Chapter 4
Continuity.

Def: Let X and Y be metric spaces.

Suppose we have:

$$f: E \subset X \to Y$$

and $p \in E'$. We say that:

$$\lim_{x \to p} f(x) = q, \quad q \in Y$$

if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that:

$$\boxed{\text{If } x \in E \text{ and } d_X(x,p) < \delta \text{ then } d_Y(f(x), q) < \varepsilon}$$

Ex: If $X = \mathbb{R}^k$, $Y = \mathbb{R}$

$f: E \subset \mathbb{R}^k \to \mathbb{R}$

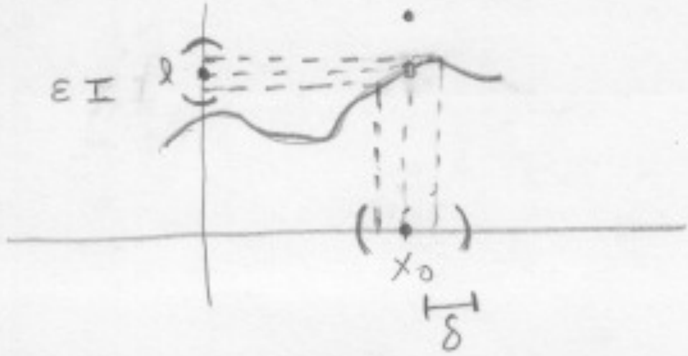
$p \in E'$, $\lim_{\vec{x} \to \vec{p}} f(\vec{x}) = l$ means:

$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. if $\vec{x} \in E$ and $|\vec{x} - \vec{p}| < \delta \Rightarrow |f(\vec{x}) - l| < \varepsilon$

Ex: $X = \mathbb{R}$, $Y = \mathbb{R}$.

$\lim_{x \to x_0} f(x) = l$ if $\forall \varepsilon > 0$, $\exists \delta$ s.t:

$$|x - x_0| < \delta \Rightarrow |f(x) - l| < \varepsilon$$



• Theorem 4.2: Let $f: E \subset X \to Y$, $p \in E'$. Then:

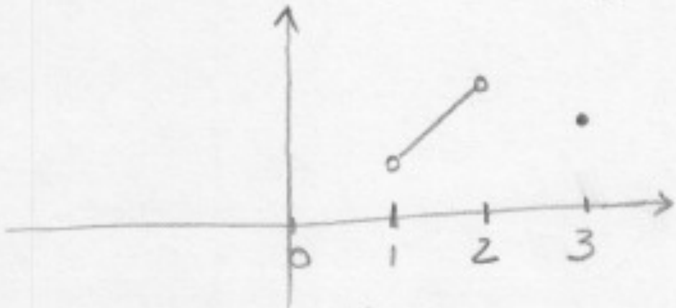
$\lim_{x \to p} f(x) = q$ if and only if $\lim_{n \to \infty} f(p_n) = q$ for every $\{p_n\}$ in E such that $p_n \to p$, $p_n \neq p$.

Corollary: If f has a limit at p, this limit is unique.

Def: The function $f: E \subset X \to Y$ is said to be continuous at $p \in E$ if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that:

$$\boxed{\text{If } x \in E \text{ and } d_X(x,p) < \delta, \text{ then } d_Y(f(x), f(p)) < \varepsilon}$$

Ex: Let $f: E \subset \mathbb{R} \to \mathbb{R}$, $E = (1,2) \cup \{3\}$.



The previous definition is true at $p = 3$, because we can choose δ so that $(-3-\delta, 3+\delta) \cap E = \{3\}$, and clearly $|f(x) - f(p)| = |f(3) - f(3)| = 0 < \varepsilon$.

f is also continuous at any $p \in (1, 2)$.

- From the definitions of limit and continuity, if $p \in E$ is a limit point then we have:

$$\boxed{f \text{ is continuous at } p \iff \lim_{x \to p} f(x) = f(p)}$$

Definition: $f: E \subset X \to Y$ is said to be continuous on E if f is continuous at every point $p \in E$.

Ex: $X = \mathbb{R}$, $Y = \mathbb{R}$, $f(x) = \sin \frac{1}{x}$.

$\lim_{x \to 0} \sin \frac{1}{x}$ does not exist.

Indeed, let $x_n = \frac{1}{n\pi}$, $n = 1, 2, \dots$. Then $x_n \to 0$ but $f(x_n) = \sin(n\pi) = 0$, so $f(x_n) \to 0$

On the other hand, let $y_n = \frac{1}{\frac{\pi}{2} + 2\pi n}$, $n = 1, 2, \dots$.

We have $y_n \to 0$ but $f(y_n) = \sin\left(\frac{\pi}{2} + 2\pi n\right) = 1$, So $f(y_n) \to 1$.

Since $1 \neq 0$, from theorem 4.2, the limit does not exist.

- Ex : Squeeze theorem: Let $A \subset \mathbb{R}$, $f, g, h : A \to \mathbb{R}$, and let $c \in A'$. If

$$f(x) \le g(x) \le h(x) \qquad \forall x \in A,\ x \ne c,$$

and if $\lim_{x \to c} f(x) = L = \lim_{x \to c} h(x)$, then:

$$\lim_{x \to c} g(x) = L$$

Proof: Let $\{x_n\}$, $x_n \to c$ a sequence in A

$$f(x_n) \le g(x_n) \le h(x_n)$$

$$\limsup_{n \to \infty} f(x_n) \le \limsup_{n \to \infty} g(x_n) \le \limsup_{n \to \infty} h(x_n)$$

$$\Rightarrow \qquad L \le \limsup_{n \to \infty} g(x_n) \le L$$

$$\Rightarrow \quad \limsup_{n \to \infty} g(x_n) = L.$$

Similarly, taking lim inf in the inequality, we get $\liminf_{n \to \infty} g(x_n) = L$.

Since $\liminf_{n \to \infty} g(x_n) = \limsup_{n \to \infty} g(x_n) = L$, it follows that $\lim_{n \to \infty} g(x_n) = L$. Now, from Theorem 4.2 we conclude that $\lim_{x \to c} g(x) = L$.

Theorem 4.4, Let $f: E \subset X \to \mathbb{R}$, $g: E \subset X \to \mathbb{R}$, $p \in E'$. If $\lim_{x \to p} f(x) = A$ and $\lim_{x \to p} g(x) = B$ Then:

(a) $\lim_{x \to p} (f+g)(x) = A+B$

(b) $\lim_{x \to p} (fg)(x) = AB$

(c) $\lim_{x \to p} \left(\frac{f}{g}\right)(x) = \frac{A}{B}$, $B \neq 0$.

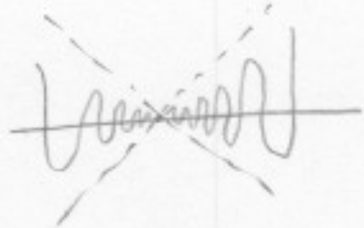
Theorem 4.9: Let f and g be continuous on a metric space X. Then $f+g$, fg, and $\frac{f}{g}$ are continuous on X

Ex: $\Phi_i: \mathbb{R}^k \to \mathbb{R}$
$\Phi_i(\vec{x}) = x_i$, $\vec{x} = (x_1, \dots, x_k)$ is continuous, since
$|\Phi_i(\vec{x}) - \Phi_i(\vec{y})| = |x_i - y_i| \leq |x - y|$ shows that with $\delta = \varepsilon$, the definition of continuity is true for Φ_i.

Ex: From Theorem 4.9 and previous example, polynomials in $\mathbb{R}^k$ are continuous, quotients of polynomials are continuous too. For example, the functio $f(x,y): \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$, $f(x,y) = \frac{2x^2 y}{x^2+y^2}$ is continuous at any point of $\mathbb{R}^2 \setminus \{(0,0)\}$.

Ex: From $||\vec{x}| - |\vec{y}|| \leq |\vec{x} - \vec{y}|$, $\forall \vec{x}, \vec{y} \in \mathbb{R}^k$, it follows that $f: \mathbb{R}^k \to \mathbb{R}$, $f(\vec{x}) = |\vec{x}|$ is continuous on $\mathbb{R}^k$.

Ex: Show that $\lim_{x \to 0} x \sin \frac{1}{x} = 0$.



We have:

$$0 \le \left| x \sin \frac{1}{x} \right| \le |x|$$

Since $\lim_{x \to 0} |x| = 0$, by the squeeze lemma we conclude that $\lim_{x \to 0} x \sin \frac{1}{x} = 0$.

Ex: Let $X = \mathbb{R}^2$, $Y = \mathbb{R}$. Let $E = \mathbb{R}^2 \setminus \{(0,0)\}$

Consider $f : E \subset \mathbb{R}^2 \to \mathbb{R}$,

$$f(x,y) = \frac{2x^2 y}{x^2 + y^2}$$

Show that $\lim_{(x,y) \to (0,0)} f(x,y) = 0$.

Solution: Note that:

$$0 \le \left| \frac{2x^2 y}{x^2 + y^2} \right| \le \left| \frac{2\cancel{x^2} y}{\cancel{x^2}} \right| = |2y|$$

Thus, given $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{2}$, then:

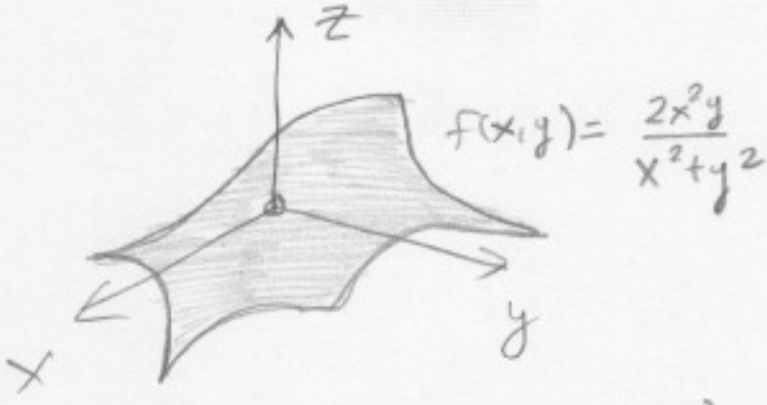
if $\|(x,y) - (0,0)\| = \sqrt{x^2 + y^2} < \delta$ then $|y| < \delta$

and

$$\left| f(x,y) - 0 \right| = \left| \frac{2x^2 y}{x^2 + y^2} \right| \le 2|y| < 2\delta = 2\frac{\varepsilon}{2} = \varepsilon$$

We have proved:

$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\underbrace{\|(x,y) - (0,0)\|}_{d((x,y),(0,0))} < \delta \Rightarrow \underbrace{|f(x,y) - 0|}_{d(f(x,y),0)} < \varepsilon$, which yields that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$. We can also use the squeeze theorem. Note that $(0,0)$ is not in E, but it is in E'.



We can define $f(x,y)$ at $(0,0)$ to make it continuous at $(0,0)$. Let:

$$f(x,y): \mathbb{R}^2 \to \mathbb{R}$$
$$f(x,y) = \begin{cases} \frac{2x^2y}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

Since
$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0,$$ then f is continuous at $(0,0)$.

Ex: Let $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ be defined as:

$$f(x,y) = \frac{x^2}{x^2+y^2}$$

Show that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Solution: By Theorem 4.2, it is enough to find two sequences converging to $(0,0)$, say $(s_n,t_n) \to (0,0)$, $(p_n,q_n) \to 0$, but $\{f(s_n,t_n)\}$ and $\{f(p_n,q_n)\}$ have different limits.

Let $(s_n,t_n) = (\frac{1}{n}, 0)$, $n = 1, \ldots$

$$\Rightarrow f(\tfrac{1}{n}, 0) = \frac{1/n^2}{1/n^2 + 0} = 1 \to 1 \text{ as } n \to \infty.$$

Let $(p_n,q_n) = (0, \frac{1}{n})$, $n = 1, 2, \ldots$

$$f(0, \tfrac{1}{n}) = 0 \longrightarrow 0 \text{ as } n \to \infty.$$

Since $0 \neq 1$, $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist. Clearly, the function can not be defined at $(0,0)$ in order to make it continuous at $(0,0)$, since there is a jump at $(0,0)$.

The composition of continuous functions is continuous:

<u>Theorem 4.7</u>; Let X, Y, Z metric spaces. Let $f: E \subset X \to Y$, $g: f(E) \to Z$. Define:

$$h(x) = g(f(x)), \quad x \in E$$

If f is continuous at $p \in E$, and g is continuous at $f(p)$, then h is continuous at p.

<u>Proof</u>: Let $\varepsilon > 0$. Since g is continuous at $f(p)$, $\exists \eta > 0$ s.t:

(1) If $y \in f(E)$ and $d_Y(y, f(p)) < \eta \Rightarrow d_Z(g(y), g(f(p))) < \varepsilon$

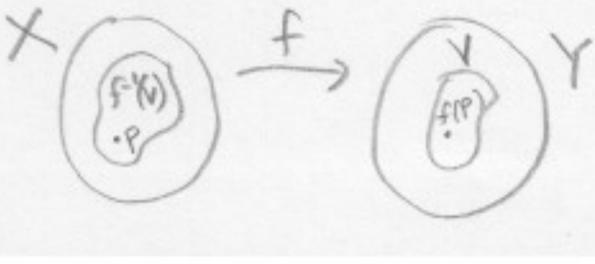
Since f is continuous at p, $\exists \delta > 0$ s.t:

(2) $x \in E$, $d_X(x, p) < \delta \Rightarrow d_Y(f(x), f(p)) < \eta$.

From (1) and (2):

If $x \in E$, $d_X(x,p) < \delta \Rightarrow d_Z(h(x), h(p)) = d_Z(g(f(x)), g(f(p))) <$

<u>Theorem 4.8</u>. X, Y metric spaces. A function $f: X \to Y$ is continuous on X if and only if $f^{-1}(V)$ is open in X for every open set V in Y.



$f^{-1}(V) = \{x \in V : f(x) \in V\}$.

Proof :

$\Longrightarrow$: Suppose $f: X \to Y$ is continuous.
Let V be an open set in Y. We need to show that $f^{-1}(V)$ is open in X. Let $p \in f^{-1}(V)$. Then $f(p) \in V$ and, since V is open, $\exists \varepsilon > 0$ such that:

$$N_\varepsilon(f(p)) = \{y \in Y : d_Y(y, f(p)) < \varepsilon\} \subset V \quad (1)$$

Since f is continuous at p, $\exists \delta > 0$ such that:

$$d(x, p) < \delta \Rightarrow d_Y(f(x), f(p)) < \varepsilon \quad (2)$$

From (1) and (2) it follows that $N_\delta(p) \subset f^{-1}(V)$.
Indeed, if $x \in N_\delta(p) \Rightarrow d(x, p) < \delta \Rightarrow$
$d_Y(f(x), f(p)) < \varepsilon \Rightarrow f(x) \in N_\varepsilon(f(p)) \subset V \Rightarrow f(x) \in V$
$\Rightarrow x \in f^{-1}(V)$. ∎

$\Longleftarrow$: Suppose now that $f^{-1}(V)$ is open in X for every open set V in Y. Fix $p \in X$, and let $\varepsilon > 0$.
Define:

$$V := N_\varepsilon(f(p)) = \{y \in Y : d_Y(y, f(p)) < \varepsilon\} \quad (1)$$

Since $f^{-1}(V)$ is open, $f^{-1}(V)$ is open. Hence, $\exists \delta$ s.t:

$$N_\delta(p) \subset f^{-1}(V) \quad (2)$$

From (1) and (2):

$$d_X(x, p) < \delta \Rightarrow x \in N_\delta(p) \Rightarrow x \in f^{-1}(V) \Rightarrow f(x) \in V$$
$$\Rightarrow d_Y(f(x), f(p)) < \varepsilon$$

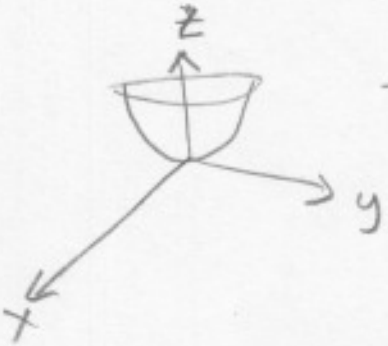
which means that f is continuous at p.

<u>Corollary</u>: $f: X \to Y$ is continuous if and only if $f^{-1}(C)$ is closed in X for every closed set C in Y.

Indeed, E closed $\Leftrightarrow$ E^c open and $f^{-1}(E^c) = (f^{-1}(E))^c$. $\square$

<u>Ex</u>: In applications, very often we have functions $f: X \to Y$, with $X = \mathbb{R}^k$, $Y = \mathbb{R}$, or $X = \mathbb{R}$, $Y = \mathbb{R}^k$.

$f(x,y) = x^2 + y^2$
$f: \mathbb{R}^2 \to \mathbb{R}$



The graph is a surface in $\mathbb{R}^3$.

$f(t) = (\sin t, \cos t, t)$
$f: \mathbb{R} \to \mathbb{R}^3$

Parametrization of a curve in $\mathbb{R}^3$

<u>Theorem 4.10</u>: Let $\vec{f}: X \to \mathbb{R}^k$, $\vec{f}(\vec{x}) = (f_1(x), \dots, f_k(x))$
$f_i: X \to \mathbb{R}$.

(a) $\vec{f}$ is continuous $\Leftrightarrow$ f_i is continuous, $i = 1, 2, \dots k$.

(b) $\vec{f}, \vec{g}$ continuous $\Rightarrow$ $\vec{f} + \vec{g}$, $\vec{f} \cdot \vec{g}$ are continuous.

Ex: Let
$\vec{f}: X \to \mathbb{R}^k$ be continuous

Then; $\phi: X \to \mathbb{R}$ given by
$\Phi(x) = |\vec{f}(x)|$
is a continuous real function on X because it is the composition of continuous functions.

$$\vec{x} \longrightarrow \vec{f}(x) \longrightarrow |\vec{f}(x)|$$

f is continuos. $\quad |\cdot| : \mathbb{R}^k \to \mathbb{R}$ is continuou

Ex: In applications, a very important case is $X = \mathbb{R}^k$, $Y = \mathbb{R}^k$. In this case, $f: \mathbb{R}^k \to \mathbb{R}^k$ is called a vector field.

$$\vec{V}(x,y,z) = (v_1(x,y,z), v_2(x,y,z), v_3(x,y,z))$$

can represent the velocity field of a fluid.