Lesson 25.

Ch 6.2: Solution of Initial Value Problems

- The Laplace transform is named for the French mathematician Laplace, who studied this transform in 1782.
- The techniques described in this chapter were developed primarily by Oliver Heaviside (1850-1925), an English electrical engineer.
- In this section we see how the Laplace transform can be used to solve initial value problems for linear differential equations with constant coefficients.
- The Laplace transform is useful in solving these differential equations because the transform of f' is related in a simple way to the transform of f, as stated in Theorem 6.2.1.

Theorem 6.2.1

- Suppose that f is a function for which the following hold:
  (1) f is continuous and f' is piecewise continuous on $[0, b]$ for all $b > 0$.
  (2) $|f(t)| \leq Ke^{at}$ when $t \geq M$, for constants a, K, M, with $K, M > 0$.
- Then the Laplace Transform of f' exists for $s > a$, with

$$L\{f'(t)\} = sL\{f(t)\} - f(0)$$

- **Proof** (outline): For f and f' continuous on $[0, b]$, we have

$$\lim_{b\to\infty}\int_0^b e^{-st} f'(t)dt = \lim_{b\to\infty}\left[e^{-st} f(t)\Big|_0^b - \int_0^b (-s)e^{-st} f(t)dt \right]$$

$u = e^{-st}$ $dv = f'(t)dt$

$du = -se^{-st}$ $v = f(t)$

$$= \lim_{b\to\infty}\left[e^{-sb} f(b) - f(0) + s\int_0^b e^{-st} f(t)dt \right] = sL(f(t)) - f(0)$$

- Similarly for f' piecewise continuous on $[0, b]$, see text.

The Laplace Transform of f'

- Thus if f and f' satisfy the hypotheses of Theorem 6.2.1, then

$$L\{f'(t)\} = sL\{f(t)\} - f(0)$$

- Now suppose f' and f'' satisfy the conditions specified for f and f' of Theorem 6.2.1. We then obtain

$$\begin{aligned} L\{f''(t)\} &= sL\{f'(t)\} - f'(0) \\ &= s\left[sL\{f(t)\} - f(0)\right] - f'(0) \\ &= s^2 L\{f(t)\} - sf(0) - f'(0) \end{aligned}$$

- Similarly, we can derive an expression for $L\{f^{(n)}\}$, provided f and its derivatives satisfy suitable conditions. This result is given in Corollary 6.2.2

Corollary 6.2.2

- Suppose that f is a function for which the following hold:
  - (1) $f, f', f'', \ldots, f^{(n-1)}$ are continuous, and $f^{(n)}$ piecewise continuous, on $[0, b]$ for all $b > 0$.
  - (2) $|f(t)| \le Ke^{at}$, $|f'(t)| \le Ke^{at}$, …, $|f^{(n-1)}(t)| \le Ke^{at}$ for $t \ge M$, for constants a, K, M, with K, $M > 0$.

Then the Laplace Transform of $f^{(n)}$ exists for $s > a$, with

$$L\{f^{(n)}(t)\} = s^n L\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0)$$

$$y'' + 5y' + 6y = 0, \quad y(0) = 2, \ y'(0) = 3$$

Example 1: Laplace Tranform Method

- Assume that our IVP has a solution ϕ and that $\phi'(t)$ and $\phi''(t)$ satisfy the conditions of Corollary 6.2.2. Then

$$L\{y'' + 5y' + 6y\} = L\{y''\} + 5L\{y'\} + 6L\{y\} = L\{0\} = 0$$

and hence

$$\left[s^2 L\{y\} - sy(0) - y'(0)\right] + 5\left[sL\{y\} - y(0)\right] + 6L\{y\} = 0$$

- Letting $Y(s) = L\{y\}$, we have

$$\left(s^2 + 5s + 6\right)Y(s) - (s+5)y(0) - y'(0) = 0 \qquad L\{y\} = Y(s)$$

- Substituting in the initial conditions, we obtain

$$\left(s^2 + 5s + 6\right)Y(s) - 2(s+5) - 3 = 0$$

- Thus

$$L\{y\} = Y(s) = \frac{2s+13}{(s+3)(s+2)}$$

Example 1: Partial Fractions (3 of 4)

- Using partial fraction decomposition, $Y(s)$ can be rewritten:

$$\frac{2s+13}{(s+3)(s+2)} = \frac{A}{(s+3)} + \frac{B}{(s+2)}$$

$$2s+13 = A(s+2) + B(s+3)$$

$$2s+13 = (A+B)s + (2A+3B)$$

$$A+B=2,\ 2A+3B=13$$

$$A=-7,\ B=9$$

- Thus

$$L\{y\} = Y(s) = -\frac{7}{(s+3)} + \frac{9}{(s+2)}$$

$$y(t) \xrightarrow{\mathcal{L}} Y(s)$$

$$y(t) = \mathcal{L}^{-1}\left[-\frac{7}{s+3} + \frac{9}{s+2}\right]$$

$$y(t) \xleftarrow[\mathcal{L}^{-1}]{} Y(s)$$

$$= \mathcal{L}^{-1}\left(\frac{-7}{s+3}\right) + \mathcal{L}^{-1}\left(\frac{9}{s+2}\right) = -7\,\mathcal{L}^{-1}\left(\frac{1}{s+3}\right) + 9\,\mathcal{L}^{-1}\left(\frac{1}{s+2}\right) = -7e^{-3t} + 9e^{-2t}$$

Example 1: Solution (4 of 4)

- Recall from Section 6.1:

$$L\left\{e^{at}\right\} = F(s) = \int_0^\infty e^{-st} e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt = \frac{1}{s-a}, \quad s > a$$

- Thus

$$Y(s) = -\frac{7}{(s+3)} + \frac{9}{(s+2)} = -7L\{e^{-3t}\} + 9L\{e^{-2t}\}, \quad s > -2,$$

- Recalling $Y(s) = L\{y\}$, we have

$$L\{y\} = L\{-7e^{-3t} + 9e^{-2t}\}$$

and hence

$$y(t) = -7e^{-3t} + 9e^{-2t}$$

Example 2

- Find the inverse Laplace Transform of the given function.

$$Y(s) = \frac{2}{s}$$

- To find $y(t)$ such that $y(t) = L^{-1}\{Y(s)\}$, we first rewrite $Y(s)$:

$$Y(s) = \frac{2}{s} = 2\left(\frac{1}{s}\right)$$

- Using Table 6.2.1,

$$L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{2}{s}\right\} = 2L^{-1}\left\{\frac{1}{s}\right\} = 2(1) = 2$$

- Thus

$$y(t) = 2$$

Example 3

- Find the inverse Laplace Transform of the given function.

$$Y(s) = \frac{3}{s-5}$$

- To find $y(t)$ such that $y(t) = L^{-1}\{Y(s)\}$, we first rewrite $Y(s)$:

$$Y(s) = \frac{3}{s-5} = 3\left(\frac{1}{s-5}\right)$$

- Using Table 6.2.1,

$$L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{3}{s-5}\right\} = 3L^{-1}\left\{\frac{1}{s-5}\right\} = 3e^{5t}$$

- Thus

$$y(t) = 3e^{5t}$$

Example 4

- Find the inverse Laplace Transform of the given function.

$$Y(s) = \frac{6}{s^4}$$

- To find $y(t)$ such that $y(t) = L^{-1}\{Y(s)\}$, we first rewrite $Y(s)$:

$$Y(s) = \frac{6}{s^4} = \frac{3!}{s^4}$$

- Using Table 6.2.1,

$$L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{3!}{s^4}\right\} = t^3$$

- Thus

$$y(t) = t^3$$

Example 5

- Find the inverse Laplace Transform of the given function.

$$Y(s) = \frac{8}{s^3}$$

- To find $y(t)$ such that $y(t) = L^{-1}\{Y(s)\}$, we first rewrite $Y(s)$:

$$Y(s) = \frac{8}{s^3} = \left(\frac{8}{2!}\right)\left(\frac{2!}{s^3}\right) = 4\left(\frac{2!}{s^3}\right)$$

- Using Table 6.2.1,

$$L^{-1}\{Y(s)\} = L^{-1}\left\{4\left(\frac{2!}{s^3}\right)\right\} = 4L^{-1}\left\{\frac{2!}{s^3}\right\} = 4t^2$$

- Thus

$$y(t) = 4t^2$$

Example 6

- Find the inverse Laplace Transform of the given function.

$$Y(s) = \frac{4s+1}{s^2+9}$$

- To find $y(t)$ such that $y(t) = L^{-1}\{Y(s)\}$, we first rewrite $Y(s)$:

$$Y(s) = \frac{4s+1}{s^2+9} = 4\left[\frac{s}{s^2+9}\right] + \frac{1}{3}\left[\frac{3}{s^2+9}\right]$$

- Using Table 6.2.1,

$$L^{-1}\{Y(s)\} = 4L^{-1}\left\{\frac{s}{s^2+9}\right\} + \frac{1}{3}L^{-1}\left\{\frac{3}{s^2+9}\right\} = 4\cos 3t + \frac{1}{3}\sin 3t$$

- Thus

$$y(t) = 4\cos 3t + \frac{1}{3}\sin 3t$$

Example 9

- For the function $Y(s)$ below, we find $y(t) = L^{-1}\{Y(s)\}$ by using a partial fraction expansion, as follows.

$$Y(s) = \frac{3s+1}{s^2+s-12} = \frac{3s+1}{(s+4)(s-3)} = \frac{A}{s+4} + \frac{B}{s-3}$$

$$3s+1 = A(s-3) + B(s+4)$$

$$3s+1 = (A+B)s + (4B-3A)$$

$$A+B = 3, \quad 4B-3A = 1$$

$$A = 11/7, \quad B = 10/7$$

$$Y(s) = \frac{11}{7}\left[\frac{1}{s+4}\right] + \frac{10}{7}\left[\frac{1}{s-3}\right] \Rightarrow y(t) = \frac{11}{7}e^{-4t} + \frac{10}{7}e^{3t}$$

Example 12: Nonhomogeneous Problem (1 of 2)

- Consider the initial value problem

$$y'' + y = \sin 2t, \quad y(0) = 2, \; y'(0) = 1$$

- Taking the Laplace transform of the differential equation, and assuming the conditions of Corollary 6.2.2 are met, we have

$$\left[s^2 L\{y\} - sy(0) - y'(0)\right] + L\{y\} = 2/(s^2+4)$$

- Letting $Y(s) = L\{y\}$, we have

$$\left(s^2+1\right)Y(s) - sy(0) - y'(0) = 2/(s^2+4)$$

- Substituting in the initial conditions, we obtain

$$\left(s^2+1\right)Y(s) - 2s - 1 = 2/(s^2+4)$$

- Thus

$$Y(s) = \frac{2s^3 + s^2 + 8s + 6}{(s^2+1)(s^2+4)}$$

$$(s^2+1)Y(s) = \frac{2}{s^2+4} + 2s + 1$$

$$Y(s) = \frac{2}{(s^2+4)(s^2+1)} + \frac{2s+1}{s^2+1}$$

$$= \frac{2 + 2s^3 + s^2 + 8s + 4}{(s^2+4)(s^2+1)}$$

Example 12: Solution (2 of 2)

- Using partial fractions,

$$Y(s) = \frac{2s^3 + s^2 + 8s + 6}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

- Then

$$2s^3 + s^2 + 8s + 6 = (As+B)(s^2+4) + (Cs+D)(s^2+1)$$
$$= (A+C)s^3 + (B+D)s^2 + (4A+C)s + (4B+D)$$

- Solving, we obtain $A = 2$, $B = 5/3$, $C = 0$, and $D = -2/3$. Thus

$$Y(s) = \frac{2s}{s^2+1} + \frac{5/3}{s^2+1} - \frac{2/3}{s^2+4}$$

- Hence

$$y(t) = 2\cos t + \frac{5}{3}\sin t - \frac{1}{3}\sin 2t$$