Sections 4.1 and 4.2

Consider the nth order, linear, homogeneous ODE:

$$y^{(n)} + p_1(t) y^{(n-1)} + \cdots + p_{n-1}(t) y' + p_n(t) y = 0 \quad (*)$$

A set $\{y_1, \ldots, y_n\}$ of solutions with $W(y_1, \ldots, y_n) \neq 0$ on the interval I is called a fundamental set of solutions.

As in the case of second order equations:

$W(y_1, \ldots, y_n) \neq 0$ if and only if $\{y_1, \ldots, y_n\}$ are linearly independent.

The space of solutions of $(*)$ is a vector space of dimension n that is generated by the base $\{y_1, \ldots, y_n\}$ of the space. Hence, all solutions of $(*)$ can be expressed as a linear combination of the fundamental set of solutions.

Therefore, the general solution of $(*)$ is:

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$$

Recall that $\{y_1, \ldots, y_n\}$ being linearly independent means:

$c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t) = 0$ if and only if $c_1 = c_2 = \cdots = c_n = 0$

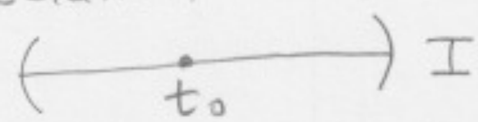
We have the following existence and uniqueness theorem:

Theorem: Consider the nth order, linear, non-homogeneous initial value problem:

$$y^{(n)}(t) + p_1(t)\, y^{(n-1)} + \cdots + p_{n-1}(t)\, y'(t) + p_n(t)\, y = g(t)$$

$$y(t_0) = y_0,\ y'(t_0) = y_0',\ \ldots,\ y^{(n-1)}(t_0) = y_0^{(n-1)}$$

If the functions $p_1, \ldots, p_n$ and g are continuous on an open interval I, then there exists exactly one solution $y = \phi(t)$ that satisfies the initial value problem. This solution exists throughout the interval I.



Ex: Determine an interval on which the solution is sure to exist:

$$t^2 y^{(4)} + t\, y^{(3)} + 5y = \sin t$$

$$y'''' + \frac{1}{t} y''' + \frac{5}{t^2} y = \frac{\sin t}{t^2}$$

We can choose $I = (0, \infty)$ or $I = (-\infty, 0)$

Ex: Verify that the given functions are solutions of the differential equation, and determine their Wronskian.

$$-t^2 y''' + t y'' = 0$$

$$y_1(t) = 1, \quad y_2(t) = t, \quad y_3(t) = t^3$$

We first notice that $y''' - \frac{1}{t} y'' = 0$, so by the existence theorem we can put our initial condition in $I = (0, \infty)$ or $I = (-\infty, 0)$

$$y_1(t) = 1, \quad y_1' = 0, \; y_1'' = 0, \quad y_1''' = 0$$

$$-t^2 y_1''' + t y_1'' = -t^2(0) + t(0) = 0 \checkmark$$

$$y_2(t) = t \quad y_2' = 1 \quad y_2'' = 0 \quad y_2''' = 0$$

$$-t^2 y_2''' + t y_2'' = -t^2(0) + t(0) = 0 \checkmark$$

$$y_3(t) = t^3, \quad y_3' = 3t^2, \quad y_3'' = 6t \quad y_3''' = 6$$

$$-t^2 y_3''' + t y_3'' = -t^2(6) + t(6t) = -6t^2 + 6t^2 = 0 \checkmark$$

Hence, y_1, y_2, y_3 are solutions. We recall:

$$W(y_1, \ldots, y_n) = \det \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ y_1'' & y_2'' & \cdots & y_n'' \\ & \vdots & & \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}$$

This is an $n \times n$ matrix. Indeed, recall that for $n=2$ (and hence $n-1=1$), corresponding to the case of second order equations we have:

$$W(y_1, y_2) = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}.$$

Hence:

$$W(1, t, t^3) = \det \begin{pmatrix} 1 & t & t^3 \\ 0 & 1 & 3t^2 \\ 0 & 0 & 6t \end{pmatrix}$$

$$= 6t \neq 0 \quad \text{on } I = (0, \infty), \text{ or } I(-\infty, 0).$$

$\Rightarrow \{1, t, t^3\}$ is a fundamental set of solutions for $-t^2 y''' + t y'' = 0$, and the general solution to this equation is:

$$y(t) = c_1 y_1 + c_2 y_2 + c_3 y_3$$

or $$\boxed{y(t) = c_1 + c_2 t + c_3 t^3}$$

We consider the simplest case of a linear, homogeneous equation with constant coefficients:

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y = 0 \qquad (**)$$

We use the notation:

$$L(y) := a_0 y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y$$

With this notation, we can simplify $(*)$ to:

$$L(y) = 0$$

As with second order, we look for a solution of the form $y(t) = e^{rt}$;

$$y(t) = e^{rt}, \ y' = re^{rt}, \ y'' = r^2 e^{rt}, \dots, y^{(n)} = r^n e^{rt}$$

We plug in $(*)$:

$$e^{rt} \underbrace{(a_0 r^n + a_1 r^{n-1} + \dots + a_{n-1} r + a_0)}_{p(r)} = 0$$

$p(r)$ is called the characteristic polynomial. Since $e^{rt} \neq 0$ for every t, we need r so that:

$$p(r) = 0$$

or

$$a_0 r^n + a_1 r^{n-1} + \dots + a_{n-1} r + a_0 = 0.$$

By the fundamental theorem of algebra, a polynomial of degree n has n roots $r_1, r_2, \ldots, r_n$ and $p(r)$ can be factored as:

$$p(r) = a_0 (r-r_1)(r-r_2)\cdots(r-r_n)$$

The difference with second order is that roots can repeat more than 2 times, complex roots also can repeat.

Recall that our theory says that we need to find n linearly independent functions $y_1, \ldots, y_n$; that is, $W(y_1, \ldots, y_n) \neq 0$.

For the case when the roots of $p(r)$ are real and unequal, then we have right away the following n solutions:

$$y_1(t) = e^{r_1 t}, \quad y_2(t) = e^{r_2 t}, \quad \ldots, \quad y_n(t) = e^{r_n t}$$

It can be proven that in this case $W(y_1, \ldots, y_n) \neq 0$, and therefore we conclude that the general solution to the equation (**) is:

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \cdots + c_n e^{r_n t}.$$

Example 1: Distinct Real Roots (1 of 3)

- Consider the initial value problem

$$y^{(4)} + 2y''' - 13y'' - 14y' + 24y = 0$$

$$y(0) = 1, y'(0) = -1, y''(0) = 0, y'''(0) = -1$$

- Assuming exponential soln leads to characteristic equation:

$$y(t) = e^{rt} \Rightarrow r^4 + 2r^3 - 13r^2 - 14r + 24 = 0$$

$$\Leftrightarrow (r-1)(r+2)(r-3)(r+4) = 0$$

- Thus the general solution is

$$y(t) = c_1 e^{t} + c_2 e^{-2t} + c_3 e^{3t} + c_4 e^{-4t}$$

$p(r) = (r-1)(r^3 + 3r^2 - 10r - 24)$

$$\begin{array}{r|l} & r^3 + 3r^2 - 10r - 24 \\ r-1 & r^4 + 2r^3 - 13r^2 - 14r + 24 \\ & \cancel{-r^4} + r^3 \\ & 3r^3 - 13r^2 - 14r + 24 \\ & -3r^3 + 3r^2 \\ & -10r^2 - 14r + 24 \end{array}$$

$-24r + 24$

24

$\pm 1 \quad \pm 3 \quad \pm 6 \quad \pm 24$

$\pm 2 \quad \pm 4 \quad \pm 12$

± 8

Ex: $r^4 + 2r^3 - 13r^2 - 14r + 24 = 0$

$p(1) = 1 + 2 - 13 - 14 + 24 = 0$

$r = 1$

$$
\begin{array}{r l}
 & r^3 + 3r^2 - 10r - 24 \\
r-1 & \overline{)\, \cancel{r^4} + 2r^3 - 13r^2 - 14r + 24} \\
 & -\cancel{r^4} + r^3 \\
\hline
 & \cancel{3r^3} - 13r^2 - 14r + 24 \\
 & -\cancel{3r^3} + 3r^2 \\
\hline
 & -\cancel{10r^2} - 14r + 24 \\
 & \cancel{10r^2} - 10r \\
\hline
 & -\cancel{24r} + \cancel{24} \\
 & \cancel{24r} - \cancel{24} \\
\hline
 & 0
\end{array}
$$

$p(r) = (r-1)(r^3 + 3r^2 - 10r - 24) = (r-1)\,q(r)$

$q(-2) = (-8 + 12 + 20 - 24)\ \cancel{(-2-1)} = 0$

$$
\begin{array}{r l}
 & r^2 + r - 12 \\
r+2 & \overline{)\, \cancel{r^3} + 3r^2 - 10r - 24} \\
 & -\cancel{r^3} - 2r^2 \\
\hline
 & \cancel{r^2} - 10r - 24 \\
 & -\cancel{r^2} - 2r \\
\hline
 & -\cancel{12r} - \cancel{24} \\
 & \cancel{12r} + \cancel{24} \\
\hline
 & 0
\end{array}
$$

$p(r) = (r-1)(r+2)(r^2 + r - 12) = (r-1)(r+2)(r+4)(r-3)$

$$y(t) = c_1 e^{t} + c_2 e^{-2t} + c_3 e^{3t} + c_4 e^{-4t}$$

Example 1: Solution (2 of 3)

- The initial conditions

$$y(0) = 1, y'(0) = -1, y''(0) = 0, y'''(0) = -1$$

yield

$$\begin{aligned} c_1 + c_2 + c_3 + c_4 &= 1 \\ c_1 - 2c_2 + 3c_3 - 4c_4 &= -1 \\ c_1 + 4c_2 + 9c_3 + 16c_4 &= 0 \\ c_1 - 8c_2 + 27c_3 - 64c_4 &= -1 \end{aligned}$$

- Solving, $c_1 = \frac{1}{2}, c_2 = \frac{4}{5}, c_3 = -\frac{11}{70}, c_4 = -\frac{1}{7}$

- Hence $y(t) = \frac{1}{2}e^{t} + \frac{4}{5}e^{-2t} - \frac{11}{70}e^{3t} - \frac{1}{7}e^{-4t}$

Example 1: Graph of Solution (3 of 3)

- The graph of the solution is given below. Note the effect of the largest root of characteristic equation.

$$y(t) = \frac{1}{2}e^{t} + \frac{4}{5}e^{-2t} - \frac{11}{70}e^{3t} - \frac{1}{7}e^{-4t}$$