Section 1.3:

Matrices, determinants and cross product.

In order to define the cross product, we first review how to compute the determinant of a matrix.

2x2 matrices

Consider the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

note curly brackets to represent a matrix

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

To simplify notation, we replace "det" with two vertical lines:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Note: Recall the multiplication of matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

3x3 matrices:

For 3x3, we usually evaluate det A by "cofactors".

Ex:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

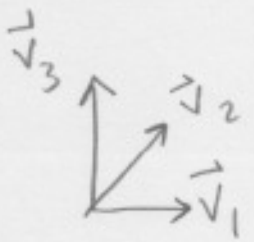
$$= 1\,(45-48) - 2\,(36-42) + 3\,(32-35)$$

$$= -3 + 12 - 9 = 0$$

The 3 columns (or rows) of the matrix are linearly dependent. Hences, they can not span $\mathbb{R}^3$.

3 vectors in $\mathbb{R}^3$ are linearly independent if and only if the determinant of the matrix formed with the 3 vectors is <u>nonzero</u>.

In this case, the 3 linearly independent vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ span $\mathbb{R}^3$; that is, any



vector $\vec{v}$ in $\mathbb{R}^3$ can be writen as a linear combination $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$, for some numbers c_1, c_2, c_3.

Cross product:

Definition: Let $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$. Then the cross product of $\vec{a}$ and $\vec{b}$ is defined as:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \vec{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \vec{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \vec{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= \vec{i}\,(a_2 b_3 - a_3 b_2) - \vec{j}\,(a_1 b_3 - a_3 b_1) + \vec{k}\,(a_1 b_2 - a_2 b_1)$$

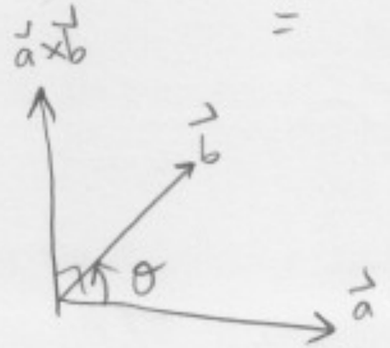
Notice that $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

Ex: Find $(3\vec{i} - \vec{j} + \vec{k}) \times (\vec{i} + 2\vec{j} - \vec{k})$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & 1 \\ 1 & 2 & -1 \end{vmatrix} = \vec{i} \begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} - \vec{j} \begin{vmatrix} 3 & 1 \\ 1 & -1 \end{vmatrix} + \vec{k} \begin{vmatrix} 3 & -1 \\ 1 & 2 \end{vmatrix}$$

$$= \vec{i}\,(1-2) - \vec{j}\,(-3-1) + \vec{k}\,(6+1)$$

$$= -\vec{i} + 4\vec{j} + 7\vec{k}$$



To see that $\vec{a} \times \vec{b}$ is orthogonal to the plane spanned by $\vec{a}$ and $\vec{b}$ we compute first the formula $(\vec{a}\times\vec{b})\cdot\vec{c}$

Let $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$, $\vec{c} = (c_1, c_2, c_3)$

$$(\vec{a}\times\vec{b})\cdot\vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \cdot \vec{c}$$

$$= \left(\vec{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \vec{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \vec{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right) \cdot (c_1, c_2, c_3)$$

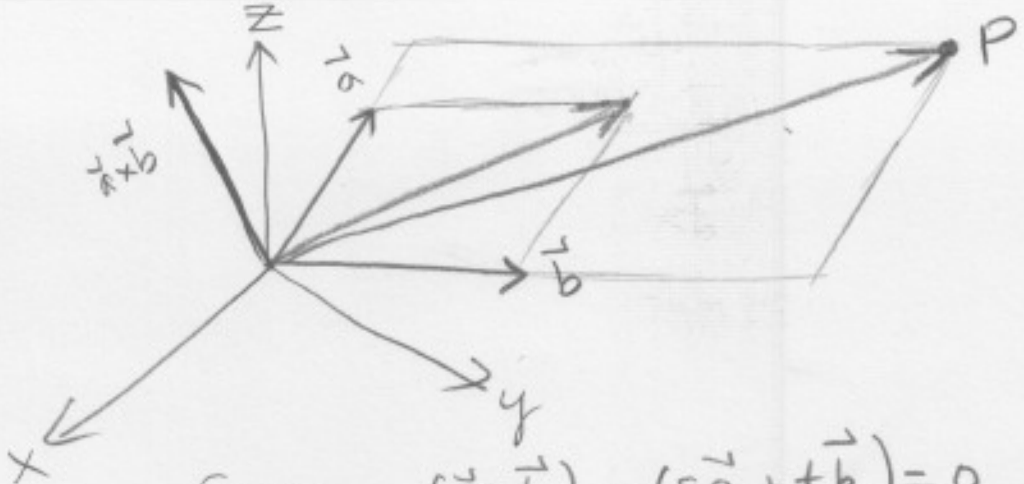
$$= c_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + c_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

* If $\vec{c} = \vec{a}$ then $(\vec{a}\times\vec{b})\cdot\vec{c} = 0$
* If $\vec{c} = \vec{b}$ then $(\vec{a}\times\vec{b})\cdot\vec{c} = 0$
* If $\vec{c} = s\vec{a} + t\vec{b}$ then $(\vec{a}\times\vec{b})\cdot\vec{c} = 0$.

Recall that a point P belongs to the plane spanned by $\vec{a}$ and $\vec{b}$ if and only if P is the tip of a vector of the form:

$s\vec{a} + t\vec{b}$, where s, t are any numbers.



Since $(\vec{a}\times\vec{b})\cdot(s\vec{a}+t\vec{b})=0$ we conclude that $\vec{a}\times\vec{b}$ is orthogonal to the plane spanned by $\vec{a}$ and $\vec{b}$.

Compute $\|\vec{a}\times\vec{b}\|$

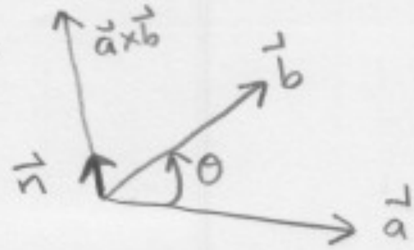
$$\|\vec{a}\times\vec{b}\|^2 = (a_2b_3-a_3b_2)^2+(a_1b_3-a_3b_1)^2+(a_1b_2-a_2b_1)^2$$
$$= (a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)-(a_1b_1+a_2b_2+a_3b_3)^2$$
$$= \|\vec{a}\|^2\|\vec{b}\|^2-(\vec{a}\cdot\vec{b})^2$$
$$= \|\vec{a}\|^2\|\vec{b}\|^2-\|\vec{a}\|^2\|\vec{b}\|^2\cos^2\theta$$
$$= \|\vec{a}\|^2\|\vec{b}\|^2\sin^2\theta$$

$$\Rightarrow \|\vec{a}\times\vec{b}\| = \|\vec{a}\|\,\|\vec{b}\|\,|\sin\theta|$$

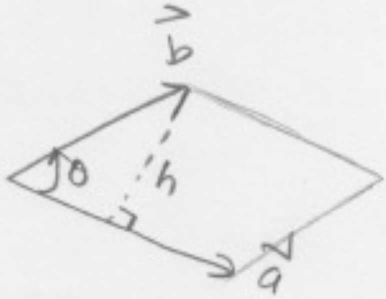
Therefore:

$$\boxed{\vec{a}\times\vec{b} = \|\vec{a}\|\,\|\vec{b}\|\,|\sin\theta|\,\vec{n}}$$

where $\vec{n}$ is a unit normal given by the right hand rule.



* Area of the paralelogram spanned by $\vec{a}$ and $\vec{b}$.

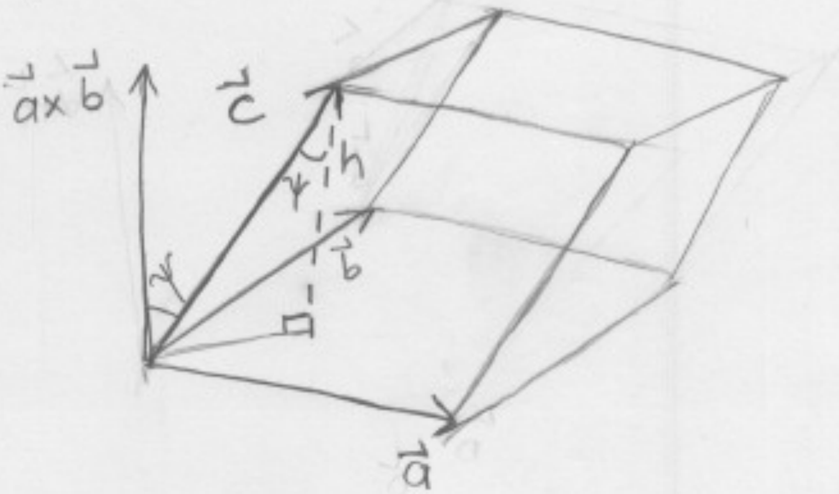


$$A = \|\vec{a}\|\, h$$

but $h = \|\vec{b}\| \sin\theta$

$$\therefore\ A = \|\vec{a}\|\,\|\vec{b}\| \sin\theta$$
$$= \cancel{\|\vec{a}\|\,\|\vec{b}\|}\ \frac{\|\vec{a}\times\vec{b}\|}{\cancel{\|\vec{a}\|\,\|\vec{b}\|}}$$
$$= \|\vec{a}\times\vec{b}\|$$

* Volume of the paralellepiped spanned by the vectors $\vec{a}, \vec{b}, \vec{c}$.

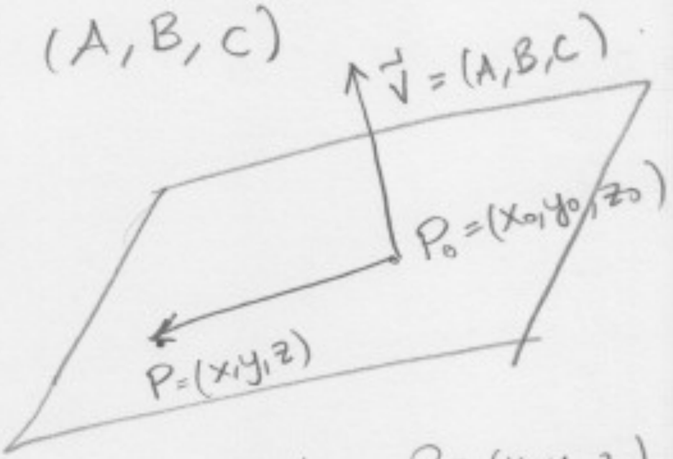


$$V = (\text{Area of base}) \times h$$
$$= \|\vec{a}\times\vec{b}\|\,\|\vec{c}\| \cos\psi; \quad \text{since } h = \|\vec{c}\| \cos\psi$$
$$= \cancel{\|\vec{a}\times\vec{b}\|\,\|\vec{c}\|}\ \frac{(\vec{a}\times\vec{b})\cdot\vec{c}}{\cancel{\|\vec{a}\times\vec{b}\|\,\|\vec{c}\|}}$$
$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

The volume is the absolute value of this determinant.

Equation of a Plane

There is a unique plane that passes through a given point $P_0 = (x_0, y_0, z_0)$ and that it is perpendicular to a given vector $\vec{v} = (A, B, C)$



A point $P = (x, y, z)$ belongs to the plane if and only if:

$$\overrightarrow{P_0P} \cdot \vec{v} = 0$$

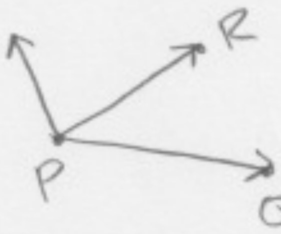
That is.

$$(x - x_0, y - y_0, z - z_0) \cdot (A, B, C) = 0$$

$$\boxed{\begin{aligned} \therefore \quad & A(x - x_0) + B(y - y_0) + C(z - z_0) = 0 \\ \text{or} \quad & Ax + By + Cz = Ax_0 + By_0 + Cz_0 \end{aligned}}$$

is the equation of such plane.

Ex: Find the equation of the plane that passes through $P=(0,0,0)$, $R=(0,4,-3)$ and $Q=(2,0,-1)$.



$$\vec{PR} = (0-0, 4-0, -3-0) = (0,4,-3)$$

$$\vec{PQ} = (2,0,-1)$$

$$\vec{PR}\times\vec{PQ} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 4 & -3 \\ 2 & 0 & -1 \end{vmatrix}$$

$$= \vec{i}(-4) - \vec{j}(6) + \vec{k}(-8) = (-4,-6,-8)$$

$$-4(x-0) - 6(y-0) - 8(z-0) = 0$$

or $\boxed{2x+3y+4z=0}$

We can use another point:

$$-4(x-2) - 6(y-0) - 8(z+1) = 0$$

$$-4x \cancel{+8} - 6y - 8z \cancel{-8} = 0$$

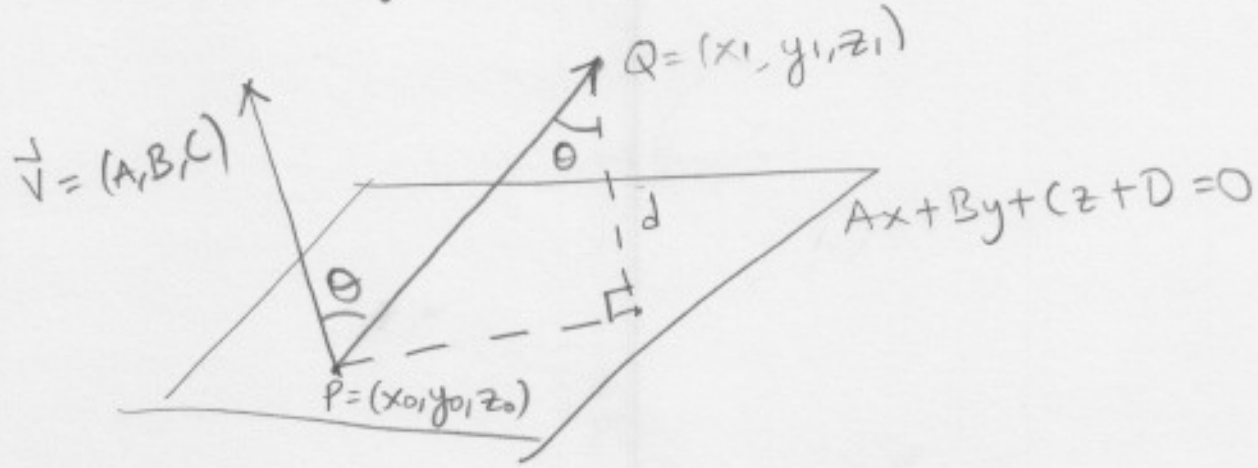
$2x+3y+4z=0$, same answer

Note: Any equation of the form:

$$Ax+By+Cz+D=0$$

is always a plane perpendicular to $\vec{v} = (A,B,C)$

We want to compute the distance from the point $Q=(x_1,y_1,z_1)$ to the plane $Ax+By+Cz+D=0$.



Let d be the distance from Q to the plane. Notice that:

$$
\begin{aligned}
d &= \|\vec{PQ}\| \cos\theta \\
&= \|\vec{PQ}\| \frac{\vec{PQ}\cdot\vec{v}}{\|\vec{PQ}\|\,\|\vec{v}\|} \quad ; \text{ Since } \vec{PQ}\cdot\vec{v} = \|\vec{PQ}\|\|\vec{v}\|\cos\theta \\
&= \frac{\vec{PQ}\cdot\vec{v}}{\|\vec{v}\|} \\
&= \frac{A(x_1-x_0)+B(y_1-y_0)+C(z_1-z_0)}{\sqrt{A^2+B^2+C^2}} \\
&= \frac{Ax_1+By_1+Cz_1-(Ax_0+By_0+Cz_0)}{\sqrt{A^2+B^2+C^2}} \\
&= \frac{Ax_1+By_1+Cz_1+D}{\sqrt{A^2+B^2+C^2}} \quad ;
\end{aligned}
$$

since $Ax_0+By_0+Cz_0+D=0$ (due to the fact that (x_0,y_0,z_0) belongs to the plane.