Lesson 6.

We have the following:

<u>Theorem 1</u>: Suppose $\mathcal{B}$ is a family of subsets of a topological space X that contains all open and all closed subsets of X. Suppose also that $\mathcal{B}$ is closed under countable unions and countable intersections. Then $\mathcal{B}$ contains all Borel sets.

<u>Proof</u>: Let

$$\mathcal{H} = \mathcal{B} \cap \{A : A^c \in \mathcal{B}\}$$

Note:

* $\mathcal{H}$ contains all closed sets

Let C a closed set. Then $\underline{C \in \mathcal{B}}$. Since C^c is open, then $C^c \in \mathcal{B}$ and hence $\underline{C \in \{A : A^c \in \mathcal{B}\}}$. Thus $C \in \mathcal{H}$

* $E \in \mathcal{H} \Rightarrow E^c \in \mathcal{H}$.

Let $E \in \mathcal{H}$. Then $E \in \mathcal{B}$ and $\underline{E^c \in \mathcal{B}}$. Now $(E^c)^c = E \in \mathcal{B}$ and hence $\underline{E^c \in \{A : A^c \in \mathcal{B}\}}$. Thus, $E^c \in \mathcal{H}$

- $\{E_i\} \in \mathcal{H} \Rightarrow \bigcup_{i=1}^{\infty} E_i \in \mathcal{H}$

Since $\mathcal{B}$ is closed under countable unions we obtain

$$\underline{\bigcup_{i=1}^{\infty} E_i \in \mathcal{B}} . \qquad (1)$$

Since $E_i \in \mathcal{H}$, then $E_i^c \in \mathcal{B}$. Since $\mathcal{B}$ is closed under countable intersection we have:

$$\bigcap_{i=1}^{\infty} E_i^c \in \mathcal{B}$$

$$\therefore \left(\bigcup_{i=1}^{\infty} E_i\right)^c \in \mathcal{B}$$

Hence:

$$\underline{\bigcup_{i=1}^{\infty} \{E_i\} \in \{A : A^c \in \mathcal{B}\}} \qquad (2)$$

From (1) and (2):

$$\bigcup_{i=1}^{\infty} E_i \in \mathcal{H}$$

We have shown that $\mathcal{H}$ is a σ-algebra that contains all open sets. Let $\mathcal{B}_s$ be the smallest σ-algebra that contains all open sets.

Recall that the elements of $\mathcal{B}_S$ are exactly all the Borel sets.

Since,

$$\mathcal{B}_s = \bigcap_{\substack{\mathcal{F} \text{ is } \sigma\text{-algebra} \\ O \in \mathcal{F},\ \forall O \text{ open}}} \mathcal{F} \longrightarrow (3)$$

we obtain

$$\mathcal{B}_s \subset \mathcal{H}$$

that is:

$$\mathcal{B}_s \subset \mathcal{B},$$

and we conclude that $\mathcal{B}$ contains all Borel sets.

<u>Corollary</u>: Let φ be a Caratheodory outer measure on a metric space X, then the Borel sets of X are φ-measurable.

<u>Proof</u>: Let $\mathcal{M}$ be the set of all φ-measurable sets. $\mathcal{M}$ satisfies the hypothesis of Theorem 1 and hence $\mathcal{B}_s \subset \mathcal{M}$.

We can also obtain this Corollorary directly from (3), since we proved earlier that $\mathcal{M}$ is a σ-algebra and hence is one of the $\mathcal{F}$'s in (3).

4.3 Lebesgue Measure.

We want to define a Carathéodory outer measure in $\mathbb{R}^n$.

Consider closed n-dimensional intervals.

$$I = \{x : a_i \le x_i \le b_i, \ i = 1, 2, \ldots, n\}.$$

and their volumes

$$v(I) = \prod_{i=1}^{n} (b_i - a_i)$$

$$\Rightarrow I = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$$
$$= I_1 \times I_2 \times \cdots \times I_n.$$

Notice that n-dimensional intervals have their edges parallel to the coordinate axes of $\mathbb{R}^n$.

Def: If $E \subset \mathbb{R}^n$ define

$$\lambda^*(E) = \inf \left\{ \sum_{I \in S} v(I) \right\}$$

where the infimum is taken over all countable collections S of closed intervals I such that

$$E \subset \bigcup_{I \in S} I$$

Thm: For a closed interval $I \subset \mathbb{R}^n$, $\lambda^*(I) = v(I)$.

Proof:

Cleary: $\lambda^*(I) \leq v(I)$

(By taking $S = \{I\}$)

Let $\varepsilon > 0$.

Let $\{I_k\}_{k=1}^{\infty}$ be a sequence of closed intervals such that

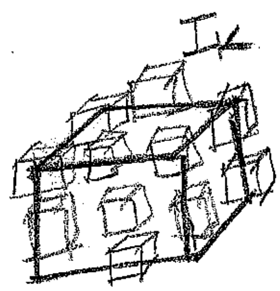
$$I \subset \bigcup_{k=1}^{\infty} I_k \qquad \sum_{k=1}^{\infty} v(I_k) < \lambda^*(I) + \varepsilon$$

(6.6)

For each K, let J_K an interval whose interior contains I_K and

$$v(J_K) \leq v(I_K) + \frac{\varepsilon}{2^K}$$

$$\Rightarrow \sum_{K=1}^{\infty} v(J_K) \leq \sum_{K=1}^{\infty} v(I_K) + \varepsilon$$



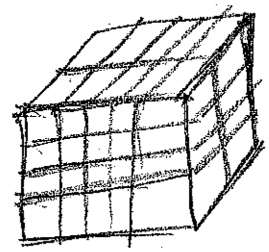
$I_K \subset \text{Int}(J_K)$

$$\Rightarrow I \subset \bigcup_{K=1}^{\infty} \text{Int}(J_K)$$

<u>Note</u>: (Problem 3.35). For each open cover $\mathcal{F}$ of a compact set in a metric space, $\exists \eta > 0$ s.t.

$x, y \in X$, $\rho(x,y) < \eta \Rightarrow \exists V \in \mathcal{F}$ s.t $x, y \in V$

Let η be the Lebesgue number for $\{\text{Int}(J_K) : K \in \mathbb{N}\}$

I 

Partition I in to subintervals
$K_1, K_2, \ldots, K_m$, diam $(K_i) < \eta$
s.t.

$$I = \bigcup_{i=1}^{m} K_i \quad \text{and} \quad v(I) = \sum_{i=1}^{m} v(K_i)$$

Each K_i is contained in the interior of some J_K, same J_{K_i}, although more than one K_i may belong to the same J_{K_i}. Let N_m be the smallest number of the J_{K_i}'s that contain the K_i's, we have $N_m \leq m$ and.

$$v(I) = \sum_{i=1}^{m} v(K_i) \leq \sum_{i=1}^{N_m} v(J_{K_i}) \leq \sum_{i=1}^{\infty} v(J_K)$$

$$\Rightarrow \quad v(I) \leq \sum_{i=1}^{\infty} v(I_k) + \varepsilon$$

$$\leq \lambda^*(I) + 2\varepsilon$$

Since ε is arbitrary:

$$v(I) \leq \lambda^*(I) \quad \blacksquare$$

<u>Thm:</u> λ^* is a Carathéodory outer measure.

<u>Proof:</u> We proof first that λ^* is an outer measure:

(i) $\lambda^*(\emptyset) = 0$ ✓

(ii) $0 \leq \lambda^*(E) \leq \infty$ ✓

(iii) $E_1 \subset E_2 \Rightarrow \lambda^*(E_1) \leq \lambda^*(E_2)$

Because:

$$E_2 \subset \bigcup_{k=1}^{\infty} I_k \Rightarrow E_1 \subset \bigcup_{k=1}^{\infty} I_k$$

and hence:

$$\left\{ \sum_{I \in S} v(I) : S = \cup I \text{ covers } E_2 \right\} \subset \left\{ \sum_{I \in S} v(I) : S = \cup I \text{ covers } E_1 \right\}$$

Hence the infimum of the left, $\lambda^*(E_1)$ is less o equal than the infimum on the right, $\lambda^*(E_2)$.