Lecture 11

Area formula

We have:

<u>Theorem 1 (Area formula for injective maps).</u> If $f: \mathbb{R}^n \to \mathbb{R}^m$ $(1 \le n \le m)$ is 1-1, Lipschitz, $E \subset \mathbb{R}^n$ Lebesgue measurable, then

$$\mathcal{H}^n(f(E)) = \int_E Jf(x)\, dx,$$

and $\mathcal{H}^n \llcorner f(\mathbb{R}^n)$ is a Radon measure on $\mathbb{R}^m$.

Note: $Jf(x)$ was defined in Lecture 8.

<u>Remark</u>: If $g: \mathbb{R}^m \to [-\infty, \infty]$ is Borel measurable on $\mathbb{R}^m$ and either $g \geq 0$ or $g \in L^1(\mathbb{R}^m, \mathcal{H}^n \llcorner f(\mathbb{R}^n))$, then $g \circ f$ is Borel measurable on $\mathbb{R}^n$ and

$$\int_{f(\mathbb{R}^n)} g(y)\, d\mathcal{H}^n(y) = \int_{\mathbb{R}^n} g(f(x))\, Jf(x)\, dx$$

Indeed, if $g \geq 0$ is a simple function, then

$$g = \sum_{i=1}^{\infty} c_i \chi_{F_i}, \quad c_i \geq 0, \quad F_i \in \mathcal{B}(\mathbb{R}^m).$$

Let:

$$E_i = f^{-1}(F_i)$$

Then:

$$g \circ f = \sum_{i=1}^{\infty} c_i \chi_{E_i}$$

Hence:

$$\int_{\mathbb{R}^m} g \, d(\mathcal{H}^n \llcorner f(\mathbb{R}^n)) = \sum_{i=1}^{\infty} c_i \, \mathcal{H}^n(F_i \cap f(\mathbb{R}^n))$$

$$= \sum_{i=1}^{\infty} c_i \int_{E_i} Jf(x)\,dx \, ; \quad \text{by the Area formula}$$

$$= \int_{\mathbb{R}^n} \left(\sum_{i=1}^{\infty} c_i \chi_{E_i} \right) Jf(x)\,dx$$

$$= \int_{\mathbb{R}^n} (g \circ f)(x)\, Jf(x)\,dx$$

Thus, it is true for any g simple function by writting $g = g^+ - g^-$. Approximating by simple functions we obtain:

$$\int_{f(\mathbb{R}^n)} g(y)\,dy = \int_{\mathbb{R}^n} (g\circ f)(x)\, Jf(x)\,dx \quad \blacksquare$$

We have:

__Lemma__: If E is a Lebesgue measurable set in $\mathbb{R}^n$ and $f: \mathbb{R}^n \to \mathbb{R}^m$ $(1 \le n \le m)$ is a Lipschitz function, then $f(E)$ is $\mathcal{H}^n$-measurable in $\mathbb{R}^m$.

__Proof__: WLOG E is bounded, so $|E| < \infty$.

E $\mathcal{L}^n$-measurable $\Rightarrow \exists \{K_i\}_{i=1}^{\infty}$ such that

$K_i \subset E$ and $|E \setminus K_i| \le \frac{\varepsilon}{2^i}$, ε fixed, $i = 1, 2, \dots$

f continuous $\Rightarrow f(K_i)$ is compact, $i=1,2,\dots$

$\Rightarrow \bigcup_{i=1}^{\infty} f(K_i)$ is a Borel set

$\Rightarrow \bigcup_{i=1}^{\infty} f(K_i)$ is $\mathcal{H}^n$-measurable in $\mathbb{R}^m$, since $\mathcal{H}^n$ is a Borel outer measure in $\mathbb{R}^m$,

Let

$$N = f(E) \setminus \bigcup_{i=1}^{\infty} f(K_i)$$

$$\begin{aligned}
\mathcal{H}^n(N) &= \mathcal{H}^n\Big(f(E) \setminus \bigcup_{i=1}^{\infty} f(K_i)\Big) \\
&\leq \mathcal{H}^n\Big(f\big(E \setminus \bigcup_{i=1}^{\infty} K_i\big)\Big) \\
&\leq (\mathrm{Lip}(f))^n \, \Big|E \setminus \bigcup_{i=1}^{\infty} K_i\Big| \\
&= (\mathrm{Lip}(f))^n \, \Big|\bigcup_{i=1}^{\infty} (E \setminus K_i)\Big| \\
&\leq (\mathrm{Lip}(f))^n \sum_{i=1}^{\infty} |E \setminus K_i| = \varepsilon\, (\mathrm{Lip}(f))^n
\end{aligned}$$

$\varepsilon \to 0$ yields $\mathcal{H}^n(N) = 0$

Note that, for an outer measure μ, if $\mu(E)=0$ then E is μ-measurable. Since $\forall F$, $\mu(F) \geq \underbrace{\mu(F \cap E)}_{=0} + \mu(F \setminus E)$.

Thus, since $\mathcal{H}^n$ is an outer measure in $\mathbb{R}^m$
$\Rightarrow N$ is $\mathcal{H}^n$-measurable. Since $f(E) = N \cup \Big(\bigcup_{i=1}^{\infty} f(K_i)\Big)$,

the union of two $\mathcal{H}^n$-measurable sets, we conclude that $f(E)$ is $\mathcal{H}^n$-measurable in $\mathbb{R}^m$.

We have:

Theorem 2 (Area formula for Linear functions):

If $T \in \mathbb{R}^m \otimes \mathbb{R}^n$ $(1 \leq n \leq m)$, then $\forall E \subset \mathbb{R}^n$:

$$\mathcal{H}^n(T(E)) = JT\,|E|$$

The proof of Theorem 2 can be found in the textbook (Theorem 8.5). We now proceed to prove the Lipschitz linearization (See Theorem 1 in Lecture 8) that we used in the study of rectifiable sets. This Lipschitz linearization makes a partition of:

$$F = \{x \in \mathbb{R}^n : 0 < Jf(x) < \infty\}$$

for $f : \mathbb{R}^n \to \mathbb{R}^m$ Lipschitz. If $x \in F$, then by the polar decomposition theorem in linear algebra we have:

$$\nabla f(x) = P_x S_x ,$$

where $S_x \in \mathrm{Sym}(n) \cap GL(n)$ and $P_x \in O(n,m)$, an orthogonal injection.

Thm (Lipschitz linearization):

$f: \mathbb{R}^n \to \mathbb{R}^m$ $(1 \le n \le m)$ Lipschitz.

Let
$$F = \{x \in \mathbb{R}^n : 0 < Jf(x) < \infty\}$$

$\Rightarrow$ $\exists \{F_k\}$, $F = \bigcup_{k=1}^{\infty} F_k$, f is 1-1 on each F_k

Moreover, $\forall t > 1$, the partition can be found with the property that, for every $k = 1, 2, \cdots$, $\exists\, S_k \in GL(n)$ such that $\forall x, y \in F_k$, $v \in \mathbb{R}^n$:

$$(*)\begin{cases} t^{-1}|S_k x - S_k y| \le |f(x) - f(y)| \le t\,|S_k x - S_k y| \\ \qquad \forall x, y \in F_k,\ v \in \mathbb{R}^n \\ t^{-1}|S_k v| \le |\nabla f(x)\, v| \le t\,|S_k v| \\ t^{-n}\, JS_k \le Jf(x) \le t^n\, JS_k \end{cases}$$

(i.e. $f|_{F_k} \circ S_k^{-1}$ is almost an isometry).

Proof: It is enough to show that F can be covered by sets F_k satisfying $(*)$ since, once this is done, we can replace each F_k with
$$F_k \setminus \bigcup_{\ell=1}^{k-1} F_\ell,$$
in order to define the desired partition of F.

Let $\varepsilon > 0$ so that:
$$t^{-1} + \varepsilon < 1 < t - \varepsilon$$

Let $\mathcal{G} = \{S_j\}_{j=1}^{\infty}$ be a dense set (in the operator norm) in $GL(n)$. (11.6)

Define:

$$F_{i,j} = \left\{ x \in F : \begin{array}{l} (t^{-1}+\varepsilon)|S_j v| \le |\nabla f(x) v| \le (t-\varepsilon)|S_j v| \quad \forall v \in \mathbb{R}^n \\ |f(x+v) - f(x) - \nabla f(x) v| \le \varepsilon |S_j v|, \ \forall v \in \mathbb{R}^n, \ |v| \le \frac{1}{i} \end{array} \right\}$$

<u>Claim</u>: $F = \bigcup_{i,j=1}^{\infty} F_{i,j}$

Indeed, let $x \in F$. By polar decomposition we have:

$\nabla f(x) = P_x S_x$, P_x orthogonal S_x symmetric and invertible.

Since $\{S_j\}$ is dense, for every $\delta > 0$, $\exists j$ such that

$\|S_x - S_j\| < \delta$ (operator norm),

which implies: (A)

$$\boxed{\|S_x S_j^{-1}\| \le 1 + \delta \|S_j^{-1}\|, \quad \|S_j S_x^{-1}\| \le 1 + \delta \|S_x^{-1}\|}$$

Because:

$\|S_x - S_j\| \le \delta \Rightarrow |(S_x - S_j)(v)| \le \delta |v| \quad \forall v$

$\Rightarrow |(S_x - S_j)(S_j^{-1} v)| \le \delta |S_j^{-1} v|$

$|S_x S_j^{-1} v| - |v| \le |(S_x S_j^{-1})v) - v| \le \delta |S_j^{-1} v|$

$\therefore \quad |S_x S_j^{-1} v| \le |v| + \delta |S_j^{-1} v|, \quad \forall v \in \mathbb{R}^n$

$\therefore \quad \| S_x S_j^{-1} v| \le 1 + \delta \, |S_j^{-1} v|, \quad \forall v \in \mathbb{R}^n, \|v\| = 1$

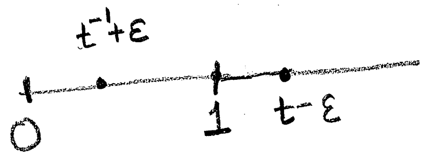
Taking sup over all $v \in \mathbb{R}^n$, $|v| = 1$ gives:

$$\|S_x S_j^{-1}\| \le 1 + \delta \, \|S_j^{-1}\|$$

In the same way we show that

$$\|S_j S_x^{-1}\| \le 1 + \delta \, \|S_x^{-1}\|.$$

We have:



$t > 1$, $\frac{1}{t} < 1$

ε small enough so that

$t^{-1} + \varepsilon \;\; 1 < t - \varepsilon$

We can choose δ small enough so that:

$$(B)\begin{cases} \|S_j S_x^{-1}\| \le 1 + \delta \, \|S_x^{-1}\| \le (t^{-1} + \varepsilon)^{-1} & ; \text{ Note that } (t^{-1}+\varepsilon)^{-1} > 1 \\ \|S_x S_j^{-1}\| \le 1 + \delta \, \|S_j^{-1}\| \le t - \varepsilon \end{cases}$$

From (B):

$$|S_x S_j^{-1}(S_j v)| \le (t - \varepsilon) |S_j v| \Rightarrow |S_x v| \le (t - \varepsilon) |S_j v| \quad \forall v \in \mathbb{R}^n$$

$$|S_j S_x^{-1}(S_x v)| \le (t^{-1} + \varepsilon)^{-1} |S_x v| \Rightarrow |S_j v| \le (t^{-1} + \varepsilon)^{-1} |S_x v|$$

$$\Rightarrow (t^{-1} + \varepsilon) |S_j v| \le |S_x v|$$

Hence, since $|\nabla f(x) v| = |P_x S_x v| = |S_x v|$ we obtain

$$\boxed{(t^{-1} + \varepsilon) |S_j v| \le |\nabla f(x) v| \le (t - \varepsilon) |S_j v| \quad \forall v \in \mathbb{R}^n}$$

Since f is differentiable at x, there exists a modulus of continuity ω_x such that, for i large enough:

$$\boxed{\omega_x\left(\frac{1}{i}\right)\|S_j^{-1}\| \le \varepsilon}$$

and:

$$|f(x+v)-f(x)-\nabla f(x)v| \le \omega_x\left(\frac{1}{i}\right)|v| \quad , \quad |v| < \frac{1}{i}$$

$$\begin{aligned}
\therefore \quad |f(x+v)-f(x)-\nabla f(x)v| &\le \omega_x\left(\frac{1}{i}\right)|v| \\
&= \omega_x\left(\frac{1}{i}\right)|S_j^{-1}S_j v| \\
&\le \omega_x\left(\frac{1}{i}\right)\|S_j^{-1}\|\,|S_j v| \\
&\le \varepsilon\,|S_j v| \quad , \quad \forall v \in \mathbb{R}^n, |v| < \frac{1}{i}
\end{aligned}$$

$\therefore$

$$\boxed{|f(x+v)-f(x)-\nabla f(x)v| \le \varepsilon\,|S_j v| \,,\ \forall v \in \mathbb{R}^n,\ |v| \le \frac{1}{i}}$$

Therefore, we have shown that

$$x \in F_{i,j}$$

which completes the proof of the claim. ∎

Note now that if $x, y \in F_{i,j}$ and $|x-y| < \frac{1}{i}$, then

$$|f(y)-f(x)| \le |\nabla f(x)(y-x)| + \varepsilon |S_j y - S_j x|$$
$$\le (t-\varepsilon)|S_j y - S_j x| + \varepsilon |S_j y - S_j x|$$
$$= t\,|S_j y - S_j x|$$

$$|f(y)-f(x)| \ge |\nabla f(x)(y-x)| - \varepsilon |S_j y - S_j x|$$
$$\ge (t^{-1}+\varepsilon)|S_j y - S_j x| - \varepsilon |S_j y - S_j x|$$
$$= t^{-1}|S_j y - S_j x|$$

$\therefore$

$$\boxed{t^{-1}|S_j y - S_j x| \le |f(y)-f(x)| \le t\,|S_j y - S_j x|}$$

Also, for every $x \in \bigcup_{i=1}^{\infty} F_{i,j}$

$$(t^{-1}+\varepsilon)|S_j v| \le |\nabla f(x) v| \le (t-\varepsilon)|S_j v|$$

$$\Rightarrow (t^{-1}+\varepsilon)^n J S_j \le J f(x) \le (t-\varepsilon)^n J S_j$$ (see Remark 8.6 in textbook).

Let now $\{x_p\}_{p=1}^{\infty}$ be a dense subset of F and we relabel the sequence of sets

$$F_{i,j} \cap B(x_p, \tfrac{1}{2i})$$

as $\{F_k\}_{k=1}^{\infty}$, and (*) holds true on each F_k. ∎

With the Lipschitz linearization we can now prove Theorem 1.

Proof of the Area formula:

We need to prove:

$$\mathcal{H}^n(f(E)) = \int_E Jf(x)\,dx \qquad (**),\ f \text{ is } 1\text{-}1$$

Since $\mathcal{H}^n(f(E)) \leq \mathrm{Lip}(f)^n |E|$, both sides of $(**)$ are zero whenever $\mathcal{H}^n(E)=0$. Thus, by Rademacher's theorem, we only need to prove $(**)$ on the set E, where f is differentiable.

Moreover, since as remarked in Lecture 8:

$$\mathcal{H}^n(f(B)) = 0, \quad \text{where } B = \{x \in \mathbb{R}^n : Jf(x)=0\}.$$

we can assume:

$$E \subset F = \{x \in \mathbb{R}^n : 0 < Jf(x) < \infty\}$$

We can now apply the Lipschitz linearization

Fix $t>1 \Rightarrow \exists \{F_k\}_{k=1}^{\infty},\ F = \bigcup_{k=1}^{\infty} F_k$.

$$\therefore\ E = \bigcup_{k=1}^{\infty} F_k \cap E$$

f is 1-1 on $\mathbb{R}^n \Longrightarrow \ f(E) = \bigcup_{k=1}^{\infty} f(F_k \cap E)$, disjoint union

Using Theorem 2, which is the Area formula for linear maps we have: (11.11)

$$\mathcal{H}^n(f(E)) = \sum_{k=1}^{\infty} \mathcal{H}^n(f(F_k \cap E))$$

$$= \sum_{k=1}^{\infty} \mathcal{H}^n\left(\left(f|_{F_k} \circ S_k^{-1}\right)(S_k(F_k \cap E))\right)$$

$$\leq \sum_{k=1}^{\infty} \operatorname{Lip}\left((f|_{F_k}) \circ S_k^{-1}\right)^n |S_k(F_k \cap E)|$$

$$\leq t^n \sum_{k=1}^{\infty} |S_k(F_k \cap E)| \quad ; \quad \text{by Lipschitz linearization (see } (*)\text{): } \operatorname{Lip}(f|_{F_k} \circ S_k^{-1}) \leq t$$

$$= t^n \sum_{k=1}^{\infty} JS_k \, |F_k \cap E| \quad ; \quad \text{by Theorem 2}$$

$$\leq t^{2n} \sum_{k=1}^{\infty} Jf(x) \, |F_k \cap E| \quad ; \quad \text{by } (*)\text{: } JS_k \leq t^n Jf(x)$$

$$= t^{2n} \sum_{k=1}^{\infty} \int_{F_k \cap E} Jf(x)\, dx$$

$$= t^{2n} \int_E Jf(x)\, dx \quad ; \quad E = \bigcup_{k=1}^{\infty} F_k \cap E \text{ disjoint union}$$

$$\therefore \boxed{\mathcal{H}^n(f(E)) \leq t^{2n} \int_E Jf(x)\, dx} \quad (D)$$

On the other hand:

$$\int_E Jf(x)\,dx = \sum_{k=1}^{\infty} \int_{E\cap F_k} Jf(x)\,dx$$

$$\le t^n \sum_{k=1}^{\infty} JS_k\,|E\cap F_k|\,;$$ since by $(*)$: $Jf(x) \le t^n JS_k$

$$= t^n \sum_{k=1}^{\infty} |S_k(E\cap F_k)|\,;$$ by Theorem 2, area formula for linear maps.

$$= t^n \sum_{k=1}^{\infty} \left| \left[S_k \circ (f|_{F_k})^{-1}\right](f(E\cap F_k)) \right|$$

$$\le t^n \sum_{k=1}^{\infty} \mathrm{Lip}\left[S_k \circ (f|_{F_k})^{-1}\right]^n \mathcal{H}^n(f(E\cap F_k))$$

$$\le t^{2n} \sum_{k=1}^{\infty} \mathcal{H}^n(f(E\cap F_k))$$; since by $(*)$: $\mathrm{Lip}\left[S_k \circ (f|_{F_k})^{-1}\right] \le t$

$$= t^{2n} \mathcal{H}^n(f(E))\,;$$ since $f(E) = \bigcup_{k=1}^{\infty} f(E\cap F_k)$ disjoint

$$\therefore \quad \boxed{\int_E Jf(x)\,dx \le t^{2n}\mathcal{H}^n(f(E))} \quad (E)$$

From (E) and (D):

$$t^{-2n}\mathcal{H}^n(f(E)) \le \int_E Jf(x) \le t^{2n}\mathcal{H}^n(f(E))$$

Letting $t \to 1^+$ we obtain $\int_E Jf(x) = \mathcal{H}^n(f(E))$. ∎