Section 4.3

Non-homogeneous equations: method of undetermined coefficients for higher order, linear, non-homogeneous ODE.

We consider the equation:

$$y^{(n)} + p_1(t)\, y^{(n-1)} + \dots + p_{n-1}(t)\, y' + p_n(t)\, y = g(t) \qquad (*)$$

or

$$L(y) = g(t),$$

where:

$$L(y) = y^{(n)} + p_1(t)\, y^{(n-1)} + \dots + p_{n-1}(t)\, y' + p_n(t)\, y$$

The general solution of $(*)$ is:

$$y(t) = y_H(t) + Y(t),$$

where $y_H(t)$ is a solution to the homogeneous equation $L(y)=0$, and $Y(t)$ is a particular solution to the non-homogeneous equation $L(y)=g(t)$

Indeed, we recall the argument that shows this fact (which is the same as for second order equations):

Suppose that $Y(t)$ is a solution to $(*)$. Hence:

$$L(Y(t)) = g(t).$$

Let $\tilde{Y}$ be any other solution of (*).
Therefore:

$$L(\tilde{Y}) = g(t).$$

We consider the difference $\tilde{Y}(t) - Y(t)$. We note that:

$$L(\tilde{Y} - Y) = L(\tilde{Y}) - L(Y) \quad ; \text{ since the equation is linear}$$

$$= g(t) - g(t) = 0.$$

Thus, $\tilde{Y} - Y$ is a solution of the homogeneous equation:

$$L(y) = 0.$$

Since the general solution of the homogeneous equation is $y_H(t) = c_1 y_1 + \cdots + c_n y_n$, it follows that there exist constants $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that:

$$\tilde{Y} - Y = \alpha_1 y_1 + \cdots + \alpha_n y_n.$$

Hence

$$\tilde{Y} = \alpha_1 y_1 + \cdots + \alpha_n y_n + Y.$$

We have shown that any solution of $L(y) = g$ is contained in the family $c_1 y_1 + \cdots + c_n y_n + Y(t)$.

Conversely, for any $c_1, \ldots, c_n$:

$$L(c_1 y_1 + \cdots + c_n y_n + Y(t)) = L(c_1 y_1 + \cdots + c_n y_n) + L(Y)$$

$$= 0 + g(t) = g(t).$$

Therefore, the general solution of (*) is $y(t) = y_H(t) + Y(t)$. ∎

In order to find $\mathbf{Y}(t)$, as with 2^{nd} order equations, we can use the method of undetermined coefficients for cases when g is a sum or product of polynomials, exponentials, and sine or cosine functions.

In order to find $y_H(t)$, we have seen that if the roots of the characteristic equation $p(r)=0$ are real and different: $r_1, \dots, r_n$, then:

$$y_H(t) = c_1 e^{r_1 t} + \dots + c_n e^{r_n t}.$$

If the characteristic equation has complex roots, then they must occur in conjugate pairs, $\lambda \pm i\mu$. Solutions corresponding to complex roots have the form:

$$e^{(\lambda+i\mu)t} = e^{\lambda t}\cos\mu t + i e^{\lambda t}\sin\mu t$$
$$e^{(\lambda-i\mu)t} = e^{\lambda t}\cos\mu t - i e^{\lambda t}\sin\mu t.$$

Using the principle of superposition, adding and substracting these complex solutions, we obtain the following two real solutions:

$$e^{\lambda t}\cos\mu t, \; e^{\lambda t}\sin\mu t$$

Repeated Roots

Suppose a root r_k of the characteristic equation $p(r)=0$ is a repeated root with multiplicity s. Then, linearly independent solutions corresponding to this repeated root have the form:

$$e^{r_k t},\ t e^{r_k t},\ t^2 e^{r_k t},\ \ldots,\ t^{s-1} e^{r_k t}.$$

If a complex root $\lambda + i\mu$ is repeated s times, then so is its conjugate $\lambda - i\mu$. There are $2s$ corresponding linearly independent solutions, derived from real and imaginary parts of:

$$e^{(\lambda+i\mu)t},\ t e^{(\lambda+i\mu)t},\ t^2 e^{(\lambda+i\mu)t},\ \ldots,\ t^{s-1} e^{(\lambda+i\mu)t},$$
$$e^{(\lambda-i\mu)t},\ t e^{(\lambda-i\mu)t},\ t^2 e^{(\lambda-i\mu)t},\ \ldots,\ t^{s-1} e^{(\lambda-i\mu)t}$$

which are:

$$\boxed{\begin{array}{l} e^{\lambda t}\cos\mu t,\ e^{\lambda t}\sin\mu t,\ t e^{\lambda t}\cos\mu t,\ t e^{\lambda t}\sin\mu t,\ \ldots \\ \ldots,\ t^{s-1} e^{\lambda t}\cos\mu t,\ t^{s-1} e^{\lambda t}\sin\mu t \end{array}}$$

Example 2: Complex Roots

- Consider the equation

$$y''' - y = 0$$

- Then

$$y(t) = e^{rt} \Rightarrow r^3 - 1 = 0 \Leftrightarrow (r-1)(r^2 + r + 1) = 0$$

- Now

$$r^2 + r + 1 = 0 \Leftrightarrow r = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{3}\,i}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

- Thus the general solution is

$$y(t) = \underbrace{c_1 e^t}_{y_1} + \underbrace{c_2 e^{-t/2} \cos\left(\sqrt{3}t/2\right)}_{y_2} + \underbrace{c_3 e^{-t/2} \sin\left(\sqrt{3}t/2\right)}_{y_3}$$

Example 3: Complex Roots (1 of 2)

- Consider the initial value problem

$$y^{(4)} - y = 0, \ \ y(0) = 7/2, y'(0) = -4, y''(0) = 5/2, y'''(0) = -2$$

- Then

$$y(t) = e^{rt} \ \Rightarrow \ r^4 - 1 = 0 \Leftrightarrow (r^2 - 1)(r^2 + 1) = 0$$

$(r+1)(r-1)(r+i)(r-i) = 0$

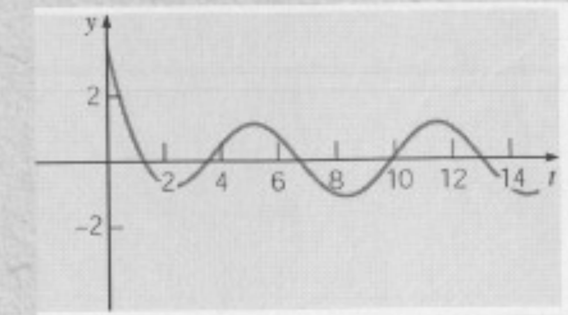
- The roots are 1, -1, i, -i. Thus the general solution is

$$y(t) = c_1 e^t + c_2 e^{-t} + c_3 \cos(t) + c_4 \sin(t)$$

- Using the initial conditions, we obtain

$$y(t) = 0e^t + 3e^{-t} + \frac{1}{2}\cos(t) - \sin(t)$$

- The graph of solution is given on right.



Example 4: Repeated Roots

- Consider the equation

$$y^{(4)} + 8y'' + 16y = 0$$

- Then

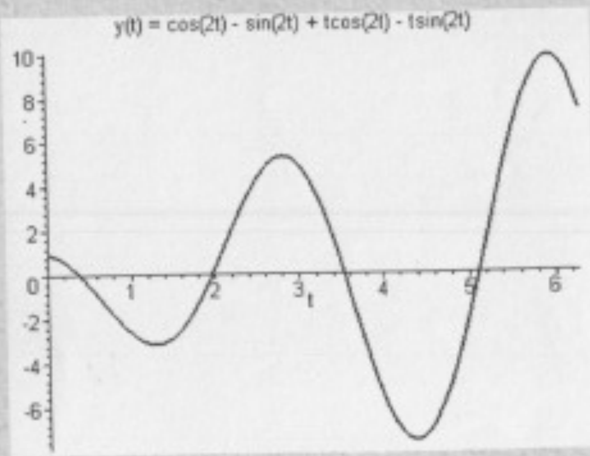
$$y(t) = e^{rt} \Rightarrow r^4 + 8r^2 + 16 = 0 \Leftrightarrow (r^2+4)(r^2+4) = 0$$

$$(r+2i)(r-2i)(r+2i)(r-2i) = 0$$

- The roots are $2i, 2i, -2i, -2i$. Thus the general solution is

$$y(t) = c_1 \cos 2t + c_2 \sin 2t + c_3 t\cos(2t) + c_4 t\sin(2t)$$

$$y(t) = c_1 y_1 + c_2 y_2 + c_3 y_3 + c_4 y_4$$



Example 1

- Consider the differential equation

$$y''' - 3y'' + 3y' - y = 4e^t$$

homogeneous equation:

$y''' - 3y'' + 3y' - y = 0$

- For the homogeneous case,

$$y(t) = e^{rt} \Rightarrow r^3 - 3r^2 + 3r - 1 = 0 \Leftrightarrow (r-1)^3 = 0$$

- Thus the general solution of homogeneous equation is

$$y_H(t) = c_1 e^t + c_2 t e^t + c_3 t^2 e^t$$

- For nonhomogeneous case, keep in mind the form of homogeneous solution. Thus begin with

$$Y(t) = At^3 e^t$$

- As in Chapter 3, it can be shown that

$$Y(t) = \frac{2}{3} t^3 e^t \Rightarrow y(t) = \underbrace{c_1 e^t + c_2 t e^t + c_3 t^2 e^t}_{y_H(t)} + \underbrace{\frac{2}{3} t^3 e^t}_{Y(t)}.$$

Example 2

- Consider the equation

$$y^{(4)} + 8y'' + 16y = 2\sin t - 3\cos t$$

- For the homogeneous case,

$$y(t) = e^{rt} \Rightarrow r^4 + 8r^2 + 16 = 0 \Leftrightarrow (r^2+4)(r^2+4) = 0$$

- Thus the general solution of homogeneous equation is

$$y(t) = c_1 \cos 2t + c_2 \sin 2t + c_3 t \cos(2t) + c_4 t \sin(2t)$$

- For the nonhomogeneous case, begin with

$$Y(t) = A\sin t + B\cos t$$

- As in Chapter 3, it can be shown that

$$Y(t) = \frac{2}{9}\sin t - \frac{1}{3}\cos t$$

$$y = \underbrace{c_1 \cos 2t + c_2 \sin 2t + c_3 t \cos 2t + c_4 t \sin 2t}_{y(t)} + \underbrace{\tfrac{2}{9}\sin t - \tfrac{1}{3}\cos t}_{Y(t)}$$

Example 3

- Consider the equation

$$y^{(4)} + 8y'' + 16y = 2\sin 2t - 3\cos 2t$$

- As in Example 2, the general solution of homogeneous equation is

$$y(t) = c_1 \cos 2t + c_2 \sin 2t + c_3 t \cos(2t) + c_4 t \sin(2t)$$

- For the nonhomogeneous case, begin with

$$Y(t) = At^2 \sin 2t + Bt^2 \cos 2t$$

- As in Chapter 3, it can be shown that

$$Y(t) = -\frac{1}{16} t^2 \sin 2t + \frac{3}{32} t^2 \cos 2t$$

$$y_N(t) = \underbrace{c_1 \cos 2t + c_2 \sin 2t + c_3 t \cos 2t + c_4 t \sin 2t}_{y_H} + \left(\frac{-1}{16} t^2 \sin 2t + \frac{3}{32} t^2 \cos 2t \right)$$

Example 4

- Consider the equation

$$y''' - 9y' = t + e^{3t}$$

- For the homogeneous case,

$$y(t) = e^{rt} \Rightarrow r^3 - 9r = 0 \Leftrightarrow r\left(r^2 - 9\right) \Leftrightarrow r(r-3)(r+3) = 0$$

- Thus the general solution of homogeneous equation is

$$y_H(t) = c_1 + c_2 e^{3t} + c_3 e^{-3t}$$

- For nonhomogeneous case, keep in mind form of homogeneous solution. Thus we have two subcases:

$$Y_1(t) = (A + Bt)t,\ Y_2(t) = Cte^{3t},$$

- As in Chapter 3, can be shown that

$$Y_1(t) = -\frac{1}{18}t^2,\ Y_2(t) = \frac{1}{18}t\,e^{3t}$$