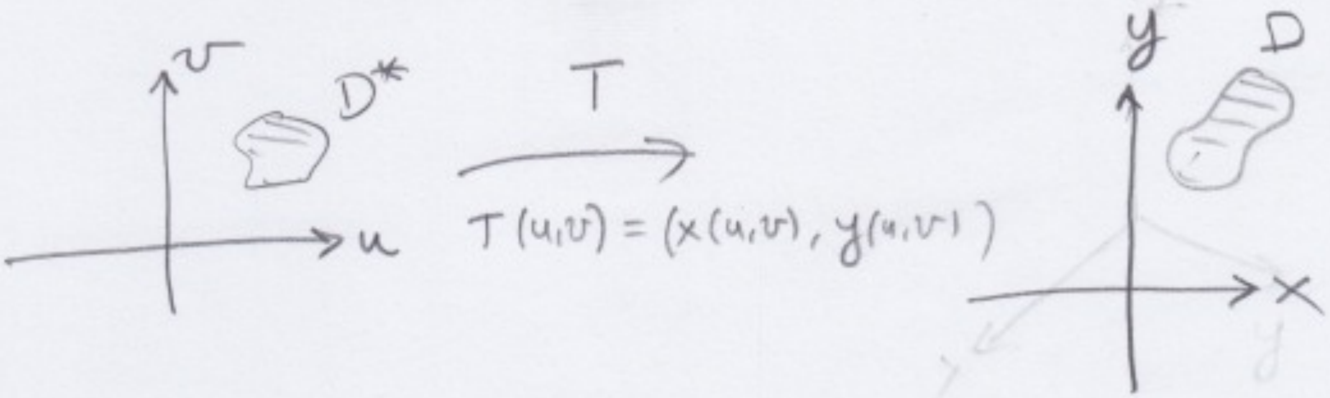
Lesson Wednesday March 25

Change of variable formula

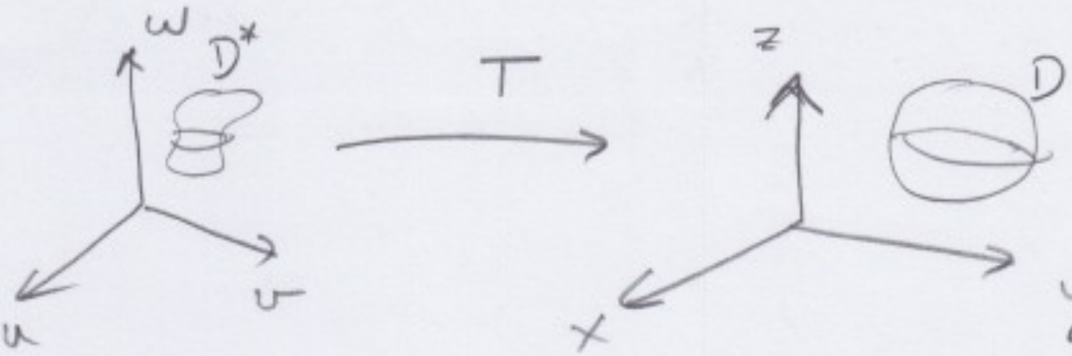


$$T(u,v) = (x(u,v), y(u,v))$$

$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

absolute value.

$$\frac{\partial(x,y)}{\partial(u,v)} = \det(DT) = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$



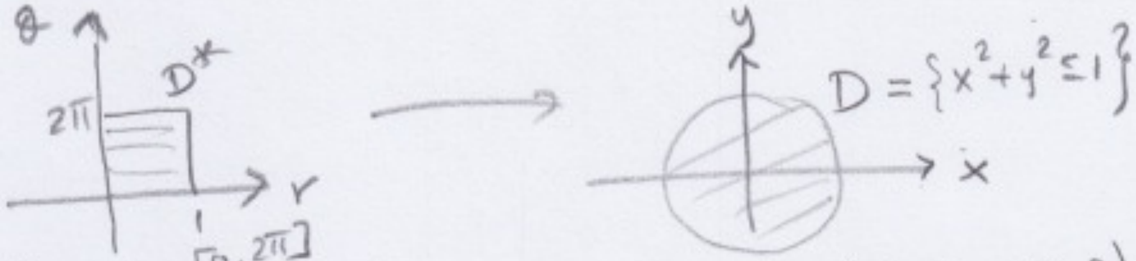
$$T(u,v,w) = (x(u,v,w), y(u,v,w), z(u,v,w))$$

$$\iiint_D f(x,y,z)\,dx\,dy\,dz = \iiint_{D^*} f(x(u,v,w), y(u,v,w), z(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

absolute value

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det(DT)$$

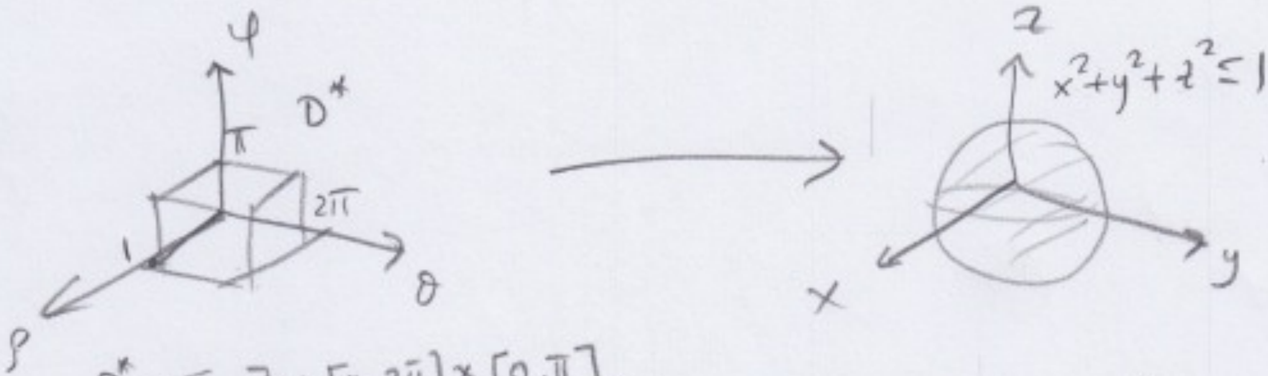
Polar Coordinates and spherical coordinates are a particular case of change of variable formula



$D^* = [0,1]\times[0,2\pi]$ $\quad T(r,\theta) = (x(r,\theta), y(r,\theta)) = (r\cos\theta, r\sin\theta)$.

$$\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = r$$

$$\iint_D f(x,y)\,dx\,dy = \int_0^1\int_0^{2\pi} f(r\cos\theta, r\sin\theta)\, r\, d\theta\, dr.$$



$D^* = [0,1]\times[0,2\pi]\times[0,\pi]$.

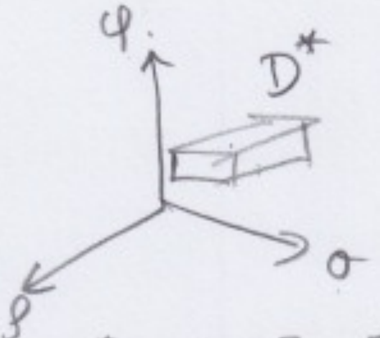
$T(\rho,\theta,\varphi) = (\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)$

$$\left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)}\right| = \rho^2\sin\varphi$$

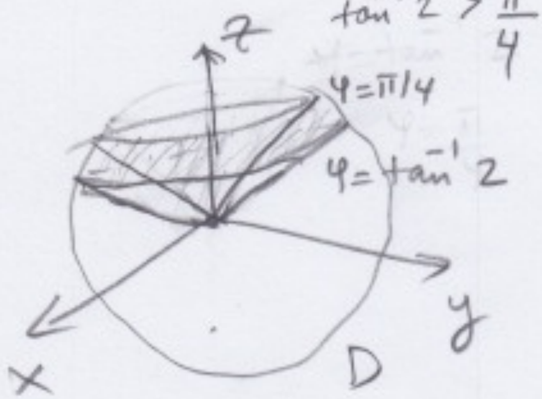
$$\iiint_D f(x,y,z)\,dx\,dy\,dz = \int_0^1\int_0^{2\pi}\int_0^{\pi} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\,\rho^2\sin\varphi\, d\varphi\, d\theta\, d\rho$$

Ex: $\iiint_D f(x,y,z)\,dx\,dy\,dz \qquad f(x,y,z) = \sqrt{x^2+y^2+z^2}$

where D is the region in first octant bounded by cones $\varphi = \frac{\pi}{4}$ and $\varphi = \tan^{-1} 2$ and the sphere $\rho = \sqrt{6}$



$\xrightarrow{T}$



$$D^* = \underset{\rho}{[0,\sqrt{6}]} \times \underset{\theta}{\left[0, \tfrac{\pi}{2}\right]} \times \underset{\varphi}{\left[\tfrac{\pi}{4}, \tan^{-1} 2\right]}$$

$$T(\rho,\theta,\varphi) = (\rho \sin\varphi \cos\theta,\ \rho \sin\varphi \cos\theta,\ \rho\cos\varphi).$$

$$\iiint_D f\,dV = \int_{\pi/4}^{\tan^{-1}2} \int_0^{\pi/2} \int_0^{\sqrt{6}} \rho \cdot \rho^2 \sin\varphi \, d\rho \, d\theta . \, d\varphi$$

$$= \int_{\pi/4}^{\tan^{-1}2} \int_0^{\pi/2} \sin\varphi \left[\frac{\rho^4}{4}\right]_0^{\sqrt{6}} d\theta \, d\varphi$$

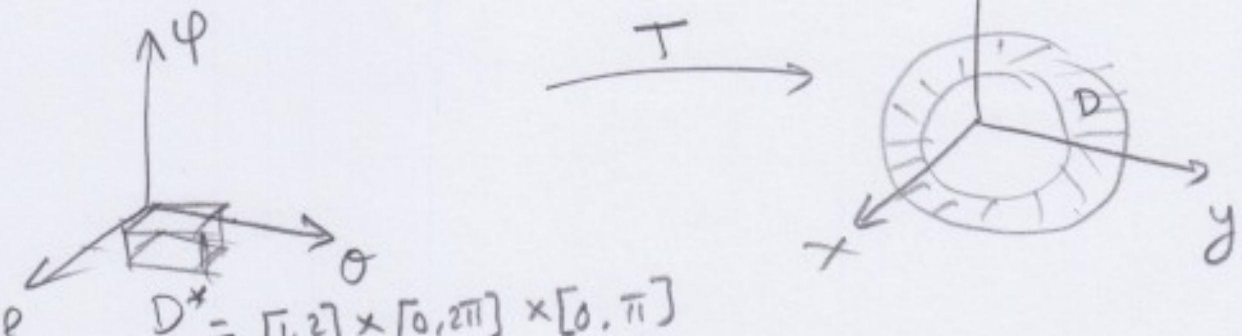
$$= \int_{\pi/4}^{\tan^{-1}2} 9 \sin\varphi \cdot \frac{\pi}{2} \, d\varphi$$

$$= \frac{9}{2}\pi \left[-\cos\varphi\right]_{\pi/4}^{\tan^{-1}2} = \frac{9}{2}\pi \left[\underbrace{-\cos(\tan^{-1}2) + \frac{1}{\sqrt{2}}}_{>0.}\right]$$

Ex: Evaluate.

$$\iiint_D \sqrt{x^2+y^2+z^2}\; e^{-(x^2+y^2+z^2)}\, dx\,dy\,dz$$

where D is the region between the two spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=4$



$$D^* = \underset{\rho}{[1,2]} \times \underset{\theta}{[0,2\pi]} \times \underset{\varphi}{[0,\pi]}$$

$$\iiint_D f\,dV = \int_0^{2\pi}\int_0^{\pi}\int_1^2 \rho\, e^{-\rho^2} \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

$$= \int_0^{2\pi} \underbrace{\int_0^{\pi}\int_1^2 \rho^3\, e^{-\rho^2} \rho^2 \underline{\sin\varphi}\, d\rho\, \underline{d\varphi}}\, d\theta$$

$$= 2\pi \left(\int_0^{\pi} \sin\varphi\, d\varphi\right)\left(\int_1^2 \rho^3 e^{-\rho^2} d\rho\right)$$

$$= 2\pi \left[-\cos\varphi\right]_0^{\pi} \int_1^2 \rho^3 e^{-\rho^2} d\rho$$

$$= 4\pi \underline{\int_1^2 \rho^3 e^{-\rho^2} d\rho}$$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

$$\int_1^2 \rho^3 e^{-\rho^2} d\rho = -\frac{1}{2}\int_1^2 \underset{u}{\rho^2}\, \underset{dv}{d\left(e^{-\rho^2}\right)}$$

$$= -\frac{1}{2}\left[\left\{\rho^2 e^{-\rho^2}\right\}_1^2 - \int_1^2 2\rho\, e^{-\rho^2} d\rho\right]$$

$$= -\frac{1}{2}\left(4e^{-4} - e^{-1} + \left[e^{-\rho^2}\right]_1^2\right) = 4\pi e^{-1} - 10\pi e^{-4}$$

Change of variables - Cylindrical coordinates.

$$T(r,\theta,z) \longrightarrow (x,y,z)$$

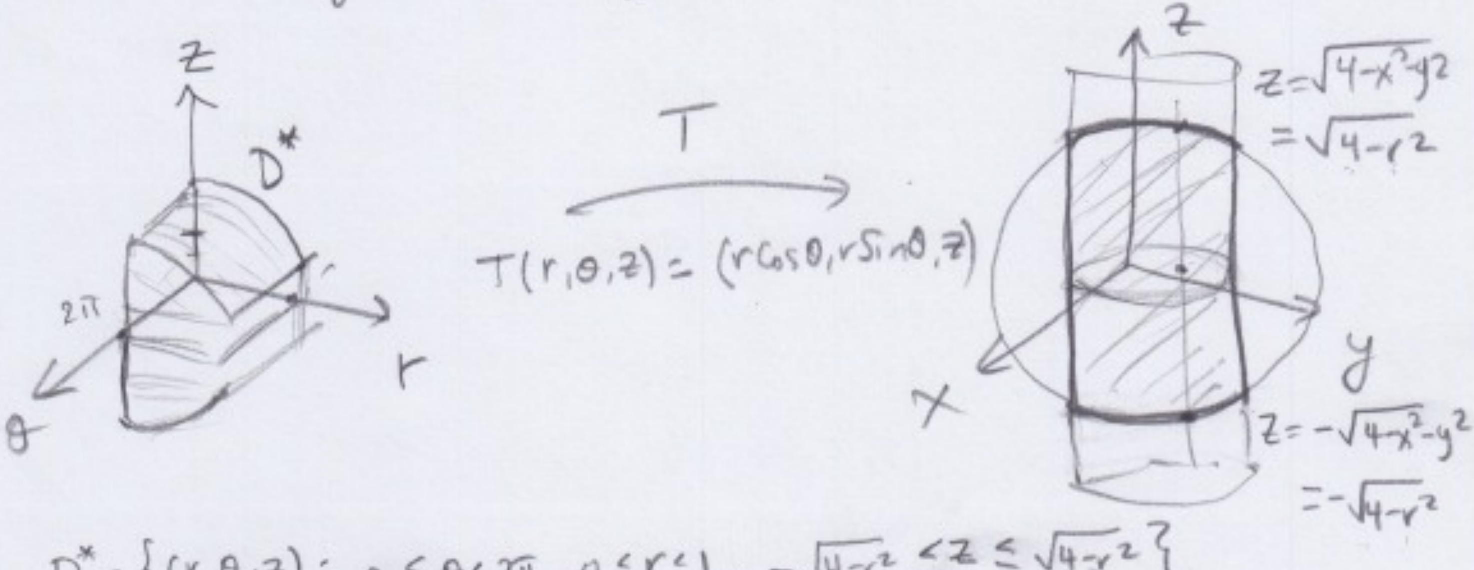
$$x = r\cos\theta,\ y = r\sin\theta,\ z = z$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \det(DT) = \det\begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r$$

$$\iiint_D f(x,y,z)\,dx\,dy\,dz = \iiint_{D^*} f(r\cos\theta, r\sin\theta, z)\, r\ dr\,d\theta\,dz$$

Ex: Integrate $f(x,y,z) = (x^2+y^2)z^2$ over the part of the cylinder $x^2+y^2 \le 1$ inside the sphere $x^2+y^2+z^2=4$



$$D^* = \{(r,\theta,z) : 0 \le \theta \le 2\pi,\ 0 \le r \le 1,\ -\sqrt{4-r^2} \le z \le \sqrt{4-r^2}\}$$

$$\iiint_D (x^2+y^2) z^2 \, dx\, dy\, dz$$

$$= \int_0^{2\pi} \int_0^1 \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r^2 z^2 \cdot r \, dz \, dr \, d\theta$$

$$= 2\pi \int_0^1 r^3 \left[\frac{z^3}{3}\right]_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} dr.$$

$$= 2\pi \int_0^1 r^3 \, \frac{2}{3} (4-r^2)^{3/2} dr.$$

$$= \frac{4}{3}\pi \int_0^1 \underline{r^3 (4-r^2)^{3/2}} \, dr \qquad \boxed{\int u\, dv = uv - \int v\, du}$$

$$= \frac{4}{3}\pi \int_0^1 \underset{u}{r^2} \, \underset{dv.}{d\left[-\frac{1}{5}(4-r^2)^{5/2}\right]}$$

$$= \frac{4}{15}\pi \left[\left[r^2 \left(-(4-r^2)^{5/2}\right)\right]_0^1 + \int_0^1 2r \, (4-r^2)^{5/2} dr \right]$$

$$= \frac{4}{15}\pi \left\{ -3^{5/2} - \left[\frac{2}{7} (4-r^2)^{7/2}\right]_0^1 \right\}$$

$$= \frac{4}{15}\pi \left\{ \frac{2}{7} 4^{7/2} - \frac{2}{7} 3^{7/2} - 3^{5/2} \right\} \cong 6.385$$