Section 7.9.

Non-homogeneous linear systems.

We want to solve:

$$\vec{x}'(t) = A\vec{x}(t) + \vec{g}(t). \quad (*)$$

The general solution of this non-homogeneous linear system is:

$$\vec{x}(t) = \vec{x}_H(t) + \vec{x}_p(t)$$

where $\vec{x}_H(t)$ is the general solution to

$$\vec{x}'(t) = A\vec{x}(t),$$

and $\vec{x}_p(t)$ is a particular solution of $(*)$.

In order to compute $\vec{x}_p(t)$ we use the method of undetermined coefficients in a similar way as with the second order linear equation:

$$ay'' + by' + cy = g(t)$$

The main difference is illustrated with the following example

Ex: Find the general solution of:

$$\vec{x}'(t) = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \vec{x}(t) + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}$$

$$\vec{g}(t) = \begin{pmatrix} 2 \\ 0 \end{pmatrix} e^{-t} + \begin{pmatrix} 0 \\ 3 \end{pmatrix} t$$

$$\det \begin{pmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{pmatrix} = (-2-\lambda)^2 = \lambda^2 + 4\lambda + 4 - 1$$

$$= \lambda^2 + 4\lambda + 3 = 0$$

$$(\lambda+3)(\lambda+1) = 0$$

$$\lambda_1 = -3 \quad \lambda_2 = -1$$

The eigenvectors are:

$$\vec{e}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ for } \lambda_1 = -3$$

$$\vec{e}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ for } \lambda_2 = -1$$

$\Rightarrow \quad \vec{x}_H(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$; since one of these terms appears in $\vec{g}(t)$, then we need to multiply by t the proposed $\vec{x}_p(t)$, but, this is where it is different from second order equations, we need to add the term without a t:

$$\vec{x}_p(t) = \vec{a}\, t e^{-t} + \vec{b} e^{-t} + \underbrace{\vec{c}\, t + \vec{d}}$$

this corresponds to $\begin{pmatrix} 0 \\ 3 \end{pmatrix} t$

constant vector times exponential multiplied by t, corresponds to $\begin{pmatrix} 2 \\ 0 \end{pmatrix} e^{-t}$

we have to add the term without the t.

Solving for the constants yields:

$$\vec{x}(t) = c_1\begin{pmatrix}1\\-1\end{pmatrix}e^{-3t} + c_2\begin{pmatrix}1\\1\end{pmatrix}e^{-t} + \begin{pmatrix}1\\1\end{pmatrix}te^{-t} - \begin{pmatrix}0\\1\end{pmatrix}e^{-t} + \begin{pmatrix}1\\2\end{pmatrix}t - \frac{1}{3}\begin{pmatrix}4\\5\end{pmatrix}.$$

<u>Ex</u>: Find the general solution of the linear system:

$$\vec{x}\,'(t) = \begin{pmatrix}2 & 1\\0 & 1\end{pmatrix}\vec{x}(t) + \begin{pmatrix}t\\0\end{pmatrix}$$

We notice that $\vec{g}(t) = t\begin{pmatrix}1\\0\end{pmatrix}$.

$$\det(A-\lambda I) = \det\begin{pmatrix}2-\lambda & 1\\0 & 1-\lambda\end{pmatrix}$$
$$= (2-\lambda)(1-\lambda) = 0$$
$$\lambda_1 = 2 \quad \lambda_2 = 1$$

For $\lambda_1 = 2$:

$$\begin{pmatrix}0 & 1\\0 & -1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$x_2 = 0 \quad x_1 = r$, any real number r.

The eigenspace is:

$$\left\{\begin{pmatrix}r\\0\end{pmatrix} = r\begin{pmatrix}1\\0\end{pmatrix} : r \text{ is any number}\right\}$$

- For $\lambda_2 = 1$

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x_1 + x_2 = 0 \qquad x_1 = -x_2$$

The eigenspace is:

$$\left\{ \begin{pmatrix} -r \\ r \end{pmatrix} = r\begin{pmatrix} -1 \\ 1 \end{pmatrix} : r \text{ is any number} \right\}$$

We have:

$$\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = 1\begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Hence;

$$\vec{x}_H(t) = c_1\begin{pmatrix} -1 \\ 1 \end{pmatrix}e^{t} + c_2\begin{pmatrix} 1 \\ 0 \end{pmatrix}e^{2t}$$

Using the method of undetermined coefficients and look for:

$$\vec{x}_p(t) = \vec{a}t + \vec{b}.$$

We need to find $\vec{a}$ and $\vec{b}$ so that:

$$\vec{x}_p'(t) = A\vec{x}_p(t) + \vec{g}(t)$$

$$\vec{x}_p'(t) = \vec{a} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}(\vec{a}t + \vec{b}) + t\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= t\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}\vec{a} + \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}\vec{b} + t\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\vec{0} = t\left[\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] + \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} - \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Hence:

$$\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$2a_1 + a_2 = -1$$

$$\boxed{a_2 = 0}$$

$$\Rightarrow \boxed{a_1 = -\frac{1}{2}}$$

$$\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 0 \end{pmatrix}$$

$$2b_1 + b_2 = -1/2$$

$$\boxed{b_2 = 0}$$

$$\boxed{b_1 = -1/4}$$

Hence:

$$\vec{x}_p(t) = \begin{pmatrix} -1/2 \\ 0 \end{pmatrix} t + \begin{pmatrix} -1/4 \\ 0 \end{pmatrix}$$

The general solution to the system is:

$$\vec{x}(t) = \vec{x}_H(t) + \vec{x}_p(t)$$

$$= c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + \begin{pmatrix} -1/2 \\ 0 \end{pmatrix} t + \begin{pmatrix} -1/4 \\ 0 \end{pmatrix}$$

<u>Ex</u>: Find the general solution of the system:

$$\vec{x}'(t) = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \vec{x}(t) + \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}$$

$$\det(A - \lambda I) = \det \begin{pmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{pmatrix}$$

$$= (2-\lambda)(-2-\lambda) + 5 = 0$$

$$-4 - \cancel{2\lambda} + \cancel{2\lambda} + \lambda^2 + 5 = 0$$

$$\lambda^2 + 1 = 0 \quad \boxed{\lambda_1 = i, \ \lambda_2 = -i}$$

$\boxed{\text{For } \lambda_1 = i :}$

$$\begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(2-i)c_1 - 5c_2 = 0$$

$$c_1 - (2+i)c_2 = 0$$

If we multiply the second equation by $(2-i)$ we obtain the first equation. If we use the first equation we obtain:

$$(2-i)c_1 = 5c_2 \qquad c_2 = \frac{(2-i)}{5} c_1$$

If $c_1 = 5 \Rightarrow c_2 = (2-i)$ so an eigenvector is

$$\begin{pmatrix} 5 \\ 2-i \end{pmatrix}$$

We can also use the second equation to obtain another eigenvector:

$$c_1 = (2+i)c_2$$

If $c_2 = 1 \Rightarrow c_1 = (2+i)$

$\Rightarrow \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$ is another eigenvector.

Notice that $\begin{pmatrix} 2+i \\ 1 \end{pmatrix}(2-i) = \begin{pmatrix} 5 \\ 2-i \end{pmatrix}$,

so as expected, one eigenvector is a multiple of the other; since the eigenspace is a vector space of dimension 1:

$$\left\{ \begin{pmatrix} 5 \\ 2-i \end{pmatrix} c \ : \ c \text{ is any complex number} \right\}$$

or

$$\left\{ \begin{pmatrix} 2+i \\ 1 \end{pmatrix} c \ : \ c \text{ is any complex number} \right\}$$

In particular notice that:

$$\begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 5 \\ 2-i \end{pmatrix} = i \begin{pmatrix} 5 \\ 2-i \end{pmatrix}$$

Also:

$$\begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 2+i \\ 1 \end{pmatrix} = i \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$$

We have:

$$\lambda_1 = i \,, \quad \vec{e}_1 = \begin{pmatrix} 5 \\ 2-i \end{pmatrix} \text{ or } \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$$

$$\lambda_2 = -i \,, \quad \vec{e}_1 = \begin{pmatrix} 5 \\ 2+i \end{pmatrix} \text{ or } \begin{pmatrix} 2-i \\ 1 \end{pmatrix}$$

$$\begin{aligned} e^{\lambda_1 t}\vec{e}_1 &= e^{it}\begin{pmatrix} 2+i \\ 1 \end{pmatrix} \\ &= (\cos t + i\sin t)\begin{pmatrix} 2+i \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 2\cos t + i\cos t + 2i\sin t - \sin t \\ \cos t + i\sin t \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + i\begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix}$$

Hence:

$$\boxed{\begin{aligned} \vec{x}_H(t) &= c_1\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + c_2\begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix} \\ &= c_1\vec{x}^{(1)}(t) + c_2\vec{x}^{(2)}(t). \end{aligned}}$$

We can also use the other eigenvector: (42)

$$e^{it}\begin{pmatrix}5\\2-i\end{pmatrix} = (\cos t + i\sin t)\begin{pmatrix}5\\2-i\end{pmatrix}$$

$$= \begin{pmatrix}5\cos t + 5i\sin t\\2\cos t - i\cos t + 2i\sin t - i^2\sin t\end{pmatrix}$$

$$= \begin{pmatrix}5\cos t\\2\cos t + \sin t\end{pmatrix} + i\begin{pmatrix}5\sin t\\2\sin t - \cos t\end{pmatrix}$$

We have:

$$\boxed{\begin{aligned}\vec{x}_H(t) &= c_1\begin{pmatrix}5\cos t\\2\cos t + \sin t\end{pmatrix} + c_2\begin{pmatrix}5\sin t\\2\sin t - \cos t\end{pmatrix}\\ &= c_1\,\vec{x}^{(3)}(t) + c_2\,\vec{x}^{(4)}(t)\end{aligned}}$$

Both solutions are correct: $\{\vec{x}^{(1)}(t), \vec{x}^{(2)}(t)\}$ is a fundamental set of solutions (i.e., they are linearly independent), and $\{\vec{x}^{(3)}(t), \vec{x}^{(4)}(t)\}$ is also a fundamental set of solutions.

Notice that:

$$2\vec{x}^{(1)}(t) + \vec{x}^{(2)}(t) = \vec{x}^{(3)}(t)$$

$$-\vec{x}^{(1)}(t) + 2\vec{x}^{(2)}(t) = \vec{x}^{(4)}(t)$$

We check now that they are fundamental set of solutions:

$$W(\vec{x}^{(1)}; \vec{x}^{(2)}) = \begin{vmatrix} 2\cos t - \sin t & 2\sin t + \cos t \\ \cos t & \sin t \end{vmatrix}$$

$$= \cancel{2\sin t \cos t} - \sin^2 t - \cancel{2\sin t \cos t} - \cos^2 t$$

$$= -1 \neq 0$$

Also:

$$W(\vec{x}^{(3)}(t), \vec{x}^{(4)}(t)) = \begin{vmatrix} 5\cos t & 5\sin t \\ 2\cos t + \sin t & 2\sin t - \cos t \end{vmatrix}$$

$$= \cancel{10\sin t \cos t} - 5\cos^2 t - \cancel{10\sin t \cos t} - 5\sin^2 t$$

$$= -5 \neq 0.$$

Recall that:

" The space of solutions is a 2-dimensional vector space, and solving the homogeneous equation means finding a base; i.e., two solutions that are linearly independent".

We now proceed to find a particular solution for the system:

$$\vec{x}\,'(t) = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \vec{x}(t) + \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}$$

We have:

$$\vec{g}(t) = \cos t \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \sin t \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\vec{x}_p(t)$ is of the form:

$$\vec{x}_p(t) = \vec{a}\, t\cos t + \vec{b}\, t\sin t + \vec{c}\cos t + \vec{d}\sin t.$$

$$\vec{x}_p\,'(t) = \vec{a}\cos t - \vec{a}\, t\sin t + \vec{b}\sin t + \vec{b}\, t\cos t - \vec{c}\sin t + \vec{d}\cos t$$

$$= (\vec{a}+\vec{d})\cos t + (\vec{b}-\vec{c})\sin t + \vec{b}\, t\cos t - \vec{a}\, t\sin t$$

We need to find $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, $\vec{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$, $\vec{d} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$ so that:

$$\vec{x}_p\,'(t) = A\vec{x}_p(t) + \vec{g}(t),$$

We want:

$$(\vec{a}+\vec{d})\cos t + (\vec{b}-\vec{c})\sin t + \vec{b}\, t\cos t - \vec{a}\, t\sin t$$

$$= A\left(\vec{a}\, t\cos t + \vec{b}\, t\sin t + \vec{c}\cos t + \vec{d}\sin t\right) + \cos t\begin{pmatrix} -1 \\ 0 \end{pmatrix} + \sin t\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

We simplify:

$$\left(\vec{a}+\vec{d}-A\vec{c}+\begin{pmatrix}1\\0\end{pmatrix}\right)\cos t+\left(\vec{b}-\vec{c}-A\vec{d}-\begin{pmatrix}0\\1\end{pmatrix}\right)\sin t$$
$$+\left(\vec{b}-A\vec{a}\right)t\cos t+\left(-\vec{a}-A\vec{b}\right)t\sin t=\begin{pmatrix}0\\0\end{pmatrix}$$

From here, we obtain:

(A) $\vec{a}+\vec{d}-A\vec{c}+\begin{pmatrix}1\\0\end{pmatrix}=\vec{0}$

(B) $\vec{b}-\vec{c}-A\vec{d}-\begin{pmatrix}0\\1\end{pmatrix}=\vec{0}$

(C) $\vec{b}-A\vec{a}=\vec{0}$

(D) $-\vec{a}-A\vec{b}=\vec{0}$

From (A) we obtain the following two equations:

$$\begin{pmatrix}a_1\\a_2\end{pmatrix}+\begin{pmatrix}d_1\\d_2\end{pmatrix}-\begin{pmatrix}2&-5\\1&-2\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix}=\begin{pmatrix}-1\\0\end{pmatrix}$$

$$\begin{pmatrix}a_1\\a_2\end{pmatrix}+\begin{pmatrix}d_1\\d_2\end{pmatrix}-\begin{pmatrix}2c_1-5c_2\\c_1-2c_2\end{pmatrix}=\begin{pmatrix}-1\\0\end{pmatrix}$$

$$\boxed{\begin{aligned}&(1)\quad a_1+d_1-2c_1+5c_2=-1\\&(2)\quad a_2+d_2-c_1+2c_2=0\end{aligned}}$$

From (B):

$$\begin{pmatrix}b_1\\b_2\end{pmatrix}-\begin{pmatrix}c_1\\c_2\end{pmatrix}-\begin{pmatrix}2&-5\\1&-2\end{pmatrix}\begin{pmatrix}d_1\\d_2\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}$$

$$\begin{pmatrix}b_1\\b_2\end{pmatrix}-\begin{pmatrix}c_1\\c_2\end{pmatrix}-\begin{pmatrix}2d_1-5d_2\\d_1-2d_2\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}$$

$$\boxed{\begin{aligned} &(3)\quad b_1 - c_1 - 2d_1 + 5d_2 = 0 \\ &(4)\quad b_2 - c_2 - d_1 + 2d_2 = 1 \end{aligned}}$$

From (C) ;

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} - \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} - \begin{pmatrix} 2a_1 - 5a_2 \\ a_1 - 2a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\boxed{\begin{aligned} &(5)\quad b_1 - 2a_1 + 5a_2 = 0 \\ &(6)\quad b_2 - a_1 + 2a_2 = 0 \end{aligned}}$$

From (D) :

$$-\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} - \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -a_1 \\ -a_2 \end{pmatrix} - \begin{pmatrix} 2b_1 - 5b_2 \\ b_1 - 2b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\boxed{\begin{aligned} &(7)\quad -a_1 - 2b_1 + 5b_2 = 0 \\ &(8)\quad -a_2 - b_1 + 2b_2 = 0 \end{aligned}}$$

We have eight equations with 8 unknowns.

As you learned in linear algebra, you put the system in an augmented matrix:

$$
\begin{array}{c}
\begin{array}{cccccccc|c}
a_1 & a_2 & b_1 & b_2 & c_1 & c_2 & d_1 & d_2 & \\
\end{array}\\
\begin{array}{cccccccc|c}
1 & 0 & 0 & 0 & -2 & 5 & 1 & 0 & -1 \\
0 & 1 & 0 & 0 & -1 & 2 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 & -2 & 5 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 & -1 & 2 & 1 \\
-2 & 5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & -2 & 5 & 0 & 0 & 0 & 0 & 0
\end{array}
\end{array}
$$

We perform row reduction to compute the solution values for $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2$. You can also use matlab to solve the system.

The solution is:

$a_1 = 2$ $\quad b_1 = -1$ $\quad c_1 = -1$ $\quad d_1 = 0$

$a_2 = 1$ $\quad b_2 = 0$ $\quad c_2 = -1$ $\quad d_2 = 0$

Hence:

The general solution of the system: (48)

$$\vec{x}'(t) = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \vec{x}(t) + \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}$$

is:

$$\vec{x}(t) = c_1 \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + c_2 \begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix}$$

$$+ \begin{pmatrix} 2 \\ 1 \end{pmatrix} t\cos t + \begin{pmatrix} -1 \\ 0 \end{pmatrix} t\sin t + \begin{pmatrix} -1 \\ -1 \end{pmatrix} \cos t \; .$$