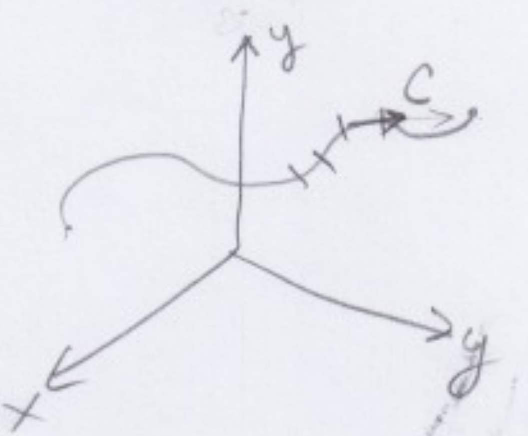
Section 7.2

Office hours: Send me an e-mail and we can set up a skype conversation.



$$\vec{r}(t) = (x(t), y(t), z(t))$$

$$\int_C f\,ds \qquad f: \mathbb{R}^3 \to \mathbb{R}$$

$$\int_C \vec{F}\cdot d\vec{r} \qquad \vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$$

$$a \le t \le b$$

f is density of wire $\Rightarrow \int_C f\,ds$ is mass of wire

$\vec{F}$ is a force $\Rightarrow \int_C \vec{F}\cdot d\vec{r}$ is the work done by particle

In practice:

$$\int_C f\,\underline{ds} = \int_a^b f(\vec{r}(t))\,\underbrace{\|\vec{r}\,'(t)\|\,dt}$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}\,\|\vec{r}\,'(t)\|\,dt$$

(2)

Ex: Evaluate $\int_C \vec{F} \cdot d\vec{r}$ → line integral when

$\vec{F} = (x, y, z^2)$ and $\vec{r}(t) = (\cos t, \sin t, t)$.

$0 \le t \le \pi$

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{\vec{r}\,'(t)}{\cancel{\|\vec{r}\,'(t)\|}} \cancel{\|\vec{r}\,'(t)\|}\, dt$$

$$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_0^\pi (\cos t, \sin t, t^2) \cdot (-\sin t, \cos t, 1)\, dt$$

$$= \int_0^\pi (\cancel{-\sin t \cos t} + \cancel{\sin t \cos t} + t^2)\, dt$$

$$= \int_0^\pi t^2\, dt = \left[\frac{1}{3} t^3\right]_0^\pi = \frac{\pi^3}{3}$$

<u>Ex</u>: Compute work done on a unit mass by force $\vec{F} = (x, y, -z)$ on curve C given by $\vec{r}(t) = (\cos t, \sin t, \sqrt{t}\sin t)$
$0 \le t \le 2\pi$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \frac{\vec{r}\,'(t)}{\cancel{\|\vec{r}\,'(t)\|}} \cancel{\|\vec{r}\,'(t)\|}\, dt$$

$$= \int_0^{2\pi} (\cos t, \sin t, -\sqrt{t}\sin t) \cdot \left(-\sin t, \cos t, \sqrt{t}\cos t + \frac{1}{2\sqrt{t}}\sin t\right) dt$$

$$= \int_0^{2\pi} \left(\cancel{-\sin t\cos t + \sin t\cos t} - t\sin t\cos t - \frac{1}{2}\sin^2 t\right) dt$$

$$\int_0^{2\pi} t\sin t\cos t\, dt = \cancel{\frac{t}{2}\sin^2 t\Big]_0^{2\pi}}^{0} - \int_0^{2\pi} \frac{1}{2}\sin^2 t\, dt$$

$u = t \qquad dv = \sin t\cos t\, dt$

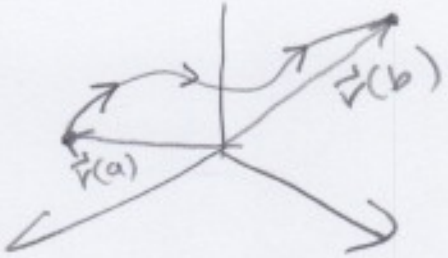
$du = dt \qquad v = \frac{1}{2}\sin^2 t$

$$\int u\, dv = uv - \int v\, du$$

$$= 0$$

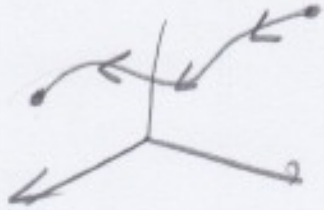
$$\int_C \vec{F}\cdot d\vec{r} = 0$$

Note that $\vec{F}$ is a gradient vector field since $\vec{F} = \nabla\left(\frac{1}{2}x^2 + \frac{1}{2}y^2 - \frac{1}{2}z^2\right)$



$\vec{r}(t)$ has an orientation

$a \le t \le b$



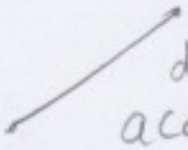
$$\int_C f\,ds = \int_a^b f(\vec{r}(t))\,\|\vec{r}\,'(t)\|\,dt$$

orientation does not matter.

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$$

orientation matters.

<u>Remark</u>: These integrals do not change under <u>re</u>parametrizations.

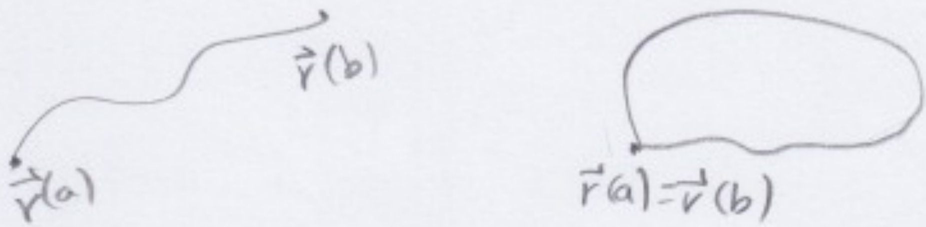


different parametrization according to different speeds

Line integrals of gradient vector fields.

If $\vec{F} = \nabla f$ for some f the we have the formula

$$\int_C \vec{F} \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$



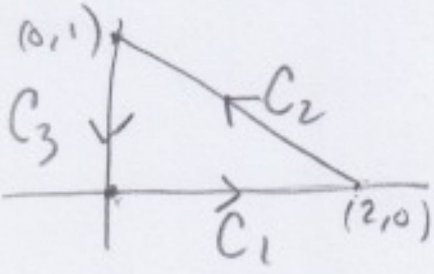
$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{\vec{r}\,'(t)}{\cancel{\|\vec{r}\,'(t)\|}} \cancel{\|\vec{r}\,'(t)\|}\, dt$$

$$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt = \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_a^b \frac{d}{dt} f(\vec{r}(t))\, dt \quad ; \text{ by chain rule}$$

$$= f(\vec{r}(b)) - f(\vec{r}(a)) \quad ; \quad \text{FTC } \int_a^b f'(t)\, dt = f(b) - f(a)$$

Ex: Let T be the triangle



$\vec{F} = (2xy, x^2)$

$C = C_1 \cup C_2 \cup C_3$

Compute $\int_C \vec{F} \cdot d\vec{r}$

Solution 1: $\vec{F} = \nabla(x^2 y)$ and C is closed,

$$\int_C \vec{F} \cdot d\vec{r} = 0$$

Solution 2: By hand. We break C into 3 integrals.

$$\int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} + \int_{C_3} \vec{F} \cdot d\vec{r}$$

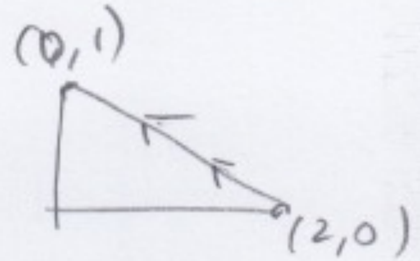
C_1: $\vec{r}(t) = (2t, 0)$, $0 \le t \le 1$

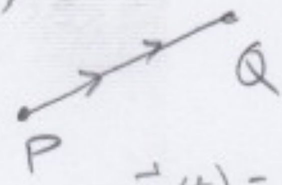
$$\int_0^1 (0, 4t^2) \cdot (2, 0)\, dt = 0$$

C_3: $\vec{r}(t) = (0, 1-t)$, $0 \le t \le 1$

$$\int_0^1 (0, 0) \cdot (0, -1)\, dt = 0$$

C_2 :





$\vec{r}(t) = P + t(Q-P)$

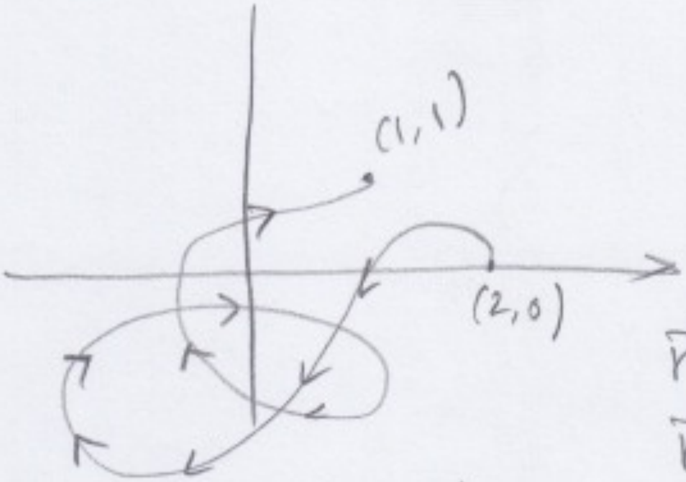
$\vec{r}(1) = \cancel{P} + Q - \cancel{P} = Q$

$$\vec{r}(t) = (2,0) + t\left[(0,1) - (2,0)\right]$$
$$= (2,0) + t(-2,1)$$
$$= (2-2t,\ t)$$

$$\int_0^1 \left(2t(2-2t), (2-2t)^2\right) \cdot (-2, 1)\, dt$$
$$= \int_0^1 \left[-4t(2-2t) + (2-2t)^2\right] dt$$
$$= \int_0^1 -8t + 8t^2 + 4(1-t)^2 \quad dt$$
$$= \left[-4t^2 + \frac{8}{3}t^3 - \frac{4}{3}(1-t)^3\right]_0^1$$
$$= -4 + \frac{8}{3} + \frac{4}{3} = -4 + \frac{12}{3} = -4+4 = 0$$

$$\Rightarrow \int_C \vec{F} \cdot d\vec{r} = 0$$

Ex: $\vec{F} = (2xy, x^2)$ and C as in picture.



$\vec{r}(b) = (1,1)$
$\vec{r}(a) = (2,0)$

Compute $\int_C \vec{F} \cdot d\vec{r}$

$$\vec{F} = \nabla(x^2y) = (2xy, x^2) \quad f(x,y) = x^2y$$

$$\int_C \vec{F} \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$
$$= f(1,1) - f(2,0)$$
$$= 1 - 0 = 1 \quad \blacksquare$$

Please e-mail me to let me know if you can see the short video on my website and/or YouTube

Office hours: Skype