Definition 4.18: Let $f: X \to Y$. We say that f is uniformly continuous on X if:
$\forall \varepsilon > 0$, $\exists \delta > 0$ such that:
If $p, q \in X$, $d_X(p, q) < \delta \implies d_Y(f(p), f(q)) < \varepsilon$

Note: Clearly, if f is uniformly continuous then f is continuous.

Theorem 4.19: Let $f: X \to Y$ continuous, X is a compact metric space. Then f is uniformly continuous on X.

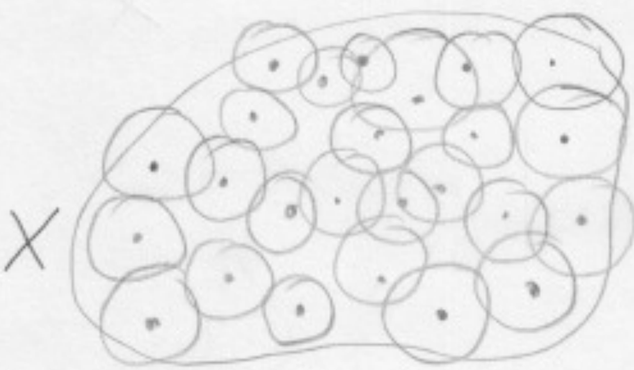
Proof: Let $\varepsilon > 0$. Since f is continuous at each $p \in X$ then $\exists\, \eta(p)$ s.t:

$$q \in X,\ d_X(p, q) < \eta \implies d_Y(f(p), f(q)) < \frac{\varepsilon}{2} \quad (*)$$

Define:

$$J(p) := \{q \in X : d_X(p, q) < \tfrac{\eta}{2}\} = N_{\frac{\eta}{2}}(p)$$

$J(p)$ is open $\{J(p)\}$ is an open cover of X



An open cover of X formed with the union of all the neighborhoods $N_{\frac{\eta}{2}}(p)$

Since X is compact, there exists a finite subcover, say $J(p_1) \cup \dots \cup J(p_n)$.

We define:

$$\delta := \frac{1}{2} \min \left[\eta(p_1), \dots, \eta(p_n) \right]$$

We will show that:

$$d_X(p,q) < \delta \Rightarrow d_Y(f(p), f(q)) < \varepsilon \quad (**).$$

Let p, q such that $d_X(p,q) < \delta$. Since:

$$p \in X \subset \bigcup_{i=1}^{n} J(p_i),$$

$\exists m$ such that $p \in J(p_m)$. Hence:

$$d_X(p, p_m) < \frac{1}{2} \eta(p_m),$$

and:

$$\begin{aligned} d_X(q, p_m) &\le d_X(q,p) + d_X(p, p_m) \\ &< \delta + \frac{1}{2}\eta(p_m) \le \frac{1}{2}\eta(p_m) + \frac{1}{2}\eta(p_m) = \eta(p_m) \end{aligned}$$

From $(*)$ we obtain:

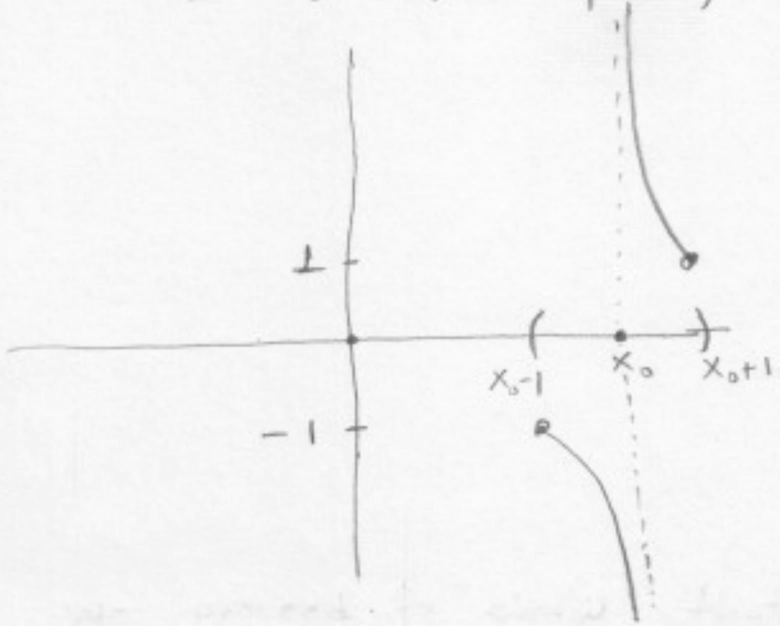
$$\begin{aligned} d_Y(f(p), f(q)) &\le d_Y(f(p), f(p_m)) + d_Y(f(p_m), f(q)) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

We have proven $(*)$. ∎

Ex: let $f : E \subset \mathbb{R} \to \mathbb{R}$,

$$f(x) = \frac{1}{x - x_0}, \quad E = (x_0 - 1, x_0 + 1) \setminus \{x_0\}$$

E is not compact, f is continuous on E



Let $\varepsilon > 0$. Suppose that there exists $\delta > 0$ such that:

$$|t - x| < \delta, \ t, x \in E \implies |f(t) - f(x)| < \varepsilon. \quad (1)$$

Let $N_1, N_2 > 0$ such that $\frac{1}{N_2} - \frac{1}{N_1} < \delta$, $N_2 - N_1 > \varepsilon$.

With $t := x_0 + \frac{1}{N_1}$ and $x = x_0 + \frac{1}{N_2}$ we have:

$$|x - t| = \left|\frac{1}{N_2} - \frac{1}{N_1}\right| < \delta$$

(We can choose such N_1, N_2 since $\frac{1}{n} \to 0$ and hence $\{\frac{1}{n}\}$ is a Cauchy sequence). But:

$$|f(x) - f(t)| = \left| \frac{1}{x_0 + \frac{1}{N_2} - x_0} - \frac{1}{x_0 + \frac{1}{N_1} - x_0} \right| = |N_2 - N_1| > \varepsilon$$

Hence:

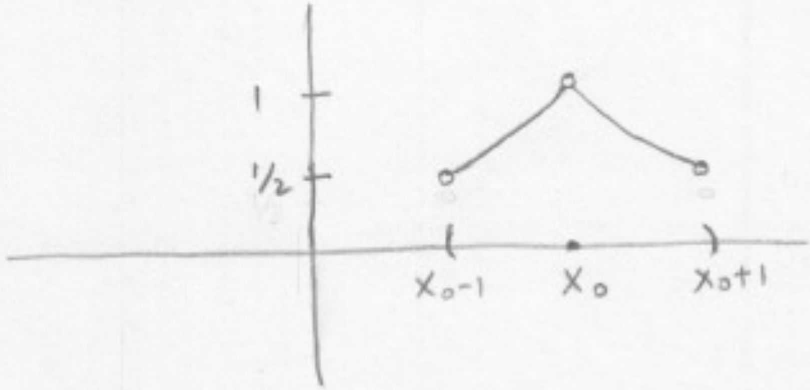
$|x - t| < \delta$ but $|f(x) - f(t)| > \varepsilon$, which contradicts (1). We conclude that f is not uniformly continuous.

Ex: Let

$g : E \subset \mathbb{R} \to \mathbb{R}, \quad E = (x_0 - 1, x_0 + 1) \setminus \{x_0\}$

$g(x) = \dfrac{1}{1 + (x - x_0)^2}$, g continuous on E.

E is not compact



g is bounded, since $\frac{1}{2} < g(x) < 1$, $x \in E$

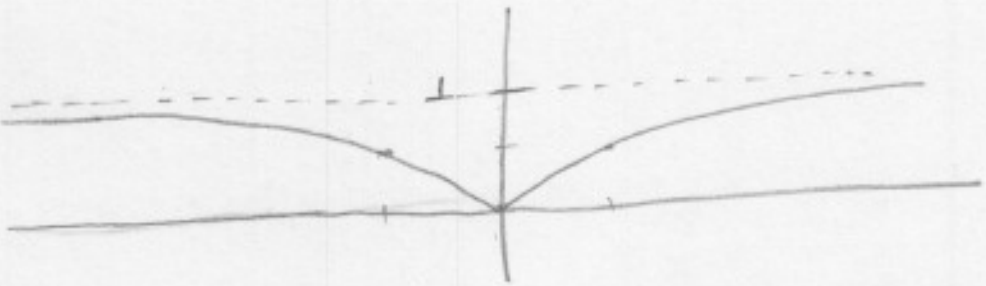
$\sup_{x \in E} g(x) = 1$

g is continuous and bounded on E, but it has no maximum on E.

Ex: Let $h : \mathbb{R} \to \mathbb{R}$.

$h(x) = \dfrac{x^2}{1 + x^2}, \quad E = \mathbb{R}$

h is continuous on $\mathbb{R}$



$h(x) = \dfrac{1}{\frac{1}{x^2} + 1}$

$\frac{1}{x^2} \to 0$ as $x \to \infty$

$\Rightarrow$ $h(x) \to 1$ as $x \to \infty$

h is bounded since $0 \le h(x) < 1$, $x \in \mathbb{R}$

E is not compact, $\sup_{x \in E} h(x) = 1$.

h is continuous and bounded on $E = \mathbb{R}$, but it has no maximum on E.

- <u>Ex</u> Let $E = \{\pm 1, \pm 2, \pm 3, \dots\}$
  E is unbounded.
  Any function $f: E \to \mathbb{R}$ is uniformly continuous on E.
  Indeed, given $\varepsilon > 0$, choosing $\delta = \frac{1}{2}$, we have that:

  $$|x-y| < \delta,\ x, y \in E \Rightarrow |f(x) - f(y) = 0 < \varepsilon.$$

  (since $x = y$ is the only case to consider).

<u>Def</u>: Let $f: E \subset \mathbb{R} \to \mathbb{R}$. If $x \in E$ and f is discontinuous at x, we say that "<u>f has a discontinuity at x</u>".

<u>Def 4.25</u>: Let f be defined on (a, b). Let $a \leq x < b$. We write:

$$f(x+) = q$$

if $f(t_n) \to q$ as $n \to \infty$ for all sequences $\{t_n\}$ in (x, b) such that $t_n \to x$

Let $a < x \leq b$. We write:

$$f(x-) = q$$

if $f(t_n) \to q$ for all sequences $\{t_n\}$ in (a, x) such that $t_n \to x$.

Clearly:

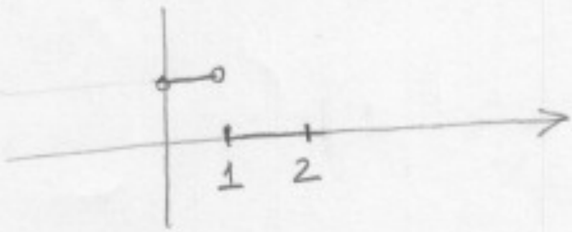
$$\lim_{t \to x} f(t) = \alpha \iff f(x+) = f(x-) = \alpha$$

<u>Definition 4.26</u>. Let $f:(a,b)\to\mathbb{R}$.
If $f(x+)$ and $f(x-)$ exists but f is discontinuous at x then we say that f has <u>a discontinuity of the first Kind</u> or a <u>simple discontinuity at x</u>.

Otherwise the discontinuity is said to be of the second Kind.

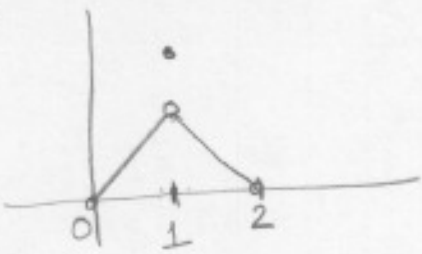
<u>Note</u>: From the previous definition we see that a simple discontinuity at x means:
(a) $f(x+)\neq f(x-)$, or (b) $f(x+)=f(x-)\neq f(x)$

<u>Ex</u>: $f(x)=\begin{cases}1, & 0<x<1\\ 0, & 1\leq x<2\end{cases}$



f has a simple discontinuity at $x=1$.

<u>Ex</u>:



f has a simple discontinuity at $x=1$

<u>Ex</u>: $f(x)=\begin{cases} 1, & x \text{ rational} \\ 0, & x \text{ irrational} \end{cases}$

f has a discontinuity of the second kind at every point x because $f(x+)$ and $f(x-)$ don't exist, since any interval $(x-\delta, x+\delta)$ always contains a rational number, and also an irrational number.

<u>Ex</u>: $f(x)=\begin{cases} x, & \text{if } x \text{ is rational} \\ 0, & x \text{ irrational.} \end{cases}$

If $x=0$, then $\lim_{x\to 0} f(x) = 0 = f(0)$. Clearly, given $\varepsilon>0$, with $\delta=\varepsilon$ we have:

$$|y-0|<\delta \Rightarrow |f(y)-f(0)| = |0-0| < \varepsilon, \text{ if } y \text{ irrational}$$

$$\Rightarrow |f(y)-f(0)| = |y| < \varepsilon, \text{ if } y \text{ is rational}$$

Hence, f is continuous at $x=0$.

If $x\neq 0$, $f(x+)$ and $f(x-)$ do not exist, because any interval $(x-\delta, x+\delta)$ contains a rational number, and also an irrational number, and hence we can choose a sequence of irrational numbers $\{t_n\}$, $t_n\to x$, or a sequence of rational numbers $\{s_n\}$, $s_n\to x$, such that $\{f(t_n)\}$ or $\{f(s_n)\}$ do not converge to any proposed limit l.

Then f has a discontinuity of the second kind at any $x\neq 0$.

<u>Ex</u>: $f(x) = \begin{cases} \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$

We have shown earlier that $\lim_{x\to 0} f(x)$ does not exist. f has a discontinuity of the second kind at $x=0$ because $f(0+)$, $f(0-)$ do not exist.

f is not continuous at 0, but f is continuous at any point $x \neq 0$, since it is the composition of two continuous functions: $x \mapsto \sin x$, and $x \to \frac{1}{x}$.