Lesson 14

We have the following Corollary of the compactness theorem (theorem 2) proved in Lecture 13.

<u>Corollary 1</u>: If $\{E_i\}_{i=1}^{\infty}$ are sets of locally finite perimeter in $\mathbb{R}^n$ with:

$$\sup_{i\in\mathbb{N}} P(E_i, B_R) < \infty, \quad \forall R>0$$

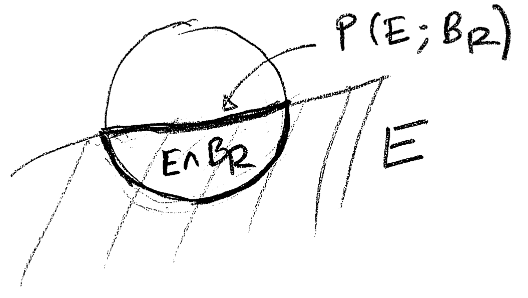
then $\exists E$ of locally finite perimeter and a subsequence $\{E_{i_k}\}_{k=1}^{\infty}$ such that:

$$E_{i_k} \to E \text{ in } L^1_{loc}(\mathbb{R}^n), \quad \mu_{E_{i_k}} \xrightarrow{*} \mu_E .$$

<u>Proof</u>:

<u>Step one:</u> If E is of locally finite perimeter and $R>0$, then:

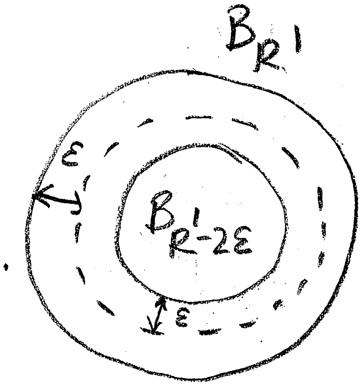
$$P(E\cap B_R) \leq P(E; B_R) + P(B_R).$$



- Given $R' < R$, let $v_\varepsilon \in C_c^\infty(B_{R'})$ be such that:

$$0 \le v_\varepsilon \le 1, \quad v_\varepsilon \longrightarrow \chi_{B_{R'}} \text{ in } L^1(\mathbb{R}^n), \quad \int_{\mathbb{R}^n} |\nabla v_\varepsilon| \xrightarrow[\varepsilon \to 0]{} P(B_{R'}).$$

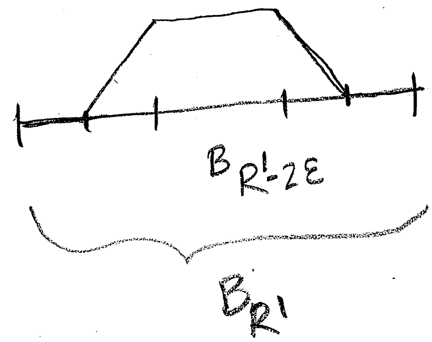
For example, we can define:



$$\tilde{v}_\varepsilon(x) = 1 - \frac{\min\{\operatorname{dist}(x, \bar{B}_{R'-2\varepsilon}), \varepsilon\}}{\varepsilon}$$

$\tilde{v}_\varepsilon$ is Lipschitz. Let $d(x) = \operatorname{dist}(x, \bar{B}_{R'-2\varepsilon})$.

$|\nabla \tilde{v}_\varepsilon(x)| \le \frac{1}{\varepsilon}$ for $\mathcal{L}^n$-a.e. $x \in \mathbb{R}^n$



By coarea formula (we will prove it next lecture) we have:

$$\int_{\mathbb{R}^n} |\nabla \tilde{v}_\varepsilon(x)|\, dx = \frac{1}{\varepsilon} \int_{0<d<\varepsilon} |\nabla d(x)|$$

$$= \frac{1}{\varepsilon} \int_0^\varepsilon \mathcal{H}^{n-1}(d^{-1}(t))\, dt$$

$$= \frac{1}{\varepsilon} \cdot \varepsilon \mathcal{H}^{n-1}(d^{-1}(t_\varepsilon)), \quad 0 < t_\varepsilon < \varepsilon$$

$$= \mathcal{H}^{n-1}(d^{-1}(t_\varepsilon)), \text{ some } 0 < t_\varepsilon < \varepsilon.$$

We have $d^{-1}(t_\varepsilon) \subset B_{R'-\varepsilon} \setminus \bar{B}_{R'-2\varepsilon}$ and:

$$\mathcal{H}^{n-1}(d^{-1}(t_\varepsilon)) \to \mathcal{H}^{n-1}(\partial B_{R'}) \text{ as } \varepsilon \to 0.$$

We can convolve $\tilde{v}_\varepsilon$ to produce a smooth function v_ε. Clearly, $0 \le v_\varepsilon \le 1$ and $v_\varepsilon \longrightarrow \chi_{B_{R'}}$ in $L^1(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} |\nabla v_\varepsilon| \to P(B_{R'})$ (A)

- Now, define
$$u_\varepsilon = \chi_E * \rho_\varepsilon$$

By Remark, Lecture 13 we have:

$\int_{\mathbb{R}^n} |\nabla u_\varepsilon| \to P(E)$; moreover, $u_\varepsilon \to \chi_E$ in $L^1(\mathbb{R}^n)$ (B)

We use (A) and (B) to compute, for every $T \in C_c^1(\mathbb{R}^n, \mathbb{R}^n)$, $|T| \le 1$:

$$\int_{E \cap B_{R'}} \operatorname{div} T = \int_{\mathbb{R}^n} \chi_{E \cap B_{R'}} \operatorname{div} T(x)\, dx$$

$$= \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} u_\varepsilon v_\varepsilon \operatorname{div} T(x)\, dx, \quad \text{Since } u_\varepsilon v_\varepsilon \to \chi_{E \cap B_{R'}} \text{ in } L^1(\mathbb{R}^n)$$

$$= -\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} T \cdot \nabla(u_\varepsilon v_\varepsilon)\, dx$$

$$\le \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} |\nabla(u_\varepsilon v_\varepsilon)|\, dx$$

$$\le \limsup_{\varepsilon \to 0} \int_{\mathbb{R}^n} |\nabla v_\varepsilon| + \int_{\mathbb{R}^n} v_\varepsilon |\nabla u_\varepsilon| \quad ; \text{ since } 0 \le u_\varepsilon \le 1, \ \nabla(u_\varepsilon v_\varepsilon) = u_\varepsilon \nabla v_\varepsilon + v_\varepsilon \nabla u_\varepsilon$$

$$= P(B_{R'}) + \limsup_{\varepsilon \to 0} \int_{\mathbb{R}^n} v_\varepsilon |\nabla u_\varepsilon| \quad ; \text{ by (A)}$$

$$= P(B_{R'}) + \limsup_{\varepsilon \to 0} \int_{B_{R'}} |\nabla u_\varepsilon| \quad ; \quad \operatorname{spt} v_\varepsilon \subset B_{R'}$$

$$\le P(B_{R'}) + P(E, \overline{B}_{R'}) \quad ; \quad \text{since } |\nabla u_\varepsilon| \mathcal{L}^n \overset{*}{\rightharpoonup} |\mu_E| \text{ and hence } \limsup_{\varepsilon \to 0} \int_{\overline{B}_{R'}} |\nabla u_\varepsilon| \le |\mu_E|(\overline{B}_{R'})$$

$$\le P(B_R) + P(E, B_R)$$

Taking sup over all T gives:

$$P(E\cap B_{R'}) \leq P(B_R) + P(E, B_R)$$

Since $E\cap B_{R'} \to E\cap B_R$ in $L^1_{loc}(\mathbb{R}^n)$ as $R'\to R$, by lower semicontinuity:

$$P(E, B_R) \leq \liminf_{R'\to R} P(E\cap B_{R'}) \leq P(B_R) + P(E, B_R)$$

We conclude step 1. ∎

Step two:

We apply the compactness theorem (theorem 2) proved in Lecture 13 to the sequence:

$$\{E_i \cap B_j\}_{i=1}^{\infty}, \quad \text{fixed } j$$

By step 1: $P(E_i\cap B_j) \leq P(B_j) + P(E_i, B_j) \leq M$, $M>0$, $\forall i$

$\Rightarrow \exists \{E_{i_\ell}\}_{\ell=1}^{\infty}$ and $F_j \subset B_j$ such that $E_{i_\ell} \to F_j$ in $L^1(\mathbb{R}^n)$. Apply the same argument to $\{E_{i_\ell}\}_{\ell=1}^{\infty}$ and B_{j+1} to obtain another subsequence $\{E_{i_{\ell_p}}\}_{p=1}^{\infty}$ and $F_{j+1} \subset B_{j+1}$ such that $E_{i_{\ell_p}} \to F_{j+1}$. Continue in this way and then take the diagonal of sets, say $\{E_{i_k}\}_{k=1}^{\infty}$. Then:

$F_j \subset F_{j+1}$ (up to null sets) and $E_{i_k} \to E := \bigcup_{j=1}^{\infty} F_j$ in $L^1_{loc}(\mathbb{R}^n)$, as $k\to\infty$.

By lower semicontinuity, E is of locally finite perimeter and $\mu_{E_{i_k}} \xrightarrow{*} \mu_E$. (see lower semicontinuity theorem in Lecture 12, Page 12.9) ∎.

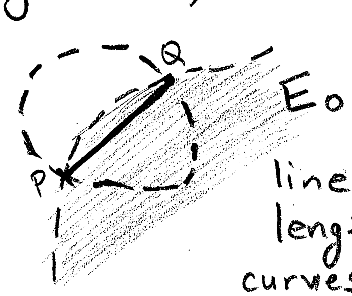
Existence of minimizers in geometric variational problems:

Problem 1: Let $A \subset \mathbb{R}^n$ open and $E_0 \subset \mathbb{R}^n$ be of finite perimeter. The Plateau-type problem in A with boundary data E_0 amounts to minimizing $P(E)$ among those sets of finite perimeter E that coincide with E_0 outside A. More precisely:

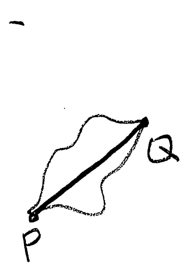
$$\gamma(A, E_0) = \inf\{P(E) : E \setminus A = E_0 \setminus A\}. \quad (*)$$

- Roughly speaking, $E \setminus A = E_0 \setminus A$ means we impose $E_0 \cap \partial A$ as a "boundary condition" for the admissible sets E in $(*)$.
- The set A may act as an obstacle.
- In general, we do not expect uniqueness.

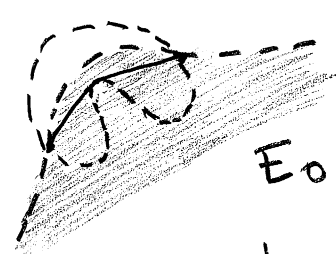
Ex 1:



line minimizes length among all curves with fixed P and Q



Ex 2:



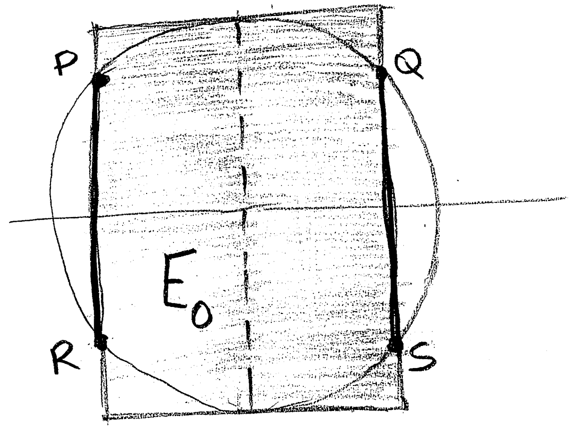
The curve that touches the obstacle minimizes length among all competitors.

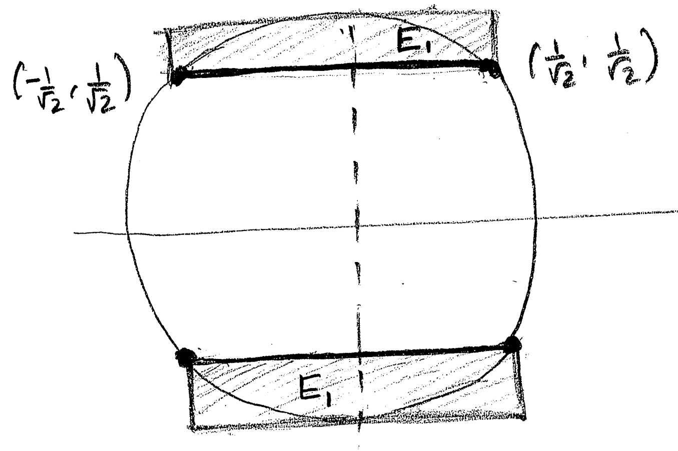
Ex: Non-uniqueness.

Let $E_0 = \{x \in \mathbb{R}^2 ;\ |x_2| < 1,\ |x_1| < \frac{1}{\sqrt{2}}\}$

$A = B(0,1)$

$E_1 = E_0 \cap \{x \in \mathbb{R}^2 : \frac{1}{\sqrt{2}} < x_2 < 1\}$





$P = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $R = (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$, $Q = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $S = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$

Both E_0 and E_1 (grey regions) are minimizers,

Proposition 1 (Existence of minimizers for the Plateau-type problem), Let $A \subset \mathbb{R}^n$ bounded open, $E_0 \subset \mathbb{R}^n$ set of finite perimeter in $\mathbb{R}^n$. Then, there exists a set of finite perimeter E such that:

$E \setminus A = E_0 \setminus A$ and $P(E) \leq P(F)$ $\forall F$, $F \setminus A = F \setminus E_0$

In particular E is a minimizer is the variational problem 1.

Proof of Proposition 1:

E_0 is itself admissible $\Rightarrow \gamma := \gamma(A, E_0) < \infty$

Let $\{E_i\}_{i=1}^{\infty}$ be a minimizing sequence; that is:

$$P(E_i) \to \gamma, \quad P(E_i) \le P(E_0), \quad E_i \setminus A = E_0 \setminus A.$$

Define:

$$M_i := E_i \Delta E_0 = (E_i \setminus E_0) \cup (E_0 \setminus E_i)$$

Applying Lemma 1, Lecture 13 ($P(E \cup F, A) + P(E \cap F, A) \le P(E,A) + P(F,A)$), we have:

$$\begin{aligned} P(E_i \Delta E_0) &= P(E_i \setminus E_0) + P(E_0 \setminus E_i) \\ &= P(E_i \cap E_0^c) + P(E_0 \cap E_i^c) \\ &\le P(E_i) + P(E_0^c) + P(E_0) + P(E_i^c); \quad \text{here } A = \mathbb{R}^n \\ &= P(E_i) + P(E_0) + P(E_0) + P(E_i); \\ &\le P(E_0) + P(E_0) + P(E_0) + P(E_0) \\ &= 4P(E_0) \end{aligned}$$

since E loc. fin. per. $\Rightarrow$ $\mathbb{R}^n \setminus E$ is loc. fin. per. and $P(E) = P(\mathbb{R}^n \setminus E)$ (see exercise 12.9 textbook)

Since $M_i \subset A$, bounded open set, we can apply the compactness theorem (Lecture 13, Page 13-6) to obtain M such that, up to extracting a subsequence, we have $M_i \to M$ in $L^1(\mathbb{R}^n)$.

Since

$$E_i = (E_0 \cup M_i) \setminus (E_0 \cap M_i)$$

and:

$$M_i \to M \quad \text{in } L^1(\mathbb{R}^n)$$

then:

$$E_i \to E \quad \text{in } L^1(\mathbb{R}^n), \quad E := (E_0 \cup M) \setminus (E_0 \cap M)$$

Note that:

$$E \setminus A = E_0 \setminus A$$

and by lower semicontinuity:

$$\gamma \le P(E) \le \liminf_{i \to \infty} P(E_i) = \gamma$$

$\therefore \ \gamma = P(E)$, E is a minimizer. ∎

Problem 2: Relative isoperimetric problems. Given an open set $A \subset \mathbb{R}^n$, the relative isoperimetric problem in A amounts to the volume-constrained minimization of the relative perimeter in A, namely:

$$\alpha(A, m) = \inf \{ P(E; A) : E \subset A, \ |E| = m \}$$

Note: The case $A = \mathbb{R}^n$ is the Euclidian isoperimetric problem, which can not be studied using the Direct-Method (compactness + lower semicontinuity).

Problem 2 is stricty related to the study of equilibrium shapes of a liquid confined in a given container (see Chapter 19 from textbook).

Proposition 2 : (Existence of relative isoperimetric sets). Let A open bounded set of finite perimeter and $m \in (0, |A|)$, then $\exists E \subset A$, set of finite perimeter such that:

$$P(E;A) = \alpha(A,m), \quad |E| = m$$

Proof: Let

$$E_t = A \cap \{x : x_1 < t\}, \; t \in \mathbb{R}. \Rightarrow \exists t \text{ s.t. }$$

$$|E_t| = m$$

$$\therefore \quad \alpha := \alpha(A, m) < \infty.$$

Let $\{E_i\}$ be a minimizing sequence; that is.

$$E_i \subset A, \quad |E_i| = m, \quad P(E_i; A) \to \alpha$$

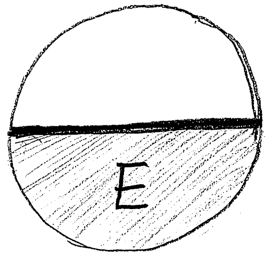
By Step one of Corollary 1 in this Lesson we have:

$$P(E_i) = P(E_i \cap A) \leq P(E_i; A) + P(A)$$

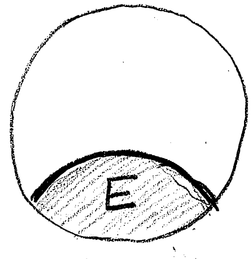
(Note: we proved this for $A = B_R$, the general case for A will be proved later; see Thm 16.3 in textbook).

Ex:

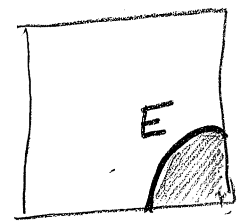
$A = B$, $m = \frac{|B|}{2}$



$A = B$, $m < \frac{|B|}{2}$



$A = Q$, m small



Hence:
$$\sup_i P(E_i) < \infty$$

A bounded, Compactness Thm from Lecture 13 $\Rightarrow$
$\exists \{E_{i_k}\}$, $\exists E \subset A$ s.t. $E_{i_k} \to E$ as $k \to \infty$. Also, $|E| = \lim_{k\to\infty} |E_{i_k}| = m$, so E is a competitor and:
$$\alpha \le P(E; A) \le \liminf_{k\to\infty} P(E_{i_k}; A) = \alpha$$

$\therefore$ $\alpha = P(E; A)$ and E is a minimizer.

Problem 3: Problems involving potential energies (prescribed mean-curvature problems) arise from the interaction between perimeter and potential energy terms. Given $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ Leb. meas., we define the potential energy of E associated with g as:

$$\mathcal{G}(E) = \int_E g(x)\,dx$$

$\Bigg\{$ Ex. g = gravitational potential, $-\nabla g$ is gravitational force field, orthogonal to level sets of g

Consider the variational problem:

$$\inf \{ P(E) + \mathcal{G}(E) : E \subset A \},$$

where A is open. See Chapter 19 for examples of physical interest. Also, we will see later that if $g \in C(A)$ and E is a minimizer, and $\partial E \cap A$ is C^2 then the mean curvature H_E of E is equal to $-g$ in A (see chapter 17).

For these problems, in order to apply the direct method, one needs to prove lower-semicontinuity results for g; for example:

<u>Proposition 3</u>: Let $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ Leb. meas. Assume $g^- \in L^1(F)$, $F \subset \mathbb{R}^n$ Leb. meas, and $E_i \to E$ in $L^1(\mathbb{R}^n)$. Then

$$\int_{E \cap F} g(x)\,dx \leq \liminf_{i \to \infty} \int_{E_i \cap F} g(x)\,dx .$$

<u>Proof</u>: Let $g = g^+ - g^-$

$$\int \lim_{i\to\infty} \chi_{E_i \cap F}\, g^-(x)dx = \int_{E \cap F} g^-(x)\,dx = \lim_{i \to \infty} \int_{E_i \cap F} g^-(x)\,dx \quad ; \quad g^- \in L^1(F),$$

using dominated convergence thm.

$$\int_{E \cap F} g^+(x)\,dx = \int \lim_{i \to \infty} \chi_{E_i \cap F}\, g^+(x)\,dx$$

$$\leq \liminf_{i \to \infty} \int_{E_i \cap F} g^+(x)\,dx \quad ; \quad \text{by Fatou's lemma}$$

$$\therefore \int_{E \cap F} g(x)dx = \int \lim_{i \to \infty} \chi_{E_i \cap F}(g^+ - g^-)dx = \int \liminf_{i\to\infty} \chi_{E_i \cap F}\, g^+ - \int \lim_{i \to \infty} \chi_{E_i \cap F}\, g^-$$

$$\leq \liminf_{i\to\infty} \int_{E_i \cap F} g^+ + \lim_{i\to\infty} \int \chi_{E_i \cap F}(-g^-) \leq \liminf_{i \to \infty} \int_{E_i \cap F} g^+ - g^-\,dx$$