Section 4.4
Divergence & Curl

For $\vec{F}(x,y,z) = (F_1(x,y,z), F_2(x,y,z), F_3(x,y,z))$

the divergence is defined by:

$$\operatorname{div} \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Similarly, if $\vec{F} = (F_1, \ldots, F_n)$ is a vector field in $\mathbb{R}^n$, its divergence is:

$$\operatorname{div} \vec{F} = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i} = \frac{\partial F_1}{\partial x_1} + \cdots + \frac{\partial F_n}{\partial x_n}$$

<u>Ex</u>: $\vec{F}(x,y,z) = (e^{x^2+y^2}, \sin xz, z^2 + \ln(x^2+y^2))$

$$\operatorname{div} \vec{F} = 2x e^{x^2+y^2} + 0 + 2z = 2z + 2x e^{x^2+y^2}$$

Often $\operatorname{div} \vec{F}$ is written as $\nabla \cdot \vec{F}$.

The curl of F, denoted as $\nabla \times \vec{F}$, is defined by:

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k}$$

Ex: $\vec{F}(x,y,z) = \left(e^{x^2+y^2}, \sin xz, z^2 + \ln(x^2+y^2)\right)$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{x^2+y^2} & \sin xz & z^2 + \ln(x^2+y^2) \end{vmatrix}$$

$$= \left(\frac{2y}{x^2+y^2} - x\cos xz\right)\vec{i} - \left(\frac{2x}{x^2+y^2} - 0\right)\vec{j} + \left(z\cos xz - 2y\, e^{x^2+y^2}\right)\vec{k}$$

The divergence and curl have important physical meaning which we will study later in the course.

Remark: If $\vec{F}$ is 2-dimensional, $\vec{F}(x,y) = (F_1(x,y), F_2(x,y))$ then:

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & 0 \end{vmatrix} = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k}$$

The quantity $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$ is called the scalar curl.

Important identities involving div and curl.

1.- div ∇f , $f: \mathbb{R}^3 \to \mathbb{R}$

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

$$\text{div } \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \Delta f,$$

Recall that Δf is called "The Laplacian of f".

$$\therefore \text{ div } \nabla f = \Delta f.$$

2.- $\nabla \times (\nabla f) = \vec{0}$

$$\nabla \times (\nabla f) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\vec{i} - \left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x}\right)\vec{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\vec{k}$$

$= \vec{0}$; since $f_{yz} = f_{zy}$, $f_{xz} = f_{zx}$, $f_{xy} = f_{yx}$

We have shown:

__Lemma__: If $\vec{F}$ is a gradient vector field (i.e. $\vec{F} = \nabla f$), then $\nabla \times \vec{F} = \vec{0}$.

This Lemma implies that if $\nabla \times \vec{F} \neq \vec{0}$, then $\vec{F}$ can not be a gradient vector field.

__Ex__: Show that $\vec{F} = (x, y, zx)$ is not a gradient vector field.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & zx \end{vmatrix} = 0\vec{i} - (z)\vec{j} + (0)\vec{k}$$
$$= (0, -z, 0) \neq \vec{0}.$$

Hence $\vec{F}$ is not a gradient vector field, by Lemma.

3.- $\operatorname{div}(\nabla \times \vec{F}) = 0$.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k}$$

$$\Rightarrow \operatorname{div}(\nabla \times \vec{F}) = \cancel{\frac{\partial^2 F_3}{\partial x \partial y}} - \cancel{\frac{\partial^2 F_2}{\partial x \partial z}} - \cancel{\frac{\partial^2 F_3}{\partial y \partial x}} + \cancel{\frac{\partial^2 F_1}{\partial y \partial z}} + \cancel{\frac{\partial^2 F_2}{\partial z \partial x}} - \cancel{\frac{\partial^2 F_1}{\partial z \partial y}}$$
$$= 0.$$

4.- $\operatorname{div}(f\vec{F}) = f \operatorname{div}\vec{F} + \vec{F}\cdot\nabla f$.

Check this identity

5. $\operatorname{div}(f\nabla g - g\nabla f) = f\Delta g - g\Delta f$

Check this identity.