Lesson 8

The Cantor Set

Let $E_0 = [0,1]$.

Remove $(\frac{1}{3}, \frac{2}{3})$ and let

$$E_1 := [0, \tfrac{1}{3}] \cup [\tfrac{2}{3}, 1].$$

Remove the middle thirds of these intervals, and let:

$$E_2 := [0, \tfrac{1}{9}] \cup [\tfrac{2}{9}, \tfrac{3}{9}] \cup [\tfrac{6}{9}, \tfrac{7}{9}] \cup [\tfrac{8}{9}, 1].$$

Continuining in this way we obtain a sequence of compact sets E_n such that

(a) $E_1 \supset E_2 \supset E_3 \supset \cdots$

(b) E_n is the union of 2^n intervals, each of length $\frac{1}{3^n}$.

Define:

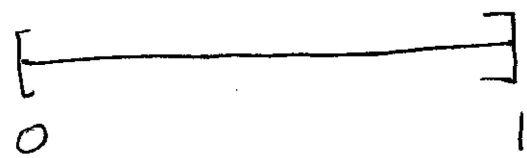
$$\boxed{C := \bigcap_{n=1}^{\infty} E_n}$$

$C \neq \phi$ since it contains $\{0,1\}$.

C is closed since it is the countable intersection of closed sets. C is also bounded. Hence $\underline{C \text{ is compact}}$.

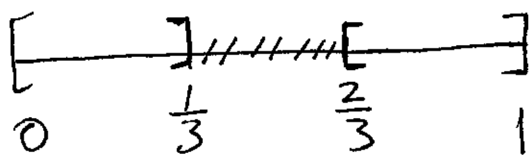
The following pictures show the sets E_n.

$E_0 =$



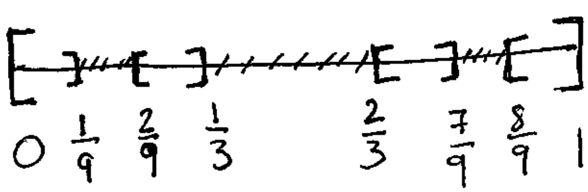
2^0 intervals

$E_1 =$



2^1 intervals of leng. $\frac{1}{3}$

$E_2 =$



2^2 intervals of length $\frac{1}{3^2}$

$\vdots$

Since each E_n is clearly measurable, the cantor set C is also measurable, as it is the countable intersection of measurable sets.

Remark: Recall that λ^* is a Caratheodory outer measure and hence all closed sets are measurable. Thus, every interval in E_n is measurable, which implies that E_n is measurable.

Since $\lambda(E_1) < \infty$ we have:

$$\lambda(C) = \lambda\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \lambda(E_n) = \lim_{n\to\infty} \frac{1}{3^n} \cdot 2^n = \lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0$$

$$\therefore \boxed{\lambda(C) = 0}$$

<u>Claim</u>: C is no-where dense in $[0,1]$.

In order to see this, we note that $\overline{C} = C$ (since C is closed) and hence $\lambda(\overline{C}) = \lambda(C) = 0$. Thus $\overline{C}$ can not contain any open interval, since this would imply $\lambda(\overline{C}) > 0$. Hence $\overline{C}$ is no-where dense in $[0,1]$.

<u>Remark</u>: Note that we have shown above that C can not contain any interval (otherwise we would have $\lambda(C) > 0$).

<u>Def</u>: Let X be a metric space. We say that $E \subset X$ is a perfect set if E is closed and if every point of E is a limit point of E

<u>Ex</u>: $[0,1] \cup \{2\}$ is not perfect

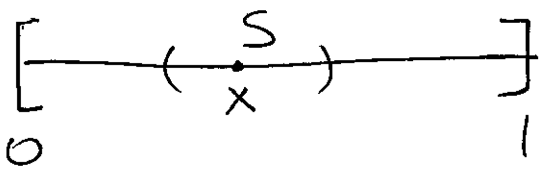
$\mathbb{R}^2$ is perfect

Any closed interval $I \subset \mathbb{R}^n$ is perfect

<u>Thm 1</u>: The Cantor set C is perfect.

<u>Proof</u>: We know that C is closed. We need to show that <u>every point of C is a limit point</u>.

Let $x \in C$ and let S be any open interval such that $x \in S$.



Let I_n be the interval of E_n which contains x (recall that $x \in \bigcap_{n=1}^{\infty} E_n$). For n large enough, $I_n \subset S$. Hence $(S \setminus \{x\}) \cap C \neq \emptyset$, and thus x is a limit point of C.

Since $x \in C$ was arbitrary we conclude that C is perfect.

<u>Thm 2</u>: Let P be a nonempty perfect set in $\mathbb{R}^n$. Then P is uncountable.

<u>Proof</u>: Since P has limit points, then P must be an infinite set. We proceed by contradiction and assume that P is countable; i.e:

$$P = \{x_1, x_2, x_3, \dots\}$$

Consider $B_{r_1}(x_1) = \{y \in \mathbb{R}^n : |y - x| < r_1\} := V_1$

Note that, since x_1 is a limit point, $B_{r_1}(x_1) \cap P$ contains an infinite number of points of P. We don't know where x_2 is, but clearly there exists $V_2 := B_{r_2}(z)$, for some $z \in P$, such that:

(i) $\bar{V}_2 \subset V_1$

(ii) $x_1 \notin \bar{V}_2$

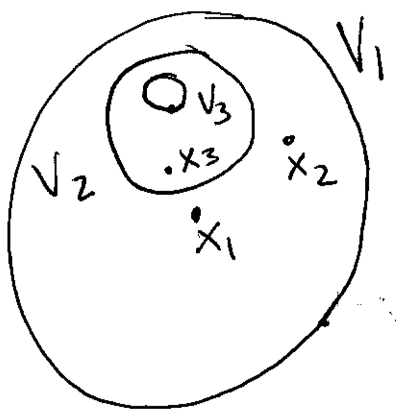
(iii) $V_2 \cap P \neq \emptyset$ (since z is a limit point, this intersection has actually an infinite number of points in P).

We proceed inductevely in this way to construct a sequence of open balls $\{V_n\}$ satisfying:

(i) $\overline{V_{n+1}} \subset V_n$

(ii) $x_n \notin \overline{V_{n+1}}$

(iii) $V_{n+1} \cap P \neq \emptyset$ (actually containing an infinite number of elements of P).



Define:

$$K_n := \overline{V_n} \cap P$$

K_n is compact since it is closed and bounded

Note that $x_n \notin K_{n+1}$ implies:

$$\bigcap_{n=1}^{\infty} K_n = \emptyset \quad (1)$$

On the other hand,

$K_n \neq \emptyset$ and $K_1 \supset K_2 \supset K_3 \supset \cdots$

Therefore:

$$\bigcap_{n=1}^{\infty} K_n \neq \emptyset,$$

which contradicts (1).

Corollary: Thm 1 and Thm 2 imply that the Cantor set C is uncountable.

Note, C is uncountable and yet $\lambda(C)=0$!

Remark: Theorem 2 is true when X is a metric space, if we assume that P is compact. In this case the closed sets $\{K_n\}$ are also compact. Instead of open balls, the sets $\{V_n\}$ are open neighborhoods of the form $N_r(x) = \{y \in X : d(y,x) < r\}$.