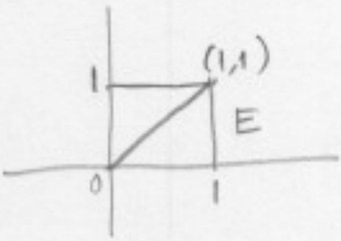
Definition: A sequence $\{p_n\}$ in a metric space X is said to be a Cauchy sequence if:

$\forall \varepsilon > 0, \exists N$ such that $d(p_n, p_m) < \varepsilon, \forall n, m \geq N$.

Def: Let $E \subset X$. The diameter of E is:

$$\text{diam } E = \sup\{ d(p,q) : p \in E, q \in E \}$$

Ex: $X = \mathbb{R}^2 \qquad E = [0,1] \times [0,1]$



$\text{diam } E = \sqrt{2}$

Remark: Let $\{p_n\}$ be a Cauchy sequence in X, and let:

$$E_N = \{p_N, p_{N+1}, \dots\},$$

then:

$$\{p_n\} \text{ is Cauchy} \iff \lim_{N \to \infty} \text{diam } E_N = 0$$

Indeed:

$\{p_n\}$ Cauchy $\iff d(p_n, p_m) < \varepsilon \;\; \forall p_n, p_m \in E_N$

$\iff \varepsilon$ is an upper bound of $\{d(p,q) : p, q \in E_N\}$

$\iff \text{diam } E_N \leq \varepsilon$; since $\text{diam } E_N$ is the smallest upper bound.

$\iff \text{diam } E_M \leq \varepsilon, \; \forall M \geq N$, since $E_M \subset E_N$

Theorem 3.10:

(a) Let $E \subset X$. Then $\operatorname{diam} \bar{E} = \operatorname{diam} E$

(b) If $K_n \supset K_{n+1}$, $n = 1, 2, \dots$ is a sequence of compact sets and if:

$$\lim_{n \to \infty} \operatorname{diam} K_n = 0$$

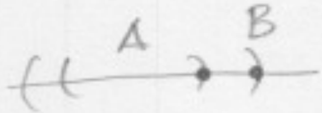
then $\bigcap_{n=1}^{\infty} K_n$ consists of exactly one point.

Proof:

(a) Since $E \subset \bar{E}$, clearly $\operatorname{diam} E \le \operatorname{diam} \bar{E}$

Indeed, note that:
If $A \subset B$, sets in $\mathbb{R}$ $\Rightarrow \sup A \le \sup B$
because:

$$b \le \sup B \quad \forall b \in B$$
$$\Rightarrow a \le \sup B \quad \forall a \in A, \text{ since } a \in B$$
$$\Rightarrow \sup B \text{ is an upper bound of } A$$
$$\Rightarrow \sup A \le \sup B; \text{ since } \sup A \text{ is the least upper bound of } A.$$



With $A = \{ d(p,q) : p, q \in E)$
$B = \{ d(p,q) : p, q \in \bar{E} \}$
$A \subset B$ since $E \subset \bar{E}$
$\Rightarrow \sup A \le \sup B$
$\Rightarrow \operatorname{diam} E \le \operatorname{diam} \bar{E}$

We now proceed to show that:

$$\operatorname{diam} \bar{E} \le \operatorname{diam} E.$$

Fix $\varepsilon > 0$. Choose $p, q \in \bar{E}$. Then $\exists p', q' \in E$ such that:

$$d(p, p') < \varepsilon, \quad d(q, q') < \varepsilon.$$

$$\Rightarrow d(p, q) \le d(p, p') + d(p', q') + d(q', q)$$
$$< 2\varepsilon + d(p', q') \le 2\varepsilon + \operatorname{diam} E$$

$$\Rightarrow d(p, q) \le 2\varepsilon + \operatorname{diam} E, \quad \forall p, q \in \bar{E}$$

$$\Rightarrow \operatorname{diam} \bar{E} \le 2\varepsilon + \operatorname{diam} E \quad (*)$$

We now take limits in both sides of (*). Let $\{\varepsilon_k\}$ a sequence such that $\varepsilon_k \to 0$.

Since (*) is true for any ε, we have:

$$\operatorname{diam} \bar{E} \le 2\varepsilon_k + \operatorname{diam} E, \quad k = 1, 2, \cdots$$

$$\lim_{k \to \infty} \operatorname{diam} \bar{E} \le \lim_{k \to \infty} (2\varepsilon_k + \operatorname{diam} E)$$

$\Rightarrow \operatorname{diam} \bar{E} \le \operatorname{diam} E$; since $\varepsilon_k \to 0$ implies that $2\varepsilon_k + \operatorname{diam} E \to \operatorname{diam} E$

Remark: Here we have used:
If $\{t_n\}$, $\{s_n\}$ are convergent sequences in $\mathbb{R}$ and $t_n \le s_n$, $n = 1, 2, 3, \ldots$ then $\lim_{n \to \infty} t_n \le \lim_{n \to \infty} s_n$.

We conclude that:

$$\operatorname{diam} \bar{E} = \operatorname{diam} E. \quad \blacksquare$$

(b) Let:

$$K = \bigcap_{n=1}^{\infty} K_n$$

We already know from the Corollary of Theorem 2.36 that:

$$\bigcap_{n=1}^{\infty} K_n \neq \emptyset$$

If K contains more than one point, then $\operatorname{diam} K > 0$. Since:

$$K \subset K_n \quad \forall n$$

$$\Rightarrow \operatorname{diam} K \leq \operatorname{diam} K_n \quad \forall n \qquad (1)$$

From (1), $\operatorname{diam} K_n$ can not converge to 0, which is a contradiction. ∎

Remarks on sequences

Ex: Suppose $\{a_n\}, \{b_n\}, \{c_n\}$ are sequences in $\mathbb{R}$ such that $\limsup_{n\to\infty} b_n = 0$, $\lim_{n\to\infty} c_n \to 0$.

Assume that:

$$|a_n| \le b_n + c_n, \quad n = 1, 2, 3, \dots \quad (1)$$

Show that $\lim_{n\to\infty} a_n = 0$.

Solution: Since we do not know at this point that $\{|a_n|\}$ is convergent, we can not take limits in both sides of (1), but we can take lim sup in both sides of (1):

$$\begin{aligned}
\limsup_{n\to\infty} |a_n| &\le \limsup_{n\to\infty} (b_n + c_n) \\
&\le \limsup_{n\to\infty} b_n + \limsup_{n\to\infty} c_n ; \quad \text{by Problem 5, chapter 3} \\
&= 0 + \limsup_{n\to\infty} c_n \\
&= \lim_{n\to\infty} c_n, \quad \text{Recall: } t_n \to t \iff \liminf_{n\to\infty} t_n = t \text{ and } \limsup_{n\to\infty} t_n = t \\
&= 0
\end{aligned}$$

$$\Rightarrow \limsup_{n\to\infty} |a_n| = 0$$

$$\Rightarrow \liminf_{n\to\infty} |a_n| \le \limsup_{n\to\infty} |a_n| = 0$$

Hence:

$$\liminf_{n\to\infty} |a_n| = \limsup_{n\to\infty} |a_n| = 0$$

Hene $\{|a_n|\}$ is convergent and:

$$\lim_{n\to\infty} |a_n| = 0$$

Therefore:

$$\lim_{n\to\infty} a_n = 0. \quad \blacksquare$$

Ex: Suppose;

$$0 \le x_n \le s_n, \quad n = 1, 2, \ldots$$

If $s_n \to 0$, show that $x_n \to 0$

Solution: $\limsup 0 \le \limsup_{n\to\infty} x_n \le \limsup_{n\to 0} s_n$

$$0 \le \limsup_{n\to\infty} x_n \le 0 \quad ; \text{ since } s_n \to 0$$

$$\Rightarrow \limsup_{n\to\infty} x_n = 0$$

But:

$$0 \le \liminf_{n\to\infty} x_n \le \limsup_{n\to\infty} x_n = 0$$

$\Rightarrow \liminf_{n\to\infty} x_n = 0$. Hence $\lim_{n\to\infty} x_n = 0$.

Theorem 3.11

(a) Every convergence sequence in a metric space X is a Cauchy sequence.

(b) Let X be a compact metric space and $\{p_n\}$ a Cauchy sequence in X. Then

$$p_n \to p, \text{ for some } p \in X.$$

(c) In $\mathbb{R}^k$, every Cauchy sequence converges.

Proof:

(a) Let $p_n \to p$, and let $\varepsilon > 0$. Then $\exists N$ such that:

$$d(p_n, p) < \frac{\varepsilon}{2}, \quad \forall n \geq N.$$

For any $n, m \geq N$:

$$d(p_n, p_m) \leq d(p_n, p) + d(p, p_m)$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$$\Rightarrow d(p_n, p_m) < \varepsilon \quad \forall n, m \geq N$$

$\Rightarrow \{p_n\}$ is Cauchy.

(b) We have $\{p_n\}$ a Cauchy sequence.

Let: $E_N = \{p_N, p_{N+1}, \dots\}$

$$\Rightarrow \lim_{N \to \infty} \operatorname{diam} \overline{E_N} = 0 \quad (*)$$

Since $\bar{E}_N$ is closed,

$\bar{E}_N \subset X$, X compact

$\Rightarrow$ $\bar{E}_N$ is compact

We have:

$$\bar{E}_N \supset \bar{E}_{N+1} \supset \ldots$$

$\Rightarrow$ $\bigcap_{i=N}^{\infty} \bar{E}_N$ has exactly one point, say $p \in X$.

Let $\varepsilon > 0$, by $(*)$, $\exists N_0$ s.t:

$$\operatorname{diam} \bar{E}_N < \varepsilon \quad \forall N \geq N_0 \qquad (2)$$

Since $p \in \bar{E}_{N_0}$ then (2) implies:

$$d(p, q) < \varepsilon, \quad \forall q \in E_{N_0} = \{p_{N_0}, p_{N_0+1}, \ldots\}$$

$$\Rightarrow d(p, p_n) < \varepsilon, \quad \forall n \geq N_0$$

We have shown that $\forall \varepsilon > 0$, $\exists N_0$ such that:

$$d(p, p_n) < \varepsilon, \quad \forall n \geq N_0.$$

We conclude $p_n \to p$. ∎

(c) Let $\{\vec{x}_n\}$ be a Cauchy sequence in $\mathbb{R}^k$.

As before, we have:

$$E_N := \{\vec{x}_N, \vec{x}_{N+1}, \cdots\}$$

and $\lim\limits_{N\to\infty} \operatorname{diam} E_N = 0$

Let $\varepsilon = 1$, then $\exists N_0$ s.t:

$$\operatorname{diam} E_N < 1, \quad \forall N \geq N_0.$$

We have $E_{N_0} = \{\vec{x}_{N_0}, \vec{x}_{N_0+1}, \vec{x}_{N_0+2}, \cdots\}$ and $\operatorname{diam} E_{N_0} < 1$. We will show that $\{\vec{x}_n\}$ is bounded. For every $n \geq N_0$:

$$\begin{aligned} |\vec{x}_n| &= |\vec{x}_n - \vec{x}_{N_0} + \vec{x}_{N_0}| \\ &\leq |\vec{x}_n - \vec{x}_{N_0}| + |\vec{x}_{N_0}| \\ &< 1 + |\vec{x}_{N_0}| \text{ ; since } |\vec{x}_n - \vec{x}_{N_0}| \leq \operatorname{diam} E_{N_0} < 1 \end{aligned}$$

Let $M = \max\{|\vec{x}_1|, |\vec{x}_2|, \dots, |\vec{x}_{N_0-1}|, 1 + |\vec{x}|_{N_0}\}$

Hence:

$$|\vec{x}_n| < M, \quad n = 1, 2, 3, \dots$$

$$\Rightarrow \{\vec{x}_n\} \text{ is bounded}$$

Hence, the range of $\{\vec{x}_n\}$ is contained in a k-cell I which is compact. From (b), $\{\vec{x}_n\}$ converges to some $\vec{x} \in I$

A Cauchy sequence $\{\vec{x}_n\}$ in $\mathbb{R}^k$ is always convergent.

<u>Def</u>: A metric space in which every Cauchy sequence converges is said to be <u>complete</u>.

<u>Ex</u>: From previous theorem, all compact metric spaces are complete.

All Euclidean spaces $\mathbb{R}^k$ are complete.

Every closed subset E of a complete metric space X is complete (since every Cauchy sequence in E is also a Cauchy sequence in X, and hence it converges to some $p \in X$, but $p \in E$ since E is closed).

<u>Ex</u>: $X = \mathbb{Q}$ metric space with distance $d(x,y) = |x-y|$.

X is not complete.