

## Section 8.3

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Ex: Faraday's Law.

Vector Calculus  $\rightarrow$  Electromagnetism $\vec{E}$ : Time-dependant electric vector field,  $\vec{E}(x, y, z, t)$ . $\vec{H}$ : Time-dependant magnetic field,  $\vec{H}(x, y, z, t)$ .

$$\int_C \vec{E} \cdot d\vec{r} = \text{Voltage around } C.$$

$$\iint_S \vec{H} \cdot \vec{n} dS = \text{magnetic flux across } S.$$

Remember other notations:

$$\int_C \vec{E} \cdot d\vec{r} \quad \text{or} \quad \int_C \vec{E} \cdot \vec{T} ds$$

$$\iint_S \vec{H} \cdot \vec{n} dS \quad \text{or} \quad \iint_S \vec{H} \cdot d\vec{S}$$

Faraday's law: Voltage around  $C$  equals the negative rate of change of magnetic flux through  $S$ :



(2)

Ex: Show that Faraday's law follows from one of Maxwell's equations.

$$\nabla \times \vec{E} = -\frac{\partial \vec{H}}{\partial t} \leftarrow \text{Maxwell eq.}$$

Answer:  $\int_C \vec{E} \cdot d\vec{r} = \iint_S (\nabla \times \vec{E}) \cdot \vec{n} \, dS$

$$= -\iint_S \frac{\partial \vec{H}}{\partial t} \cdot \vec{n} \, dS$$

$$= -\frac{\partial}{\partial t} \iint_S \vec{H} \cdot \vec{n} \, dS.$$

$$\int_C \vec{E} \cdot d\vec{r} = -\frac{\partial}{\partial t} \iint_S \vec{H} \cdot \vec{n} \, dS$$

Faraday's law

Recall:

$\vec{F}$  is a gradient vector field (or conservative vector field) if

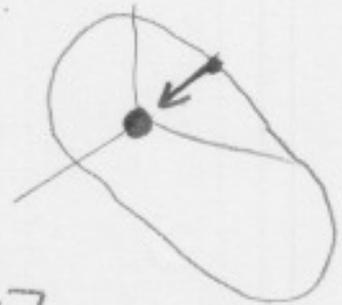
$$\vec{F} = \nabla f$$

for some function  $f$ , called a potential for  $\vec{F}$ .

Ex:  $V = \frac{GmM}{r}$ ,  $r = (x^2 + y^2 + z^2)^{1/2}$   
 $V: \mathbb{R}^3 \rightarrow \mathbb{R}$  Gravitational potential energy.

$$\vec{F} = \nabla V$$

$$\vec{F} = - \left( \frac{GmM}{r^3} \right) (x, y, z).$$



$$\frac{\partial V}{\partial x} = \frac{\partial}{\partial x} \left[ GmM (x^2 + y^2 + z^2)^{-1/2} \right]$$

$$= - \frac{GmM}{r^3} (x^2 + y^2 + z^2)^{-3/2} (2x)$$

$$= -GmM \frac{x}{r^3}$$

$$\frac{\partial V}{\partial y} = -GmM \frac{y}{r^3}$$

$$\frac{\partial V}{\partial z} = -GmM \frac{z}{r^3}$$

$$\nabla V = -GmM \left( \frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3} \right) = \vec{F}$$

Thm:  $\vec{F} = \nabla f$  if and only if  $\nabla \times \vec{F} = \vec{0}$ . (4)

If the exam, we showed that:

\* If  $\vec{F} = \nabla f$  then  $\nabla \times \vec{F} = \vec{0}$ .

$$P \Rightarrow Q$$

\* For proof of  $Q \Rightarrow P$ , see book.

Ex: Consider

$$\vec{F}(x, y, z) = (y, z \cos yz + x, y \cos yz).$$

Show that  $\vec{F}$  is irrotational and find a scalar potential for  $\vec{F}$ .

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z \cos yz + x & y \cos yz \end{vmatrix}$$

$$= \vec{i} (\cancel{\cos yz} - yz \cancel{\sin yz}) - \vec{j} (0 - 0) + \vec{k} (\cancel{1} - \cancel{1})$$

$$= \vec{i} (0) + \vec{j} (0) + \vec{k} (0) = \vec{0}$$

$$= (0, 0, 0) = \vec{0}$$

(5)

$$\vec{F}(x, y, z) = (y, z \cos yz + x, y \cos yz).$$

By thm,  $\exists f: \mathbb{R}^3 \rightarrow \mathbb{R}$  s.t.

$$\vec{F} = \nabla f.$$

$$\frac{\partial f}{\partial x} = y \quad \frac{\partial f}{\partial y} = z \cos yz + x \quad \frac{\partial f}{\partial z} = y \cos yz.$$

$$f(x, y, z) = \int y dx = xy + h_1(y, z)$$

$$\frac{\partial f}{\partial x} = y + 0 \quad \checkmark$$

$$\frac{\partial f}{\partial y} = x + \frac{\partial h_1}{\partial y} = x + z \cos yz$$

$$= \frac{\partial h_1}{\partial y} = z \cos yz.$$

$$h_1(y, z) = \int z \cos yz dy$$

$$= \sin yz + h_2(z)$$

$$f(x, y, z) = xy + \sin yz + h_2(z).$$

$$\frac{\partial f}{\partial y} = x + z \cos yz + 0 \quad \checkmark$$

$$\frac{\partial f}{\partial z} = y \cos yz + h_2'(z) = y \cos yz$$

$$h_2'(z) = 0, \quad h_2(z) = C.$$

$$f(x, y, z) = xy + \sin yz + C \quad \checkmark.$$