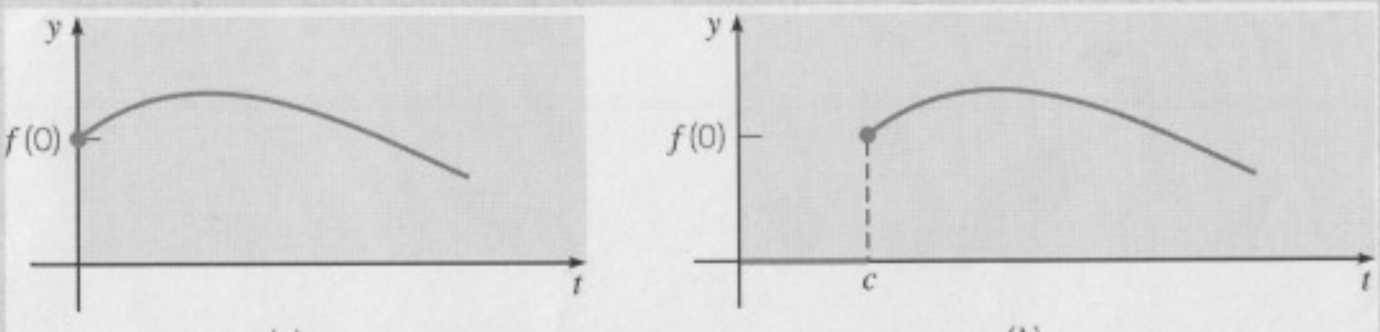
Lesson 26

Ch 6.3: Step Functions

- Some of the most interesting elementary applications of the Laplace Transform method occur in the solution of linear equations with discontinuous or impulsive forcing functions.
- In this section, we will assume that all functions considered are piecewise continuous and of exponential order, so that their Laplace Transforms all exist, for s large enough.

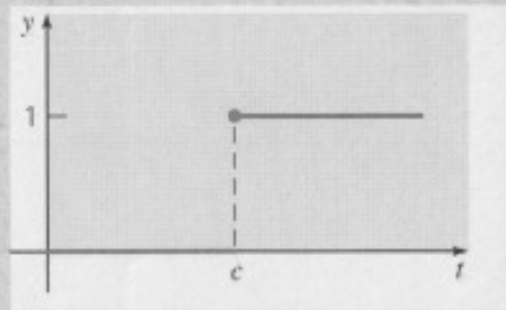


(a) (b)

Step Function definition

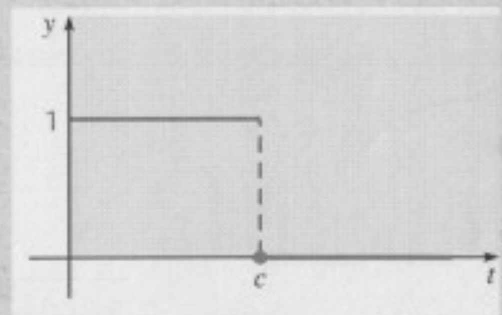
- Let $c \geq 0$. The **unit step function**, or Heaviside function, is defined by

$$u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$$



- A negative step can be represented by

$$y(t) = 1 - u_c(t) = \begin{cases} 1, & t < c \\ 0, & t \geq c \end{cases}$$



Example 1

- Sketch the graph of

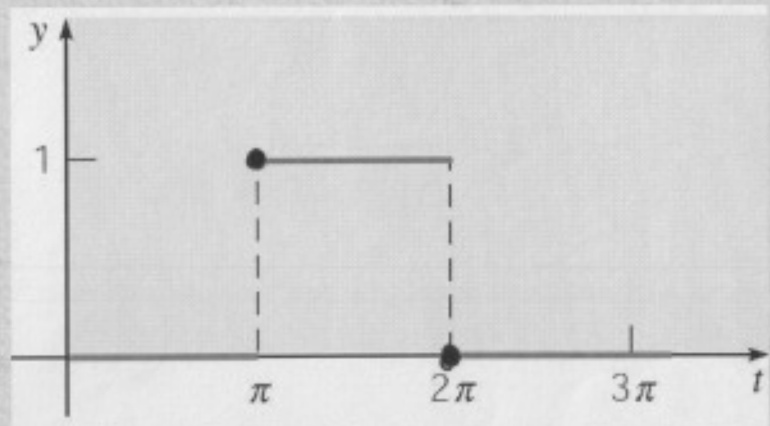
$$h(t) = u_{\pi}(t) - u_{2\pi}(t), \quad t \geq 0$$

- Solution: Recall that $u_c(t)$ is defined by

$$u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$$

- Thus

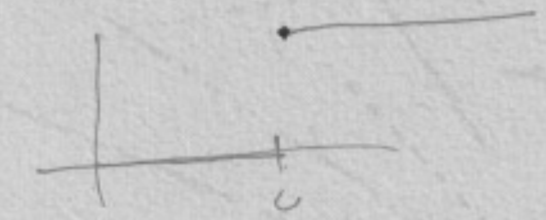
$$h(t) = \begin{cases} 0, & 0 \leq t < \pi \\ 1, & \pi \leq t < 2\pi \\ 0 & 2\pi \leq t < \infty \end{cases}$$



and hence the graph of $h(t)$ is a rectangular pulse.

Laplace Transform of Step Function

- The Laplace Transform of $u_c(t)$ is



$$L\{u_c(t)\} = \int_0^\infty e^{-st} u_c(t)\,dt = \int_c^\infty e^{-st}\,dt$$

$$= \lim_{b\to\infty} \int_c^b e^{-st}\,dt = \lim_{b\to\infty} \left[-\frac{1}{s} e^{-st} \Big|_c^b \right]$$

$$= \lim_{b\to\infty} \left[-\frac{e^{-bs}}{s} + \frac{e^{-cs}}{s} \right]$$

$$= \frac{e^{-cs}}{s}$$

$$\mathcal{L}(u_c(t)) = \frac{e^{-cs}}{s}$$

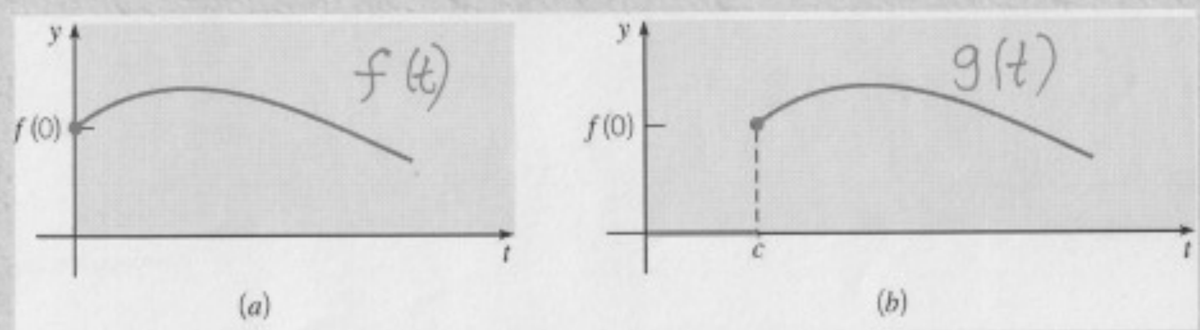
$$\mathcal{L}^{-1}\left(\frac{e^{-cs}}{s}\right) = u_c(t)$$

Translated Functions

- Given a function $f(t)$ defined for $t \geq 0$, we will often want to consider the related function $g(t) = u_c(t) f(t - c)$:

$$g(t) = \begin{cases} 0, & t < c \\ f(t-c), & t \geq c \end{cases}$$

- Thus g represents a translation of f a distance c in the positive t direction.
- In the figure below, the graph of f is given on the left, and the graph of g on the right.



Example 2

- Sketch the graph of

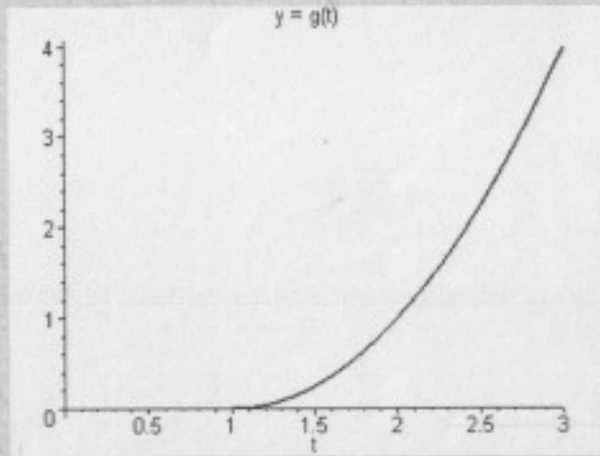
$$g(t) = f(t-1)u_1(t), \quad \text{where } f(t) = t^2,\ t \geq 0.$$

- Solution: Recall that $u_c(t)$ is defined by

$$u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$$

- Thus

$$g(t) = \begin{cases} 0, & 0 \leq t < 1 \\ (t-1)^2, & t \geq 1 \end{cases}$$



and hence the graph of $g(t)$ is a shifted parabola.

Theorem 6.3.1

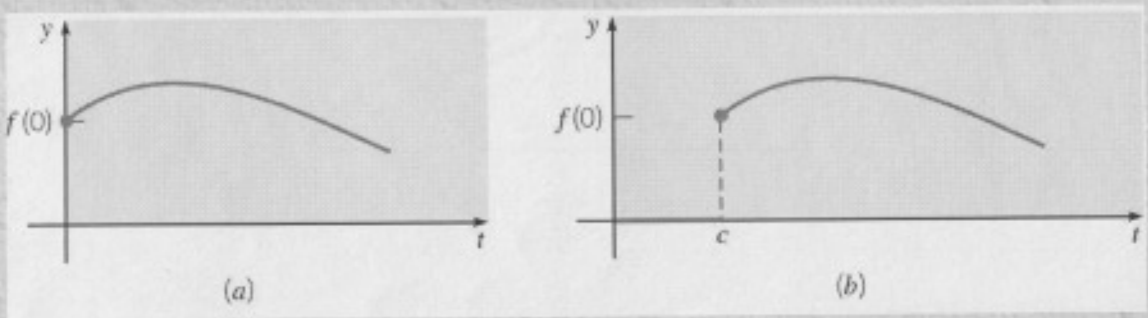
- If $F(s) = L\{f(t)\}$ exists for $s > a \geq 0$, and if $c > 0$, then

$$L\{u_c(t)f(t-c)\} = e^{-cs}L\{f(t)\} = e^{-cs}F(s)$$

- Conversely, if $f(t) = L^{-1}\{F(s)\}$, then

$$u_c(t)f(t-c) = L^{-1}\{e^{-cs}F(s)\}$$

- Thus the translation of $f(t)$ a distance c in the positive t direction corresponds to a multiplication of $F(s)$ by e^{-cs}.



Theorem 6.3.1: Proof Outline

- We need to show

$$L\{u_c(t)f(t-c)\} = e^{-cs}F(s)$$

$F(s) = \mathcal{L}(f(t))$.

- Using the definition of the Laplace Transform, we have

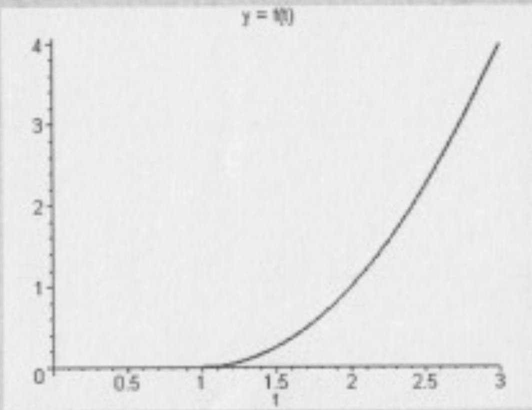
$$\begin{aligned} L\{u_c(t)f(t-c)\} &= \int_0^\infty e^{-st}u_c(t)f(t-c)\,dt \\ &= \int_c^\infty e^{-st}f(t-c)\,dt \\ &\overset{u=t-c}{=} \int_0^\infty e^{-s(u+c)}f(u)\,du \\ &= e^{-cs}\int_0^\infty e^{-su}f(u)\,du \\ &= e^{-cs}F(s) \end{aligned}$$

$u = t - c \Rightarrow t = u + c$
$du = dt$

Example 3

- Find the Laplace transform of

$$\mathcal{g}(t) = \begin{cases} 0, & 0 \le t < 1 \\ (t-1)^2, & t \ge 1 \end{cases}$$



- Solution: Note that

$$\mathcal{g}(t) = (t-1)^2 u_1(t) = u_1(t) f(t-1) \quad c = 1 \quad f(t) = t^2$$

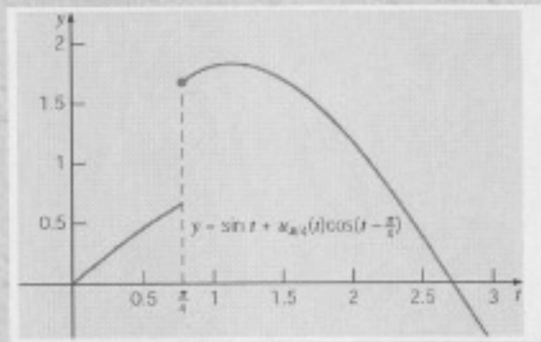
- Thus

$$L\{\mathcal{g}(t)\} = L\{u_1(t)(t-1)^2\} = e^{-s} L\{t^2\} = \frac{2e^{-s}}{s^3}$$

Example 4

- Find $L\{\mathcal{g}(t)\}$, where $\mathcal{g}$ is defined by

$$\mathcal{g}(t) = \begin{cases} \sin t, & 0 \le t < \pi/4 \\ \sin t + \cos(t - \pi/4), & t \ge \pi/4 \end{cases}$$



- Note that $\mathcal{g}(t) = \sin(t) + u_{\pi/4}(t)\cos(t - \pi/4)$, and $t \geq 0$

$$\begin{aligned} L\{\mathcal{g}(t)\} &= L\{\sin t\} + L\{u_{\pi/4}(t)\cos(t - \pi/4)\} \\ &= L\{\sin t\} + e^{-\pi s/4} L\{\cos t\} \\ &= \frac{1}{s^2+1} + e^{-\pi s/4}\frac{s}{s^2+1} \\ &= \frac{1 + s e^{-\pi s/4}}{s^2+1} \end{aligned}$$

$c = \pi/4$

$f(t) = \cos t$

$\mathcal{L}(u_c(t) f(t-c)) = e^{-sc}\mathcal{L}(f(t))$

Example 5

- Find $L^{-1}\{G(s)\}$, where

$$G(s)=\frac{3+e^{-7s}}{s^4}$$

- Solution:

$$g(t)=L^{-1}\left\{\frac{3}{s^4}\right\}+L^{-1}\left\{\frac{e^{-7s}}{s^4}\right\}$$

$$=\frac{1}{2}L^{-1}\left\{\frac{3!}{s^4}\right\}+\frac{1}{6}L^{-1}\left\{e^{-7s}\cdot\frac{3!}{s^4}\right\}$$

$$=\frac{1}{2}t^3+\frac{1}{6}u_7(t)(t-7)^3$$

$\mathcal{L}(u_c(t)f(t-c))=e^{-sc}\mathcal{L}(f(t))$

$f(t)=t^3$

$c=7$

Theorem 6.3.2

- If $F(s) = L\{f(t)\}$ exists for $s > a \geq 0$, and if c is a constant, then

$$\boxed{L\left\{e^{ct} f(t)\right\} = F(s-c), \quad s > a+c}$$

F(s) = $\mathcal{L}$(f(t))

- Conversely, if $f(t) = L^{-1}\{F(s)\}$, then

$$e^{ct} f(t) = L^{-1}\left\{F(s-c)\right\}$$

- Thus multiplication $f(t)$ by e^{ct} results in translating $F(s)$ a distance c in the positive t direction, and conversely.
- Proof Outline:

$$L\left\{\underline{e^{ct} f(t)}\right\} = \int_0^\infty e^{-st} e^{ct} f(t)dt = \int_0^\infty e^{-(s-c)t} f(t)dt = F(s-c)$$

Example 4

- Find the inverse transform of

$$G(s) = \frac{s+1}{s^2+2s+5}$$

- To solve, we first complete the square:

$$G(s) = \frac{s+1}{s^2+2s+5} = \frac{s+1}{\left(s^2+2s+1\right)+4} = \frac{(s+1)}{(s+1)^2+4} = \frac{s-(-1)}{(s-(-1))^2+4}$$

- Since

$$f(t) = L^{-1}\{F(s)\} = L^{-1}\left\{\frac{s}{s^2+4}\right\} = \cos(2t)$$

it follows that

$$L^{-1}\{G(s)\} = L^{-1}\{F(s+1)\} = e^{-t} f(t) = e^{-t}\cos(2t)$$

$\mathcal{L}(e^{ct} f(t)) = F(s-c) \quad F = \mathcal{L}(f(t))$