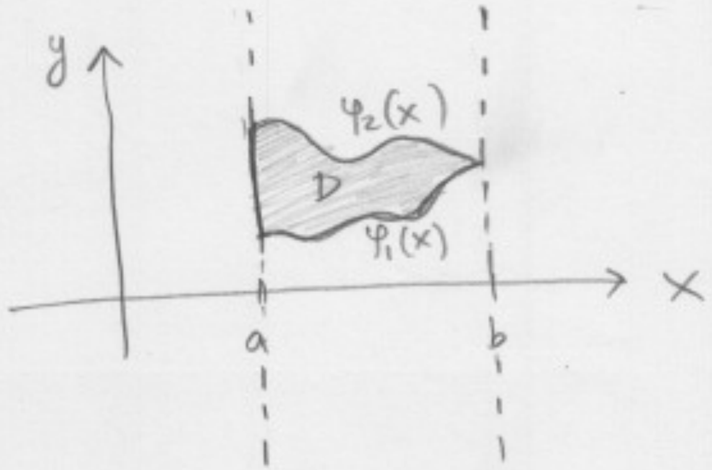
Section 5.3

The double integral over more general Regions

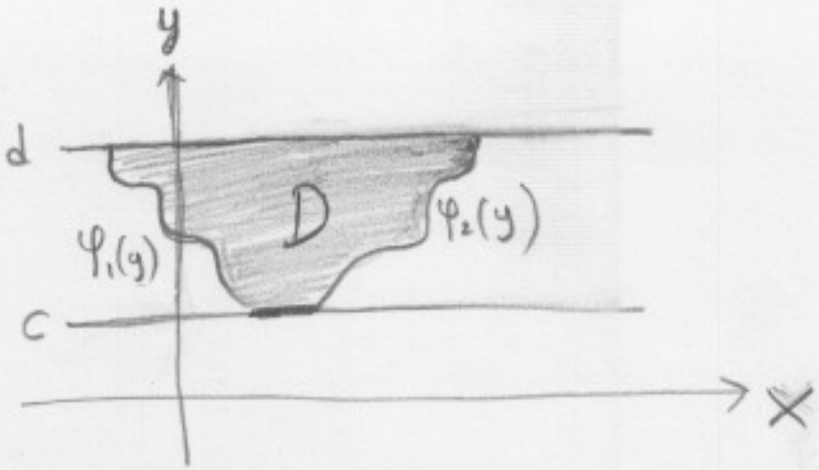
Integration over general regions.

There are two types of regions in $\mathbb{R}^2$ for which the Fubini's theorem also holds.

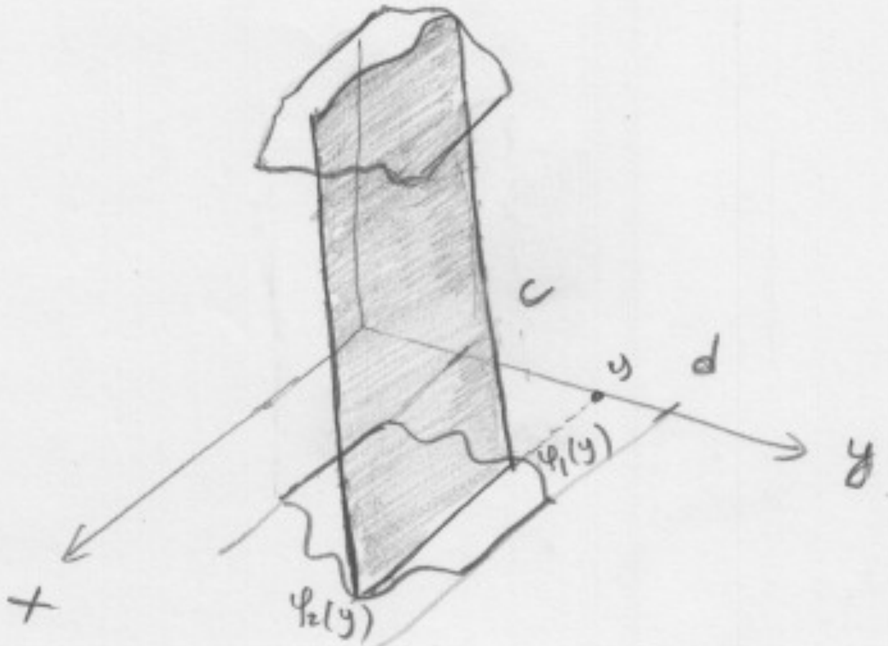
I.- Regions of Type 1: A region of Type 1 is lying between two functions $y = \varphi_1(x)$ and $y = \varphi_2(x)$



II.- Regions of Type 2: A region of Type 2 is lying between two functions $x = \varphi_1(y)$ and $x = \varphi_2(y)$



For example, if D is a region of Type 2, Fubini's theorem can be applied as follows:



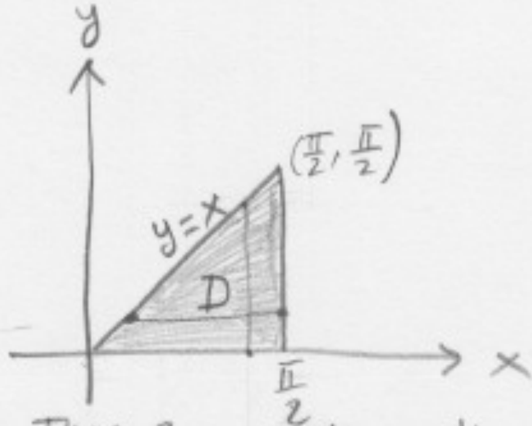
$$A(y) = \int_{\varphi_1(y)}^{\varphi_2(y)} f(x,y)\,dx$$

$$V = \iint_D f(x,y)\,dA = \int_c^d \left(\int_{\varphi_1(y)}^{\varphi_2(y)} f(x,y)\,dx \right) dy.$$

If D is a region of Type 1, a similar argument yields:

$$V = \iint_D f(x,y)\,dA = \int_a^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy \right) dx$$

<u>Ex</u>: $\displaystyle\iint_D (x^3 y + \cos x)\,dA$



D is both Type 1 and Type 2. Integrating as Type 2:

$$\int_0^{\pi/2} \int_y^{\pi/2} (x^3 y + \cos x)\,dx\,dy = \int_0^{\pi/2} \left[\frac{x^4}{4} y + \sin x \right]_y^{\pi/2} dy$$

$$= \int_0^{\pi/2} \left(\frac{\pi^4}{64} y + 1 - \frac{y^5}{4} - \sin y \right) dy$$

$$= \left[\frac{\pi^4 y^2}{128} + y - \frac{y^6}{24} + \cos y \right]_0^{\pi/2}$$

$$= \frac{\pi^6}{4(128)} + \frac{\pi}{2} - \frac{\pi^6}{24 \cdot 64} - 1 = \frac{\pi^6}{768} + \frac{\pi}{2} - 1$$

$$\frac{1}{4(128)} - \frac{1}{(3)(4)(128)} = \frac{3-1}{(3)(4)(128)} = \frac{2}{768}$$

$$\begin{array}{r} 128 \\ \times\ 6 \\ \hline 768 \end{array}$$

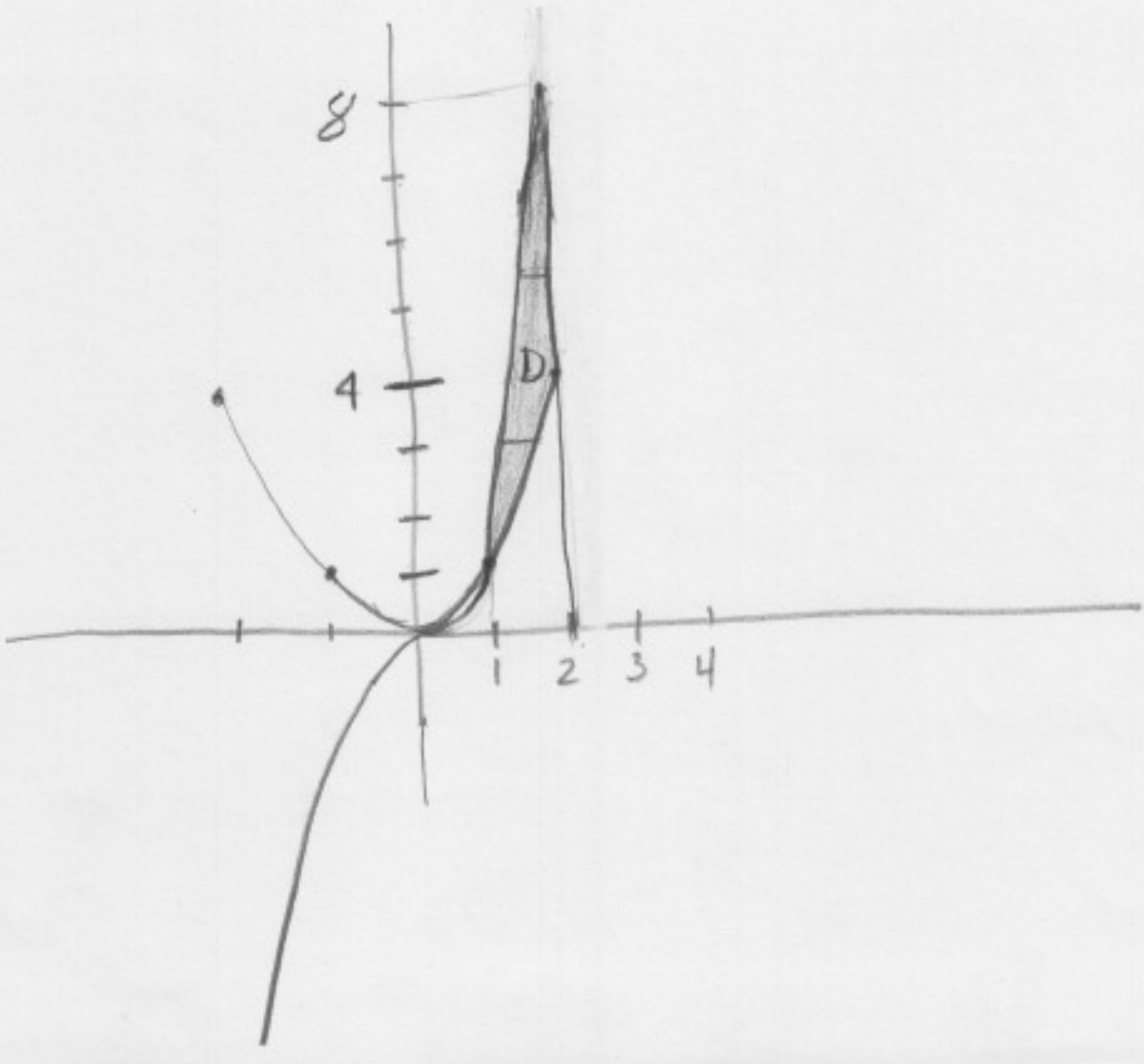
We could also integrate as Type 1:

$$\int_0^{\pi/2}\int_0^{x}\left(x^3y+\cos x\right)dy\,dx$$

Some regions D are neither Type 1 or Type 2. In this case, the integral must be broken up.

Ex. Draw the region of integration for:

$$\int_1^2\int_{x^2}^{x^3} y\,dy\,dx$$



The region D is Type 1 and Type 2.

Integrating as type 1:

$$\int_1^2 \left(\int_{x^2}^{x^3} y \, dy \right) dx .$$

Integrating as type 2:

$$\int_1^4 \left(\int_{y^{1/3}}^{y^{1/2}} y \, dx \right) dy + \int_4^8 \left(\int_{y^{1/3}}^{2} y \, dx \right) dy .$$

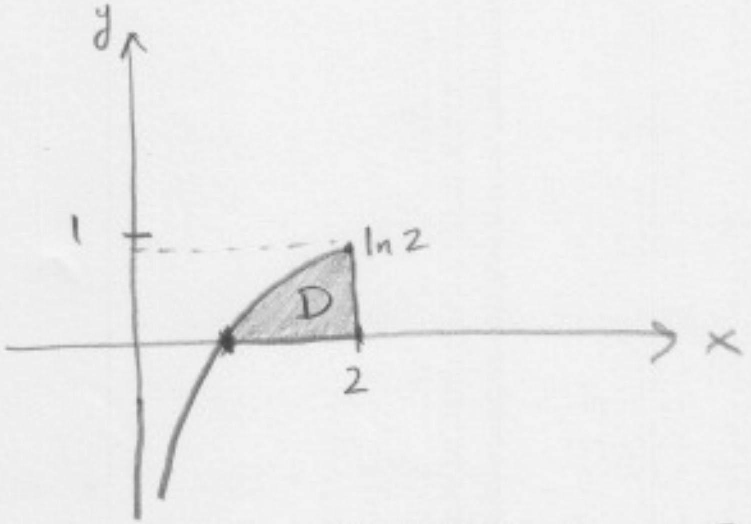
Check the both integrals give the same answer.

Section 5.4

Interchanging the order of integration.

$$\int_1^2 \int_0^{\ln x} x \, dy \, dx$$

In this integral the domain is being considered as Type 1:



But D is also of Type 2. Since $y = \ln x$, then $x = e^y$. Hence, we can change the order of integration and conside D as Type 2:

$$\int_0^{\ln 2} \left(\int_{e^y}^{2} x \, dx \right) dy$$

We compute both integrals and check that we obtain same result.

$$\int_1^2 \int_0^{\ln x} x \, dy \, dx = \int_1^2 x \ln x \; dx$$

$u = \ln x \quad dv = x$

$du = \frac{1}{x} \quad v = \frac{1}{2}x^2$

$\int u \, dv = uv - \int v \, du$

$$= \left[\tfrac{1}{2}x^2 \ln x\right]_1^2 - \int_1^2 \tfrac{1}{2}x \; dx$$

$$= 2\ln 2 - \left[\tfrac{1}{4}x^2\right]_1^2$$

$$= 2\ln 2 - 1 + \tfrac{1}{4}$$

$$= 2\ln 2 - \tfrac{3}{4}$$

Now :

$$\int_0^{\ln 2} \int_{e^y}^{2} x \; dx \; dy = \int_0^{\ln 2} \left[\tfrac{x^2}{2}\right]_{e^y}^{2} dy$$

$$= \int_0^{\ln 2} \left(2 - \frac{e^{2y}}{2}\right) dy$$

$$= \left[2y - \tfrac{1}{4}e^{2y}\right]_0^{\ln 2}$$

$$= 2\ln 2 - \tfrac{1}{4}e^{2\ln 2} + \tfrac{1}{4}$$

$$= 2\ln 2 - \tfrac{1}{4}e^{\ln 2^2} + \tfrac{1}{4}$$

$$= 2\ln 2 - \tfrac{1}{4}\cdot 2^2 + \tfrac{1}{4} = 2\ln 2 - 1 + \tfrac{1}{4}$$

$$= 2\ln 2 - \tfrac{3}{4}$$