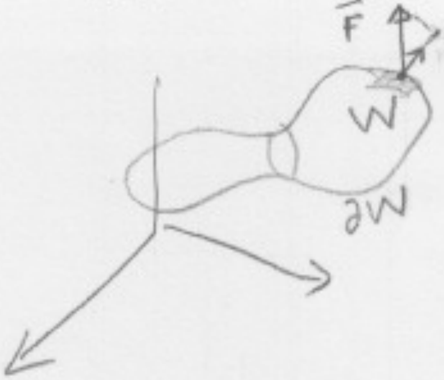
Remarks on Divergence Theorem.



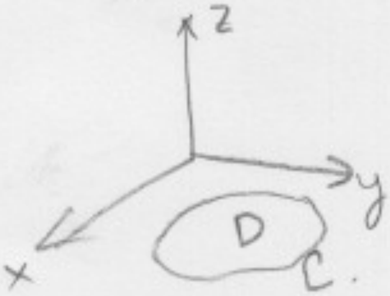
$\vec{F}$

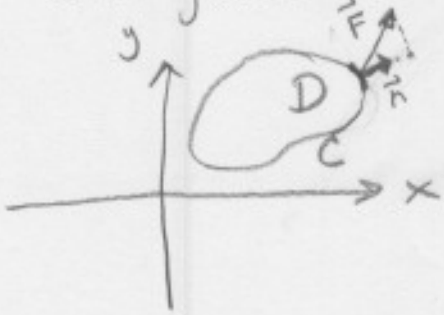
$$\iint_{\partial W} \vec{F}\cdot\vec{n}\, dS = \iiint_{W} \operatorname{div} \vec{F}\, dx\,dy\,dz$$

$$\frac{m}{sec} \quad m^2 = \frac{m^3}{sec}$$

Divergence theorem is true in any dimension

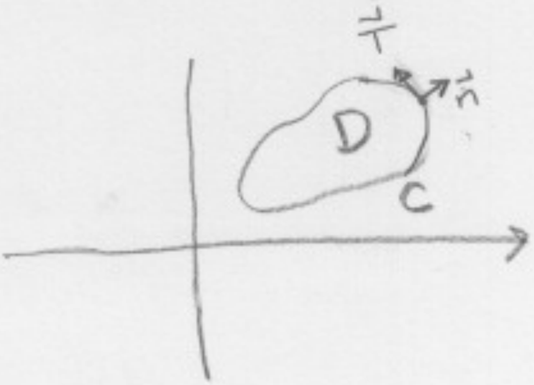
2-dimensional divergence theorem.





$$\int_C \vec{F}\cdot\vec{n}\, \underline{ds} = \iint_D \operatorname{div} \vec{F}\, dx\,dy$$

<u>Ex</u>: Use the Green's theorem to obtain the 2-dimensional divergence theorem.



$\vec{r}(t) = (x(t), y(t)), \quad a \le t \le b$

$\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} \quad \vec{F}(x,y) = (P(x,y), Q(x,y))$

$$\vec{n} = \frac{(y'(t),\ -x'(t))}{\sqrt{(x'(t))^2 + (y'(t))^2}}$$

$$\int_C \vec{F}\cdot\vec{n}\,ds = \int_a^b \left(P(x(t),y(t)),\, Q(x(t),y(t))\right)\cdot \frac{(y'(t),-x'(t))}{\cancel{\sqrt{(x')^2+(y')^2}}}\cdot \cancel{\sqrt{(x')^2+(y')^2}}\,dt$$

$$= \int_a^b P(x(t),y(t))\,y'(t)\,dt - Q(x(t),y(t))\,x'(t)\,dt.$$

$$= \int_a^b P\frac{dy}{dt}\,dt - Q\frac{dx}{dt}\,dt$$

$$= \int_C -Q\,dx + P\,dy\,,$$ in the notation of Green's theorem

Form $\vec{G}(x,y) := (-Q, P)$

Apply Green's theorem to $\vec{G}$.

by Green's theorem

$$= \int_C \vec{G}\cdot\vec{T}\,ds.$$

$$= \iint_D (\nabla\times\vec{G})\cdot\vec{n}\,dS$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -Q & P & 0 \end{vmatrix} = \vec{i}(0) - \vec{j}(0) + \vec{k}\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right).$$

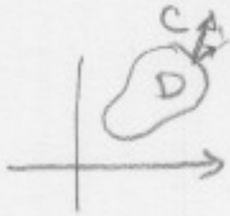
$$= \iint_D \left(0, 0, \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)\cdot(0,0,1)\,dx\,dy$$

$$= \iint_D \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\,dx\,dy.$$

$$= \iint_D \operatorname{div} F\,dx\,dy\,,$$ Recall $\vec{F} = (P, Q)$.

$$\Rightarrow \quad \int_C \vec{F}\cdot\vec{n}\,ds = \iint_D \operatorname{div} F$$

Ex: Suppose $u(x,y)$ that is harmonic in D with boundary C, which is simple closed curve,



$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{in } D.$$

If u is a potential for a vector field $\vec{F}$ (i.e. $\vec{F} = \nabla u$), show that its flux accross D must be 0.

Sol:
$$\int_C \vec{F} \cdot \vec{n}\, ds = \iint_D \operatorname{div} \vec{F}\, dx\,dy \quad ; \text{ by 2-d divergence theorem}$$

$$= \iint_D \operatorname{div}(\nabla u)\, dx\,dy$$

$$= \iint_D \Delta u\, dx\,dy = 0$$