Lecture 16

In Lecture 15, we proved a fundamental theorem to approximate sets of locally finite perimeter by smooth sets. Two corollaries of this theorem are:

<u>Corollary 1</u>: (Approximation by <u>bounded</u> sets):
If $|E|<\infty$ and $P(E)<\infty$ then $\exists \{E_k\}_{k=1}^{\infty}$, E_k <u>bounded</u> open set with smooth boundary, such that

$$E_k \to E \text{ in } L^1(\mathbb{R}^n), \quad P(E_k) \to P(E).$$

Indeed, it is sufficient to show the existence of $\{R_k\}$, $R_k \to \infty$ such that:

$$E \cap B_{R_k} \to E \quad , \quad P(E\cap B_{R_k}) \to P(E)$$

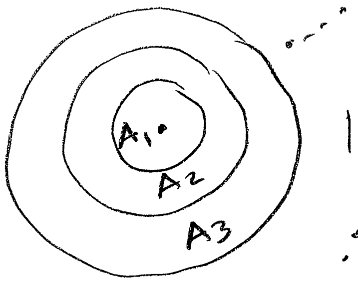
and then apply the theorem from Lecture 15 to approximate each $E \cap B_{R_k}$. We note:

$$|E|<\infty \implies |E \setminus B_R| \to 0 \text{ as } R\to\infty$$
$$\implies \boxed{E\cap B_R \to E \text{ as } R \to \infty} \quad (1)$$

Also

$$P(E)<\infty \implies \boxed{\lim_{R\to\infty} P(E; \mathbb{R}^n \setminus B_R) = 0} \quad (2)$$

Indeed, to see (1) note that:



$$|E| = \sum_{i=1}^{\infty} |E \cap A_i| < \infty$$

$$\therefore \sum_{i=j}^{\infty} |E \cap A_i| \to 0 \quad \text{as } j \to \infty$$

$$\therefore |E \cap (\mathbb{R}^n \setminus B_R)| \to 0 \quad \text{as } R \to \infty.$$

For (2) is the same;

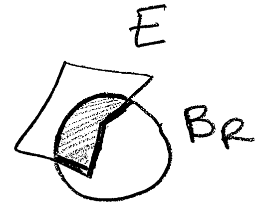
$$P(E) = \sum_{i=1}^{\infty} P(E; A_i) < \infty$$

$$\therefore \sum_{i=j}^{\infty} P(E; A_i) \to 0 \quad \text{as } j \to \infty$$

$$\therefore P(E; \mathbb{R}^n \setminus B_R) \to 0 \quad \text{as } R \to \infty.$$

Also, we have; for a.e. $R > 0$:

$$\boxed{P(E \cap B_R) = P(E; B_R) + \mathcal{H}^{n-1}(E \cap \partial B_R)} \quad (3)$$



We will prove (3) later but the idea is that for a.e. $R > 0$, E and B_R intersect transversally and hence $\mathcal{H}^{n-1}(\partial E \cap \partial B_R) = 0$. Recall that in a previous lecture, we proved the weaker version:

$$P(E \cap B_R) \leq P(E; B_R) + P(B_R).$$

Now, note that for every $\varepsilon > 0$ and every $M > 0$, $\exists R > 0$ with $R > M$ such that:

$$\mathcal{H}^{n-1}(E \cap \partial B_R) < \varepsilon$$

Indeed, if this is not true then $\exists \varepsilon_0$ and $\exists M_0 > 0$ such that:

$$\mathcal{H}^{n-1}(E \cap \partial B_r) \geq \varepsilon_0 \quad \forall r \geq M_0$$

But therefore, by the coarea formula, and choosing R such that $R > M_0$ and $R - M_0 > \frac{2|E|}{\varepsilon_0}$ we have:

$$|E| \geq |E \cap B_R| = \int_0^R \mathcal{H}^{n-1}(E \cap \partial B_r)\,dr$$

$$\geq \int_{M_0}^R \mathcal{H}^{n-1}(E \cap \partial B_r)\,dr$$

$$\geq \varepsilon_0 (R - M_0) > \frac{2|E|\varepsilon_0}{\varepsilon_0} = 2|E|,$$

and hence $|E| > 2|E|$; which is a contradiction.

From this discussion it is clear that there exists a sequence $\{R_k\}$ such that:

$$\boxed{\lim_{k \to \infty} \mathcal{H}^{n-1}(E \cap \partial B_{R_k}) = 0} \quad (4)$$

Now:

$$P(E \cap B_{R_k}) = P(E; B_{R_k}) + \mathcal{H}^{n-1}(E \cap \partial B_{R_k}) \text{ ; by (3)}$$

$$= P(E) - P(E; \mathbb{R}^n \setminus B_{R_k}) + \mathcal{H}^{n-1}(E \cap \partial B_{R_k})$$

$$\longrightarrow P(E) \text{ as } k \to \infty \text{; by (2) and (4).}$$

Thus we have:

$$E \cap B_{R_k} \longrightarrow E \quad \text{and} \quad P(E \cap B_{R_k}) \to P(E) \text{; as desired,}$$

and then by approximating each $E \cap B_{R_k}$ with a smooth open set (using convolution as in Lecture 15), we obtain a sequence of <u>bounded</u> open sets $\{E_k\}$, with smooth boundary, such that

$$E_k \longrightarrow E \text{ in } L^1(\mathbb{R}^n) \quad \text{and} \quad P(E_k) \to P(E). \quad \blacksquare$$

Corollary 2 (Approximation by polyhedra):
If $|E|<\infty$ and $P(E)<\infty$ then $\exists\{E_k\}_{k=1}^{\infty}$, a sequence of open bounded sets with polyhedral boundary such that:

$$E_k \to E \text{ in } L^1(\mathbb{R}^n),\quad P(E_k)\to P(E).$$

Indeed, we can assume that E is bounded (otherwise, we consider $\{E\cap B_{R_k}\}$ as in the previous Corollary 1 and apply corollary 2 to each $E\cap B_{R_k}$).

Thus, if E is bounded, we proceed as in the proof of the approximation theorem in Lecture 15, and consider

$$\{u_i\}_{i=1}^{\infty},\quad u_i = \chi_E * \rho_{\varepsilon_i},\quad \varepsilon_i\to 0.$$

$$\Rightarrow\quad u_i\in C_c^1(\mathbb{R}^n).$$

But every $v\in C_c^1(\mathbb{R}^n)$ can be approximated by a sequence $\{v_j\}_{j=1}^{\infty}$ of piecewise affine functions with compact support such that:

$$v_j\to v \text{ in } L^1(\mathbb{R}^n),\quad \int_{\mathbb{R}^n}|\nabla v_j|\to\int_{\mathbb{R}^n}|\nabla v|.$$

Therefore, we follow the proof in Lecture 15 with

$$\{v_i\}_{i=1}^{\infty} \text{ instead of } \{u_i\}_{i=1}^{\infty}$$

The sets $\{E_k\}$ will be selected among the sets $\{v_i>t\}$ which, for a.e. $t\in(0,1)$, are bounded open sets with polyhedral boundary. ∎

The <u>Euclidean isoperimetric problem</u> is:

$$\boxed{\inf\{P(E) : |E| = m\}, \quad m > 0 \quad (*)}$$

We Recall that:

$$\begin{aligned} \omega_n &= |B(0,1)| \\ &= \int_0^1 \mathcal{H}^{n-1}(\partial B(0,r))\, dr \\ &= \int_0^1 r^{n-1} \mathcal{H}^{n-1}(\partial B(0,1))\, dr \\ &= \mathcal{H}^{n-1}(\partial B(0,1)) \left[\frac{r^n}{n}\right]_0^1 = \frac{\mathcal{H}^{n-1}(\partial B(0,1))}{n} \end{aligned}$$

$$\therefore \quad P(B(0,1)) = n\,\omega_n \qquad (A)$$

<u>Remark 1</u>: Let $B(x,r)$ be a ball of measure m

That is:

$$m = |B(x,r)| = \omega_n r^n \implies r = \frac{m^{1/n}}{\omega_n^{1/n}}$$

Hence:

$$\begin{aligned} P(B(x,r)) &= r^{n-1} P(B(0,1)) = r^{n-1} n\,\omega_n \ ; \quad \text{by } (A) \\ &= \frac{m^{\frac{n-1}{n}}}{\omega_n^{\frac{n-1}{n}}}\, n\,\omega_n \\ &= m^{\frac{n-1}{n}}\, n\,\omega_n^{1/n} \\ &= n\,\omega_n^{1/n} |B(x,r)|^{\frac{n-1}{n}} \quad \blacksquare \end{aligned}$$

Hence, if we can prove that the infimum in (*) is attained at a ball, say $\tilde{B} = B(x,r)$, hence since $\tilde{B}$ is a minimizer:

$$P(\tilde{B}) \le P(E), \quad \forall E, \ |E| = m$$

$$\therefore \ n\omega_n^{1/n} |\tilde{B}|^{\frac{n-1}{n}} \le P(E) \ ; \ \text{by Remark 1}$$

$$\therefore \ n\omega_n^{1/n} |E|^{\frac{n-1}{n}} \le P(E) \ ; \quad \text{since } |E| = m = |\tilde{B}| .$$

Hence:

$$\boxed{n\omega_n^{1/n} |E|^{\frac{n-1}{n}} \le P(E), \ \forall E, \ |E| = m}$$

Therefore, solving (*) is equivalent to proving the following Euclidean isoperimetric inequality:

Theorem 1: If E is a Lebesgue measurable set in $\mathbb{R}^n$ with $|E| < \infty$ then

$$|E|^{\frac{n-1}{n}} n\,\omega_n^{1/n} \le P(E). \qquad (**)$$

Equality holds if and only if $|E \Delta B(x,r)| = 0$ for some $x \in \mathbb{R}^n$, $r > 0$.

In order to prove Theorem 1, we need to use the Steiner symmetrization, introduced in a previous lecture. Recall:

$$x = (z, x_n), \quad p: \mathbb{R}^n \to \mathbb{R}^{n-1}, \quad q: \mathbb{R}^n \to \mathbb{R}$$

$$x = (px, qx) \quad p(x) = z, \quad q(x) = x_n$$

Define:

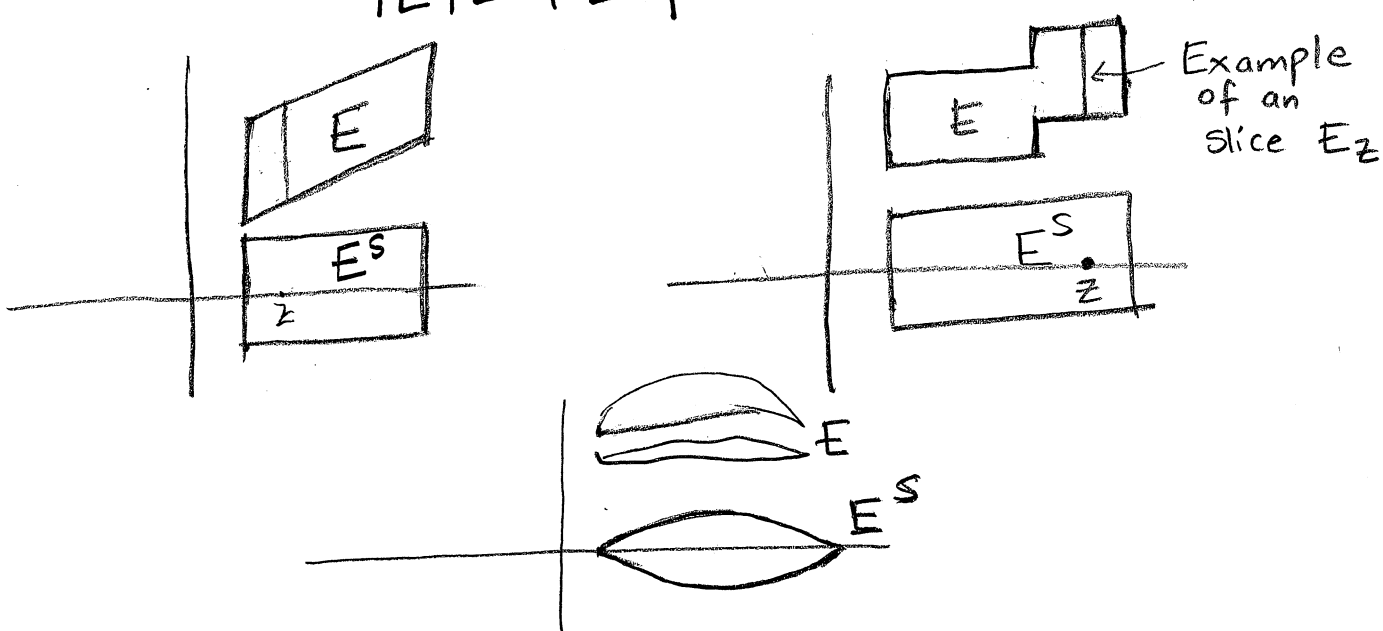
$$E_z = \{ t \in \mathbb{R} : (z, t) \in E \}$$

The Steiner symmetrization of E is:

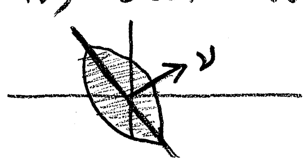
$$E^S = \left\{ x \in \mathbb{R}^n ;\ |qx| \leq \frac{\mathcal{L}^1(E_{px})}{2} \right\}$$

By Fubini's theorem:

$$|E| = |E^S|$$



These definitions and examples show symetrization with respect to e_n, but we can symmetrize with respect to any $\nu \in S^{n-1}$.



To prove Theorem 1, we need the following to results, which we will discuss next class:

<u>Theorem 2 (Steiner inequality)</u>: If E is a set of finite perimeter in $\mathbb{R}^n$, $|E|<\infty$, then E^s is a set of finite perimeter in $\mathbb{R}^n$, with

$$\boxed{P(E^s) \le P(E)}, \qquad (***)$$

and, in fact, whenever A is an open set in $\mathbb{R}^{n-1}$:

$$P(E^s; A\times\mathbb{R}) \le P(E; A\times\mathbb{R})$$

Moreover

(i) If <u>equality</u> holds in $(***)$, then:

<u>E_z is equivalent to an interval</u>, for a.e. $z\in\mathbb{R}^{n-1}$

(ii) If <u>E</u> is equivalent to a <u>convex set</u>, then equality holds in $(***)$ if and only if $\exists c\in\mathbb{R}$ such that:

$$\boxed{E \text{ is equivalent to } E^s + c e_n}$$

In order to fully exploit the necessary condition of equality (i) in the Steiner inequality, we will use the following:

<u>Lemma 1</u>: If $E\subset\mathbb{R}^n$ is of locally finite perimeter and:

E_z is equivalent to an interval for a.e. $z\in\mathbb{R}^{n-1}$

then the set of points of density one $E^{(1)}$ of E has the property that:

$(E^{(1)})_z$ is an interval, for every $z\in\mathbb{R}^{n-1}$.

Proof of the Euclidean isoperimetric inequality (Theorem 1):

Let E, $P(E)<\infty$, $|E|<\infty$

(a) Consider first the case E bounded, then $\exists \tilde{R}>0$ such that

$$E \subset B_{\tilde{R}} \quad ; \quad B_{\tilde{R}} = B(0,\tilde{R})$$

Let

$$\tilde{m} = |E| .$$

we will prove:

$$\boxed{\begin{array}{l}\text{Claim 1: Let } m, R>0 \text{ and consider the constrained isoperimetric problem} \\ \quad \inf \{P(E) : E \subset B_R,\ |E| = m\},\ m<\omega_n R^n \\ \text{then the inf is attained in a set } E, \text{ that is equivalent to a ball.}\end{array}}$$

Thus, by claim applied to $\tilde{R}$ and $\tilde{m}$, it follows that:

$$P(\tilde{B}) \leq P(E),$$

where $\tilde{B}$ is a ball minimizer given by claim 1, then by Remark 1 we have:

$$P(\tilde{B}) = n\omega^{1/n} |\tilde{B}|^{\frac{n-1}{n}} = n\omega^{1/n} \tilde{m}^{\frac{n-1}{n}} = n\omega^{1/n} |E|^{\frac{n-1}{n}},$$

and hence:

$$n\omega^{1/n} |E|^{\frac{n-1}{n}} \leq P(E),$$ which is $(**)$.

(b) If E is not bounded, then by Corollary 1, $\exists \{E_k\}$ bounded open sets with smooth boundary such that:

$$E_k \to E \text{ in } L^1(\mathbb{R}^n), \quad P(E_k) \to P(E)$$

Then, by step (a):

$$n\omega^{1/n} |E_k|^{\frac{n-1}{n}} \leq P(E_k)$$

Letting $k \to \infty$ yields

$$n\omega^{1/n} |E|^{\frac{n-1}{n}} \leq P(E)$$

(c) Suppose now that $P(E) = n\omega_n^{1/n} |E|^{\frac{n-1}{n}}$, $|E| < \infty$. Let $\tilde{m} = |E|$. If E is bounded then $\exists \tilde{R}$ s.t $E \subset B_{\tilde{R}}$. Hence, $\forall F$, $F \subset B_{\tilde{R}}$, $|F| = \tilde{m}$:

$$\begin{array}{c} |F|^{\frac{n-1}{n}} n\omega^{1/n} \leq P(F) \ ; \text{ by } (**) \\ \parallel \\ |E|^{\frac{n-1}{n}} n\omega^{1/n} \\ \parallel \\ P(E) \end{array}$$

$\therefore$ $P(E) \leq P(F)$ $\forall F \subset B_{\tilde{R}}$, $|F| = \tilde{m}$.

$\therefore$ By claim 1 $\Rightarrow$ E is equivalent to a ball.

In the case E is unbounded and $P(E) = n\omega_n^{1/n} |E|^{\frac{n-1}{n}}$. then we consider the symmetrizations E_ν^s for every $\nu \in S^{n-1}$. Since $P(E) = n\omega_n^{1/n} |E|^{\frac{n-1}{n}} = n\omega_n^{1/n} |E_\nu^s|^{\frac{n-1}{n}} \leq P(E_\nu^s)$, it follows that $P(E) \leq P(E_\nu^s)$. And, since $P(E_\nu^s) \leq P(E)$, by Theorem 2, it follows that:

$$P(E^s_\nu) = P(E).$$

Proceeding now as in the proof of claim 1, we obtain that E is equivalent to a ball. We have proved then that:

If Equality in (**) holds $\Rightarrow$ $|E \Delta B(x,r)| = 0$ some $x \in \mathbb{R}^n$, $r > 0$

Clearly, the reverse implication is also true. ∎

Proof of Claim 1:

- Since $m < \omega_n R^n$, clearly the competition class is non-empty and the existence of a minimizer E follows by the Direct method as in Lecture 14.

- We may directly assume $E = E^{(1)}$, since $\mathcal{L}^n(E \Delta E^{(1)}) = 0$. Let $\nu \in S^{n-1}$ and consider the symmetrization with respect to ν, E^s_ν. Now:

$$|E^s_\nu| = |E| = m \Rightarrow P(E) \leq P(E^s_\nu)$$

But, since by Theorem 2,

$$P(E^s_\nu) \leq P(E),$$

we obtain

$$P(E^s_\nu) = P(E)$$

• The minimizer E is convex.

Indeed, let $x, y \in E$. Define $\nu = \frac{x-y}{|x-y|}$.
Consider E^s_ν. We have seen, by previous step that

$$P(E) = P(E^s_\nu)$$

By Theorem 2 (i) $\Rightarrow$ almost every slice E_z is equivalent to an interval. Then, Lemma 1 implies that every slice $(E^1)_z$ is actually an interval. Thus, the segment with endpoints at x and y is contained in E.

• Since E is convex and $P(E) = P(E^s_\nu)$, $\forall \nu \in S^{n-1}$, then by Theorem (ii) $\exists\, c_\nu$ such that

$$E = c_\nu \nu + E^s_\nu .$$

• Define now:

$$F = -(c_{e_1} e_1 + \dots + c_{e_n} e_n) + E$$

Clearly, F is also a minimizer, $|F| = |E|$, $P(F) = P(E)$. Hence; as done above for E,

$$F = d_\nu \nu + F^s_\nu$$

But, by construction, $d_{e_K} = 0$, $K = 1, \dots, n$, 16.13

that is, F is invariant by reflection with respect to the hyperplanes:

$$\{x_k = 0\}, \quad k = 1, \dots, n$$

Therefore, F is invariant under the map

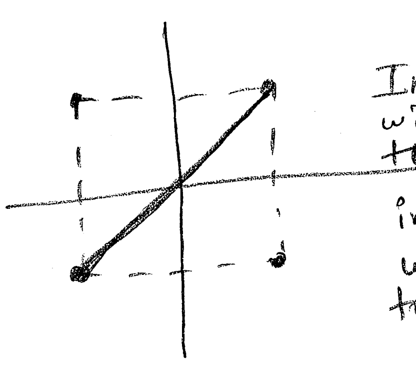
$$x \mapsto -x$$

But we have:

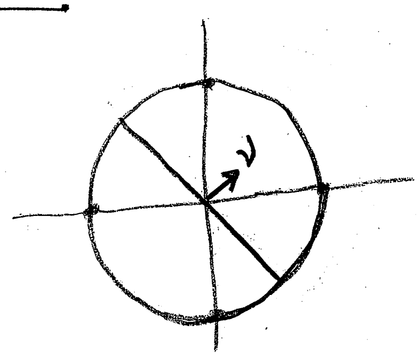
$$F = d_\nu \nu + F_\nu^s, \qquad \forall \nu \in S^{n-1}$$

and therefore, since F is invariant under $x \mapsto -x$ it follows that d_ν must be zero, for any ν. Therefore, F is invariant by reflection with respect to any hyperplane $\nu^\perp$, $\nu \in S^{n-1}$, through the origin. (Indeed, $F = F_\nu^s$ and F_ν^s is invariant by reflection with respect to $\nu^\perp$). Therefore, we conclude that:

<u>F is a ball</u>



Invariant with respect to every axis $\Rightarrow$ invariant with respect to origin



F is actually a ball because $F = F_\nu^s$, $\forall \nu$ and F_ν^s is invariant under reflection with respect to any $\nu^\perp$.