$$\Rightarrow v(I) \leq \sum_{i=1}^{\infty} v(I_k) + \varepsilon$$

$$\leq \lambda^*(I) + 2\varepsilon$$

Since ε is arbitrary:

$$v(I) \leq \lambda^*(I) \quad \blacksquare$$

<u>Thm:</u> λ^* is a Carathéodory outer measure.

<u>Proof:</u> We proof first that λ^* is an outer measure:

(i) $\lambda^*(\emptyset) = 0$ ✓

(ii) $0 \leq \lambda^*(E) \leq \infty$ ✓

(iii) $E_1 \subset E_2 \Rightarrow \lambda^*(E_1) \leq \lambda^*(E_2)$

Because:

$$E_2 \subset \bigcup_{k=1}^{\infty} I_k \Rightarrow E_1 \subset \bigcup_{k=1}^{\infty} I_k$$

and hence:

$$\left\{ \sum_{I \in S} v(I) : S = \cup I \text{ covers } E_2 \right\} \subset \left\{ \sum_{I \in S} v(I) : S = \cup I \text{ covers } E_1 \right\}$$

Hence the infimum of the right, $\lambda^*(E_1)$ is less o equal than the infimum on the left $\lambda^*(E_2)$.

(iv) Let $\{A_i\}$ be a countable collection of arbitrary sets in $\mathbb{R}^n$ and let

$$A = \bigcup_{i=1}^{\infty} A_i$$

If $\lambda^*(A_j) = \infty$ for some j the clearly

$$\lambda^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \lambda^*(A_i)$$

We now assume $\lambda^*(A_i) < \infty \ \forall i$.

Let $\varepsilon > 0$.

By definition of λ^*, for every $i = 1, 2, \ldots$,

$$A_i \subset \bigcup_{j=1}^{\infty} I_j^{(i)}$$

$$\sum_{j=1}^{\infty} v(I_j^{(i)}) \leq \lambda^*(A_i) + \frac{\varepsilon}{2^i}$$

$$\Rightarrow \lambda^*(A) \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} v(I_j^{(i)})$$

$$\leq \sum_{i=1}^{\infty} \left(\lambda^*(A_i) + \frac{\varepsilon}{2^i}\right) = \sum_{i=1}^{\infty} \lambda^*(A_i) + \varepsilon.$$

$\blacksquare$

We now verify that λ^* is Carathéodory.
Let $A, B \subset \mathbb{R}^n$ arbitrary sets with

$$d(A,B) > 0.$$

Clearly:

$$\lambda^*(A \cup B) \leq \lambda^*(A) + \lambda^*(B).$$

We need to prove:

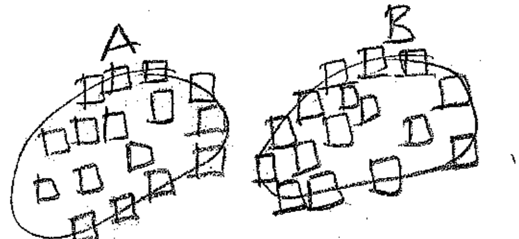
$$\lambda^*(A \cup B) \geq \lambda^*(A) + \lambda^*(B).$$

Let $\varepsilon > 0$. Then:

$$A \cup B \subset \bigcup_{k=1}^{\infty} I_k$$

such that:

$$\sum_{k=1}^{\infty} v(I_k) \leq \lambda^*(A \cup B) + \varepsilon,$$



Subdivide each I_k, if necessary, into smaller intervals so that

$$\operatorname{diam}(I_k) < d(A,B)$$

$$\Rightarrow \lambda^*(A) + \lambda^*(B) \leq \sum_{k=1}^{\infty} v(I_k') + \sum_{k=1}^{\infty} v(I_k'')$$

$$= \sum_{k=1}^{\infty} v(I_k)$$

$$\leq \lambda^*(A \cup B) + \varepsilon$$

Since ε is arbitrary we conclude:

$$\lambda^*(A) + \lambda^*(B) \leq \lambda^*(A \cup B). \blacksquare$$

Define

$$\lambda = \lambda^*|_{\mathcal{M}}$$

where $\mathcal{M}$ is the sets of all Lebesgue measurable sets.

λ is called the Lebesgue measure.

$\mathcal{M}$ is "large". It contains all closed set, open sets and Borel sets.

<u>Thm</u>: The following five conditions are equivalent for Lebesgue outer measure, λ^*, on $\mathbb{R}^n$

(i) $E \subset \mathbb{R}^n$ is λ^*-measurable

(ii) For every $\varepsilon > 0$, $\exists U$ open

$$E \subset U$$

such that $\lambda^*(U \setminus E) < \varepsilon$

(iii) There is a G_δ set $U \supset E$ such that $\lambda^*(U \setminus E) = 0$

(iv) For each $\varepsilon > 0$, $\exists$ F closed

$$F \subset E$$

such that $\lambda^*(E \setminus F) < \varepsilon$

(v) There is a F_σ set F such that $\lambda^*(E \setminus F) = 0$

Proof:

(i) $\Rightarrow$ (ii)

Assume first $\lambda(E) < \infty$.

Let $\varepsilon > 0$. Then there exists a countable collection:

$$E \subset \bigcup_{k=1}^{\infty} I_k$$

$$\sum_{k=1}^{\infty} v(I_k) < \lambda^*(E) + \frac{\varepsilon}{2}$$

For each k, let I_k' be an open interval:

$$I_k \subset I_k' \quad \text{and} \quad v(I_k') < v(I_k) + \frac{\varepsilon}{2^{k+1}}$$

Def $\quad U = \bigcup_{k=1}^{\infty} I_k'$

$$\Rightarrow \lambda(U) \leq \sum_{k=1}^{\infty} v(I_k') \leq \sum_{k=1}^{\infty} v(I_k) + \frac{\varepsilon}{2}$$

$$< \lambda^*(E) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$\Rightarrow \lambda^*(U \setminus E) = \lambda^*(U) - \lambda^*(E) < \varepsilon \quad \blacksquare$$

If $\lambda(E) = \infty$,

Def.

$$E_i = E \cap B(i)$$

$\Rightarrow \exists\, U_i \supset E_i$ s.t

$$\lambda(U_i \setminus E_i) < \frac{\varepsilon}{2^i}$$

Def: $U = \bigcup_{i=1}^{\infty} U_i$

$$\Rightarrow \quad U \setminus E = \bigcup_{i=1}^{\infty} (U_i \setminus E_i)$$

$$\Rightarrow \lambda(U \setminus E) \leq \sum_{i=1}^{\infty} \lambda(U_i \setminus E_i)$$

$$< \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} = \varepsilon$$

$$\Rightarrow \quad \lambda(U \setminus E) < \varepsilon. \quad \blacksquare$$

(ii) $\Rightarrow$ (iii)

For each i, $\exists U_i$ open $U_i \supset E$

$$\lambda^*(U_i \setminus E) < \frac{1}{i}$$

Def $\quad U = \bigcap_{i=1}^{\infty} U_i$

$$\Rightarrow \lambda^*(U \setminus E) = \lambda^*\left[\bigcap_{i=1}^{\infty}(U_i \setminus E)\right]$$

$$= \lim_{i\to\infty} \lambda^*(U_i \setminus E)$$

$$\leq \lim_{i\to\infty} \frac{1}{i} = 0.$$

(iii) $\Longrightarrow$ (i)

We have $E = U \setminus (U \setminus E)$

U is measurable since U is G_δ (a countable intersection of open sets, which are measurable).

$U \setminus E$ is measurable because $\lambda^*(U \setminus E) = 0$

$\Rightarrow$ E is measurable

(i) $\Rightarrow$ (iv)

(iv) $\Rightarrow$ (v)

(v) $\Rightarrow$ (i)