Applications of Stokes' theorem and Divergence Theorem.

Conservation Laws.

Let $\vec{v}$ be a C^1 vector field, where now $\vec{v}(t, x, y, z)$ may depend upon time as well as position. Let $\rho(t, x, y, z)$ be the mass density of a fluid having velocity $\vec{v}$. Conservation of mass says that, for any region Ω, the rate of change of total mass in Ω equals the rate at which mass flows in to Ω.

$$\frac{d}{dt} \iiint_{\Omega} \rho \, dx\,dy\,dz = -\iint_{\partial\Omega} \vec{J}\cdot\vec{n}\, dS.$$

$$J = \rho \vec{v}$$

$$\iiint_{\Omega} \frac{\partial \rho}{\partial t}\, dx\,dy\,dz = -\iiint_{\Omega} \operatorname{div} \vec{J}\, dx\,dy\,dz.$$

$$\iiint_{\Omega} \left(\frac{\partial \rho}{\partial t} + \operatorname{div} \vec{J}\right) dx\,dy\,dz = 0.$$

Since this is true for every Ω, it must be that the integrand is 0; that is,

$$\boxed{\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \vec{v}) = 0}$$ ← one of the Euler's equation

In the special case when $\rho =$ constant, $\frac{\partial \rho}{\partial t} = 0$, we have $\operatorname{div}(\rho \vec{v}) = 0$, again, since ρ is constant, the equation simplifies to:

$$\boxed{\operatorname{div} \vec{v} = 0}$$ (incompressible fluid)

Finally, suppose that besides $\rho =$ constant, the flow is irrotational; i.e, $\nabla \times \vec{v} = 0$.

$$\nabla \times \vec{v} = 0 \implies \vec{v} = \nabla \varphi.$$

Thus, $\operatorname{div} \vec{v} = 0$ simplifies to:

$$\operatorname{div}(\nabla \varphi) = 0$$

$$\therefore \quad \Delta \varphi = 0,$$

Hence, φ is a harmonic function.

Heat equation

Let $T(t,x,y,z)$ be a C^2 function giving temperature. Then $\nabla \vec{T}$ points in the direction of increasing temperature, so the vector giving heat flow would be

$$\vec{F} = -\nabla \vec{T}$$

If ρ_0 (constant) is the mass density and c is the specific heat, then $c\rho_0 T$ gives the energy per unit volume, and the energy flux is $\vec{J} = k\vec{F}$, where K is the conductivity.

Using conservation of energy as with conservation of mass, for arbitrary Ω,

$$\frac{d}{dt}\iiint_{\Omega} c\rho_0 T \, dx\,dy\,dz = -\iint_{\partial\Omega} \vec{J}\cdot\vec{n}\, dS.$$

$$\iiint_{\Omega} c\rho_0 \frac{\partial T}{\partial t}\, dx\,dy\,dz = -\iiint_{\Omega} k \operatorname{div} \vec{F}\, dx\,dy\,dz$$

$$\iiint_{\Omega} c\rho_0 \frac{\partial T}{\partial t} + K \operatorname{div}(-\nabla T)\; dx\,dy\,dz = 0.$$

$$c\rho_0 \frac{\partial T}{\partial t} = K \operatorname{div}(\nabla T) \Rightarrow \boxed{\frac{\partial T}{\partial t} = \frac{K}{c\rho_0}\Delta T} \text{ heat equation}$$

If T is time indep. $\Rightarrow \Delta T = 0$

Maxwell's equations

$\vec{E}$ electric field

$\vec{H}$ magnetic field.

$\vec{J}_c$ conduction current density.

ρ charge density

(1) $\operatorname{div} \vec{E} = \rho$. (Gauss' law).

(2) $\operatorname{div} \vec{H} = 0$ (no magnetic sources)

(3) $\nabla \times \vec{E} + \frac{\partial \vec{H}}{\partial t} = \vec{0}$ (Faraday's law)

(4) $\nabla \times \vec{H} - \frac{\partial \vec{E}}{\partial t} = \vec{J}_c$ (Ampere's law).

(1), (2), (3), (4) $\longrightarrow$ Wave equation

$$\frac{\partial^2 \phi}{\partial t^2} = \Delta \phi$$

Partial Differential equations (ODE)

Isaac Newton: "All in nature reduces to differential equations".

Max Planck: "Present day physics, as far as it is theoretically organized, is completely governed by a system of space-time differential equations".