Section 3.5, continuation

Method of undetermined coefficients

Recall from previous class that if we are solving:

$$ay'' + by' + cy = g(t) = g_1(t) + g_2(t), \quad (**)$$

then the general solution is:

$$y(t) = c_1 y_1 + c_2 y_2 + Y(t),$$

where $y_H(t) = c_1 y_1 + c_2 y_2$ is the general solution to the corresponding homogeneous equation:

$$ay'' + by' + cy = 0,$$

and $Y(t)$ is a particular solution to the non-homogeneous equation; that is:

$$aY'' + bY' + cY = g(t).$$

If $g(t)$ is the sum of $g_1(t)$ and $g_2(t)$, then, if Y_1 and Y_2 are solutions to:

$$aY_1'' + bY_1' + cY_1 = g_1(t),$$
$$aY_2'' + bY_2' + cY_2 = g_2(t),$$

we have: $Y = Y_1 + Y_2$

Indeed, Y_1+Y_2 is a solution to the non-homogeneous equation (**) since:

$$a(Y_1+Y_2)'' + b(Y_1+Y_2)' + c(Y_1+Y_2) =$$

$$a(Y_1''+Y_2'') + b(Y_1'+Y_2') + cY_1 + cY_2 =$$

$$(aY_1''+bY_1'+cY_1) + (aY_2''+bY_2'+cY_2) =$$

$$= g_1(t) + g_2(t) = g(t)$$

<u>Note</u>: If $g(t) = g_1(t)+g_2(t)+\dots+g_n(t)$, then $Y(t) = Y_1+Y_2+\dots+Y_n$, where Y_i solves:

$$aY_i'' + bY_i' + cY_i = g_i(t) \quad (***)$$

The method of undetermined coefficients only works if g_i is an exponential, polynomial, sine or cosine, or a multiplication of these function.

The proof of this theorem is explained in page 139 of your book. Given $g_i(t)$, the theorem gives the form of $Y_i(t)$ that guarantees a solution for the different constants. There is a summary of the method in page 138, which I explained in class with multiple examples.

The particular solution of $ay''+by'+cy=g_i(t)$.

$g_i(t)$	$Y_i(t)$
$P_n(t)=a_0t^n+a_1t^{n-1}+\dots+a_n$	$t^s(A_0t^n+A_1t^{n-1}+\dots+A_n)$
$P_n(t)e^{\alpha t}$	$t^s(A_0t^n+A_1t^{n-1}+\dots+A_n)e^{\alpha t}$
$P_n(t)e^{\alpha t}\sin\beta t$ or $P_n(t)e^{\alpha t}\cos\beta t$	$t^s\left[(A_0t^n+A_1t^{n-1}+\dots+A_n)e^{\alpha t}\cos\beta t + (B_0t^n+B_1t^{n-1}+\dots+B_n)e^{\alpha t}\sin\beta t\right]$

s is the smallest non-negative integer ($s=0, 1,$ or 2) that will ensure that no term in $Y_i(t)$ is a solution of the corresponding homogeneous eq

Ex: Determine a suitable form for $Y(t)$ if the method of undetermined coefficients is to be used (do not solve the equation).

$$y''+3y'=\underbrace{2t^4}_{g_1}+\underbrace{t^2e^{-3t}}_{g_2}+\underbrace{\sin 3t}_{g_3}$$

$$y''+3y'=0$$

$$r^2+3r=0 \quad r(r+3)=0 \quad r_1=0,\ r_2=-3$$

$$y_H(t)=c_1+c_2e^{-3t}.$$

$$Y(t)=t\,(A_0t^4+A_1t^3+A_2t^2+A_3t+A_4)$$
$$+\,t\,(B_0t^2+B_1t+B_2)e^{-3t}$$
$$+\,D\sin 3t+E\cos 3t$$

Ex: Find the form of Y (do not solve the equation):

$$y'' + y = t(1+\sin t)$$

We have:

$$y'' + y = t + t\sin t = g_1(t) + g_2(t)$$

$$y'' + y = 0$$

$$r^2 + 1 = 0 \qquad r = \pm i$$

$$y_H(t) = C_1 \cos t + C_2 \sin t$$

$$Y(t) = \underbrace{A_0 t + A_1}_{Y_1(t)} + \underbrace{t\left[(B_0 t + B_1)\sin t + (D_0 t + D_1)\cos t\right]}_{Y_2(t)}$$

Ex: Find the form of $Y(t)$ (do not solve the equation).

$$y'' + 9y = t^2 e^{3t} + 6$$

$$y'' + 9 = 0, \quad r^2 + 9 = 0 \qquad r = \pm 3i$$

$$y_H(t) = C_1 \cos 3t + C_2 \sin 3t$$

$$Y(t) = (At^2 + Bt + C)e^{3t} + D.$$

Ex: Find the general solution to:

$$y'' + 2y' + y = 2e^{-t}$$

$$y(t) = y_H(t) + Y(t),$$

where $y_H(t)$ is the general solution of:

$$y'' + 2y' + y = 0$$

$$r^2 + 2r + 1 = 0$$

$$(r+1)^2 = 0, \quad r = -1, \text{ repeated root}$$

$$\Rightarrow y_H(t) = c_1 e^{-t} + c_2 t e^{-t}. \quad (1)$$

Following the theorem:

$$Y(t) = At^2 e^{-t}$$

Note: if you choose $Y(t) = Ae^{-t}$, then we have:

$$Y'' + 2Y' + Y = 0 \neq 2e^{-t},$$

because Ae^{-t} is a solution to the homogeneous equation (see (1) above).

If you choose $Y(t) = Ate^{-t}$, the same thing happens:

$$Y'' + 2Y' + Y = 0 \neq 2e^{-t},$$

because Ate^{-t} solves the homogeneous equation.

Hence, the correct form is:

$$Y(t) = At^2 e^{-t}.$$

$$Y(t) = At^2 e^{-t}$$

$$Y'(t) = 2Ate^{-t} - At^2 e^{-t}$$

$$Y''(t) = 2Ae^{-t} - 2Ate^{-t} - 2Ate^{-t} + At^2 e^{-t}$$

$$= 2Ae^{-t} - 4Ate^{-t} + At^2 e^{-t}$$

We plug:

$$Y''(t) + 2Y' + Y = 2Ae^{-t} \cancel{- 4Ate^{-t}} + \cancel{At^2 e^{-t}}$$

$$\cancel{+ 4Ate^{-t}} - \cancel{2At^2 e^{-t}} + \cancel{At^2 e^{-t}}$$

$$= 2Ae^{-t} = 2e^{-t}$$

Hence $A = 1$

$$\Rightarrow Y(t) = t^2 e^{-t}$$

$\Rightarrow$ The general solution of $y'' + 2y' + y = 2e^{-t}$ is:

$$y(t) = y_H(t) + Y(t)$$

$$\boxed{y(t) = c_1 e^{-t} + c_2 t e^{-t} + t^2 e^{-t}}$$

<u>Ex</u>: Find the general solution of:

$$y'' + 2y' = \underbrace{3}_{g_1} + \underbrace{4\sin 2t}_{g_2}$$

$$y'' + 2y' = 0 \qquad r^2 + 2r = 0 \qquad r(r+2) = 0$$

$$r_1 = 0, \quad r_2 = -2$$

$$\Rightarrow y_H(t) = c_1 e^{0t} + c_2 e^{-2t} = c_1 + c_2 e^{-t}$$

We have:

$$Y(t) = At + B\sin 2t + C\cos 2t$$

$$\Rightarrow Y'(t) = A + 2B\cos 2t - 2C\sin 2t$$

$$Y''(t) = -4B\sin 2t - 4C\cos 2t$$

We plug:

$$Y'' + 2Y' = -4B\sin 2t - 4C\cos 2t + 2A + 4B\cos 2t - 4C\sin 2t = 3 + 4\sin 2t$$

Hence:

$$2A + (-4B - 4C)\sin 2t + (4B - 4C)\cos 2t = 3 + 4\sin 2t$$

$$\Rightarrow \quad 2A = 3 \Rightarrow A = 3/2$$

$$-4B - 4C = 4$$

$$4B - 4C = 0 \Rightarrow B = C$$

$$\Rightarrow -8B = 4 \Rightarrow B = -\tfrac{1}{2} = C$$

Hence:

$$Y(t) = \frac{3}{2}t - \frac{1}{2}\sin 2t - \frac{1}{2}\cos 2t$$

The general solution of $y'' + 2y' = 3 + 4\sin 2t$ is:

$$y(t) = y_H(t) + Y(t)$$

$$\boxed{y(t) = c_1 + c_2 e^{-2t} + \frac{3}{2}t - \frac{1}{2}\sin 2t - \frac{1}{2}\cos 2t}$$

<u>Ex</u>: Find the general solution of:

$$y'' - 2y' - 3y = -3te^{-t}$$

$$y'' - 2y' - 3y = 0$$

$$r^2 - 2r - 3 = 0$$

$$(r-3)(r+1) = 0$$

$$r_1 = 3 \quad r_2 = -1$$

$$y_H(t) = c_1 e^{3t} + c_2 e^{-t}$$

$$Y(t) = t(At+B)e^{-t},$$ by theorem for the method of undetermined coefficient we need to multiply by t.

$$Y(t) = (At^2 + Bt)e^{-t}$$

$$Y'(t) = (2At+B)e^{-t} - (At^2+Bt)e^{-t}$$

$$Y''(t) = 2Ae^{-t} - (2At+B)e^{-t} - (2At+B)e^{-t} + (At^2+Bt)e^{-t}$$

$$= 2Ae^{-t} - 2(2At+B)e^{-t} + (At^2+Bt)e^{-t}$$

We plug:

$$Y''(t) - 2Y' - 3Y = 2Ae^{-t} - 2(2At+B)e^{-t} + \cancel{(At^2+Bt)e^{-t}}$$

$$- 2(2At+B)e^{-t} + \cancel{2(At^2+Bt)e^{-t}}$$

$$\cancel{-3(At^2+Bt)e^{-t}}$$

$$= 2Ae^{-t} - 4(2At+B)e^{-t} = -3te^{-t}$$

We need:

$$2Ae^{-t} - 8Ate^{-t} - 4Be^{-t} = -3te^{-t}$$

$$\Rightarrow (2A-4B)e^{-t} - 8Ate^{-t} = -3te^{-t}$$

Hence:

$$2A - 4B = 0$$

$$-8A = -3, \quad A = 3/8$$

$$\Rightarrow 4B = 2A \quad \Rightarrow B = \tfrac{1}{2}A = \tfrac{1}{2}\left(\tfrac{3}{8}\right) = \tfrac{3}{16}$$

We found:

$$Y(t) = \left(\frac{3}{8}t^2 + \frac{3}{16}t\right)e^{-t}$$

The general solution of $y'' - 2y' - 3y = -3te^{-t}$ is:

$$y(t) = y_H(t) + Y(t)$$

$$\boxed{y(t) = c_1 e^{3t} + c_2 e^{-t} + \left(\frac{3}{8}t^2 + \frac{3}{16}t\right)e^{-t}}$$