$$\varphi(A \cup B) = \varphi(A) + \varphi(B)$$

whenever A, B are arbitrary subsets of X with

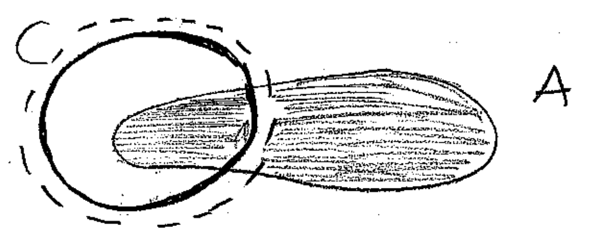
$$d(A,B) > 0$$

Recall: $d(A,B) = \inf\{\rho(a,b) : a \in A, b \in B\}$

<u>Thm</u>: If φ is a Carathéodory outer measure on a metric space X, then all closed sets are φ-measurable.

<u>Proof</u>: Let $C \subset X$ be a closed set. Let A be an arbitrary set. If $\varphi(A) = \infty$ the result is clear, so we assume $\varphi(A) < \infty$. We need to prove:

$$\varphi(A) \geq \varphi(A \cap C) + \varphi(A \setminus C)$$



Consider, for each $j = 1, 2, \dots$, the sets

$$C_j = \left\{x : d(x, C) \leq \frac{1}{j}\right\}$$

Since

$$\varphi((A \setminus C_j) \cup (A \cap C)) = \varphi(A \setminus C_j) + \varphi(A \cap C)$$

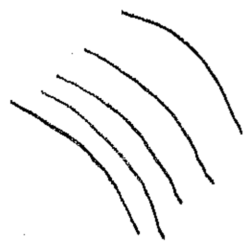
$$\Rightarrow \varphi(A) \geq \varphi(A \setminus C_j) + \varphi(A \cap C)$$

We now show that:

$$\lim_{j\to\infty} \varphi(A\setminus C_j) = \varphi(A\setminus C)$$

Define:

$$T_i = A \cap \left\{x : \frac{1}{i+1} < d(x, C) \le \frac{1}{i}\right\}$$



We have:

$$A\setminus C = (A\setminus C_j) \cup \left(\bigcup_{i=j}^{\infty} T_i\right) \quad \forall j$$

$x \in A\setminus C$

$\iff x \notin C$

$\iff d(x, C) > 0$ (Since C is closed)

$\iff \exists i$ s.t. $\frac{1}{i+1} < d(x, C) \le \frac{1}{i}$

$i \ge j$ or $d(x, C) > \frac{1}{j}$

$\iff x \in (A\setminus C_j) \cup \left(\bigcup_{i=j}^{\infty} T_i\right)$

$$\Rightarrow \varphi(A\setminus C) \le \varphi(A\setminus C_j) + \sum_{i=j}^{\infty} \varphi(T_i)$$

(We have to use subadditivity since we don't know if the sets C_j, T_i are measurable).

Since φ is a <u>Carathéodory</u> Outer measure and $d(T_i, T_j) > 0$ if $|i-j| \ge 2$. Thus

$$\sum_{i=1}^{m} \varphi(T_{2i}) = \varphi\left(\bigcup_{i=1}^{m} T_{2i}\right) \le \varphi(A) < \infty,$$

$$\sum_{i=1}^{m} \varphi(T_{2i-1}) = \varphi\left(\bigcup_{i=1}^{m} T_{2i-1}\right) \le \varphi(A) < \infty.$$

$$\Rightarrow \sum_{i=1}^{\infty} \varphi(T_i) < \infty$$

$$\Rightarrow \lim_{j\to\infty} \left(\sum_{i=j}^{\infty} \varphi(T_i) \right) = 0$$

$$\Rightarrow \varphi(A\setminus C) \le \limsup_{j\to\infty} \varphi(A\setminus C_j) + 0$$

and, since

$$A \setminus C_j \subset A \setminus C$$

$$\Rightarrow \varphi(A \setminus C_j) \le \varphi(A \setminus C)$$

$$\Rightarrow \limsup_{j\to\infty} \varphi(A \setminus C_j) \le \varphi(A \setminus C)$$

$$\Rightarrow \varphi(A \setminus C) = \limsup_{j\to\infty} \varphi(A \setminus C_j)$$

Now, from

$$\varphi(A) \ge \varphi(A \setminus C_j) + \varphi(A \cap C)$$

$$\varphi(A) - \varphi(A \cap C) \ge \varphi(A \setminus C_j)$$

$$\varphi(A) - \varphi(A \cap C) \ge \limsup_{i\to\infty} \varphi(A \setminus C_j)$$

$$\varphi(A) - \varphi(A \cap C) \ge \varphi(A \setminus C)$$

$$\Rightarrow \boxed{\varphi(A) \ge \varphi(A \setminus C) + \varphi(A \cap C)}$$

<u>Note</u>: Actually, since $\{\varphi(A \setminus C_j)\}$ is an increasing sequence of numbers, then $\lim_{j\to\infty} \varphi(A \setminus C_j)$ exists and we have shown $\varphi(A \setminus C) = \lim_{j\to\infty} \varphi(A \setminus C_j)$. So it is OK if we replace limsup by lim in proof.

__Def:__ In a topological space, the elements of the smallest σ-algebra that contains all open sets are called __Borel sets__.

__Thm:__ If φ is a Carathéodory outer measure on a metric space X, then the Borel sets of X are φ-measurable

__Proof:__ The set of all measurable sets $\mathcal{M}$ is a σ-algebra that contains all closed sets (and hence all open sets). Thus $\mathcal{M}$ contains all Borel sets.