Theorem 6.17 : Assume that the monotonically increasing function α satisfies the additional property that $\alpha' \in \mathcal{R}$ on $[a,b]$. Let f be a bounded real function on $[a,b]$. Then:

$f \in \mathcal{R}(\alpha)$ if and only if $f\alpha' \in \mathcal{R}$. In this case:

$$\int_a^b f\,d\alpha = \int_a^b f(x)\alpha'(x)\,dx$$

Proof : Let $\varepsilon > 0$.

Since $\alpha' \in \mathcal{R}$, the criterion of integrability implies:

$$\boxed{\exists P \text{ s.t. } U(P,\alpha') - L(P,\alpha') < \varepsilon} \quad (*)$$

Note: $\alpha(x) = x$ in $\int_a^b \alpha'(x)\,dx$

Since $\alpha'(x)$ exists for every x, then α is continuous on $[a,b]$. We can apply the mean value theorem to get $t_i \in [x_{i-1}, x_i]$ such that:

$$\boxed{\alpha(x_i) - \alpha(x_{i-1}) = \alpha'(t_i)\Delta x_i \quad (1)}$$

If $s_i \in [x_{i-1}, x_i]$, Theorem 6.7 (b) and $(*)$ yield:

$$\boxed{\sum_{i=1}^{n} |\alpha'(s_i) - \alpha'(t_i)|\,\Delta x_i < \varepsilon \quad (2)}$$

Since:

$$\sum_{i=1}^{n} f(s_i)\Delta\alpha_i = \sum_{i=1}^{n} f(s_i)\alpha'(t_i)\Delta x_i$$

we obtain:

$$\left| \sum_{i=1}^{n} f(s_i)\Delta\alpha_i - \sum_{i=1}^{n} f(s_i)\alpha'(s_i)\Delta x_i \right|$$

$$= \left| \sum_{i=1}^{n} f(s_i)\, \alpha'(t_i)\Delta x_i - \sum_{i=1}^{n} f(s_i)\alpha'(s_i)\Delta x_i \right|, \text{ by (}$$

$$= \left| \sum_{i=1}^{n} f(s_i)\left(\alpha'(t_i) - \alpha'(s_i)\right)\Delta x_i \right|$$

$$\leq \sum_{i=1}^{n} |f(s_i)|\ |\alpha'(t_i) - \alpha'(s_i)|\,\Delta x_i$$

$$\leq M\varepsilon, \text{ by (2) and } |f(x)| \leq M,\ x \in [a,b].$$

We have shown:

$$\boxed{\left| \sum_{i=1}^{n} f(s_i)\Delta\alpha_i - \sum_{i=1}^{n} f(s_i)\alpha'(s_i)\Delta x_i \right| \leq M\varepsilon \quad (3)}$$

From (3):

$\sum_{i=1}^{n} f(s_i)\Delta\alpha_i \leq U(P, f\alpha') + M\varepsilon$, and this is true for every $s_i \in [x_{i-1}, x_i]$, $i = 1, 2, \dots, n$. Therefore:

$\sum_{i=1}^{n} M_i \Delta\alpha_i \leq U(P, f\alpha') + M\varepsilon$; that is:

$$\boxed{U(P, f, \alpha) \leq U(P, f\alpha') + M\varepsilon}$$

From (3) we also have:

$$\sum_{i=1}^{n} f(s_i)\alpha'(s_i)\Delta x_i \le M\varepsilon + U(P,f,\alpha);$$ which

is true for every $s_i \in [x_{i-1}, x_i]$. Hence:

$$\boxed{U(P, f\alpha') \le U(P,f,\alpha) + M\varepsilon}$$

We have shown that:

$$-M\varepsilon \le U(P,f,\alpha) - U(P,f\alpha') \le M\varepsilon$$

$$\Rightarrow \quad \boxed{|U(P,f,\alpha) - U(P,f\alpha')| \le M\varepsilon} \quad (4)$$

Since (*) remains true for any refinement of P, then (4) remains true for any refinement of P. Therefore:

$$\left| \overline{\int_a^b} f(x)\,d\alpha - \overline{\int_a^b} f(x)\alpha'(x)\,dx \right| \le M\varepsilon.$$

Proceeding in a similar way from (3) we can prove:

$$\left| \underline{\int_a^b} f(x)\,d\alpha - \underline{\int_a^b} f(x)\alpha'(x)\,dx \right| \le M\varepsilon.$$

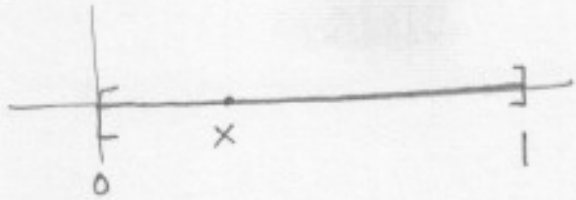
Since ε is arbitrary, we conclude that:

$$\overline{\int_a^b} f(x)\,d\alpha = \overline{\int_a^b} f(x)\alpha'(x)\,dx, \quad \underline{\int_a^b} f(x)\,d\alpha = \underline{\int_a^b} f(x)\alpha'(x)\,dx.$$

Clearly, it follows that:

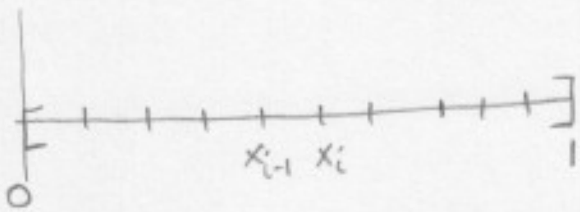
$$f \in \mathcal{R}(\alpha) \iff f\alpha' \in \mathcal{R}. \quad \blacksquare$$

Ex: Consider a straight wire of unit length:



Suppose that $\rho(x)$, $0 \leq x \leq 1$, gives the density of the wire at x. Assume that $\rho(x)$ is continuous. Compute the mass of the wire.

Solution: Consider a partition P of the wire:



$$P = \{x_0, x_1, \ldots, x_n\}.$$

Let α_i the mass of the piece $[x_{i-1}, x_i]$. Then:

$$\alpha_i \approx \underset{gr/cm \cdot cm}{M_i \Delta x_i}, \qquad M_i = \{\sup \rho(x) : x_{i-1} \leq x \leq x_i\}$$

Let $\alpha(x)$ be the mass of piece $[0, x]$, $0 \leq x \leq 1$.

$$\alpha(1) = \sum_{i=1}^{n} \alpha_i \cong \sum_{i=1}^{n} M_i \Delta x_i \Rightarrow \alpha(1) \cong U(P, \rho)$$

Since $\rho(x)$ is continuous, then $\rho \in \mathcal{R}$. Hence:

$$\alpha(1) = \inf\{U(P,\rho) : P \text{ is any partition}\}$$
$$= \sup\{L(P,\rho) : P \text{ is any partition}\} = \int_a^b \rho(x)\,dx.$$

We have:

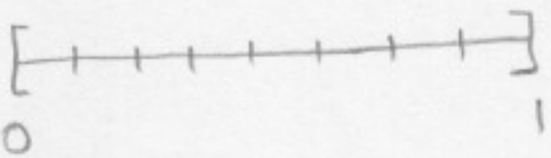
$$\alpha(x) = \int_0^x \rho(t)\, dt, \qquad \alpha(x) = \text{mass of piece } [0,x].$$

We will prove in Theorem 6.20 that, since $\rho(x)$ is continuous, then $\alpha(x)$ is differentiable, and:

$$\alpha'(x) = \rho(x).$$

Def: The moment of inertia is a quantity expressing a body's tendency to resist angular acceleration. It is the sum of the products of the mass of each particle in the body with the square of its distance from the axis of rotation.

Ex: Find a formula to compute the moment of inertia of the wire.



Let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[0,1]$

Let I_i be the moment of inertia of the piece $[x_{i-1}, x_i]$. Then:

$$I_i \cong \tilde{M}_i \alpha_i \qquad \begin{aligned} \alpha_i &= \text{mass of } [x_{i-1}, x_i] \\ &= \alpha(x_i) - \alpha(x_{i-1}) = \Delta\alpha_i \end{aligned}$$

$$\tilde{M}_i = \sup\{x^2 : x_{i-1} \le x \le x_i\}$$

Let I be the total inertia. Then:

$$I = \sum_{i=1}^{n} I_i$$
$$\approx \sum_{i=1}^{n} \tilde{M}_i \, \Delta\alpha_i$$

$$\Rightarrow \quad I \cong U(P, x^2, \alpha)$$

Also : $\quad I \cong L(P, x^2, \alpha)$.

Since $x \mapsto x^2$ is a continuous continuous function, and $x \mapsto \alpha(x)$ is increasing we have that:

$$x^2 \in \mathcal{R}(\alpha).$$

Hence:

$$I = \sup \{ L(P, x^2, \alpha) : P \text{ is any partition} \}$$
$$= \inf \{ U(P, x^2, \alpha) : P \text{ is any partition} \}$$
$$= \int_0^1 x^2 \, d\alpha$$

As an application of Theorem 6.17 we see that, since $\alpha'(x) = \rho(x) \quad 0 \le x \le 1$:

$$\int_0^1 x^2 \, d\alpha = \int_0^1 x^2 \rho(x) \, dx .$$

We conclude:

$$I = \int_0^1 x^2 \rho(x) \, dx . \quad \blacksquare$$

Integration and Differentitiation.

<u>Theorem 6.20</u> : Let $f \in \mathcal{R}$ on $[a,b]$. Define:

$$F(x) = \int_a^x f(t)\, dt, \quad a \le x \le b.$$

Then F is continuous on $[a,b]$. Moreover, if f is continuous at $x_0 \in [a,b]$, then F is differentiable at x_0, and:

$$F'(x_0) = f(x_0)$$

<u>Proof</u> : Since $f \in \mathcal{R}$, f is bounded; i.e. $\exists M$ s.t:

$$|f(t)| \le M, \quad a \le t \le b.$$

If $a \le x \le y \le b$, then:

$$\begin{aligned} |F(y) - F(x)| &= \left| \int_a^y f(t)\, dt - \int_a^x f(t)\, dt \right| \\ &= \left| \int_x^y f(t)\, dt \right| \text{; by Theorem 6.12 (c)} \\ &\le \int_x^y |f(t)|\, dt \\ &\le M\,|y-x| \quad \text{by Theorem 6.12 (d)} \end{aligned}$$

Let $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{M}$.

If $|x-y| < \delta \Rightarrow |F(y) - F(x)| \le M|y-x| < M \cdot \frac{\varepsilon}{M} = \varepsilon$

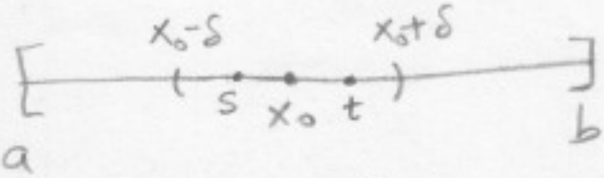
$\Rightarrow$ F is uniformly continuous.

Suppose now that f is continuous at x_0. Let $\varepsilon > 0$. Then, $\exists\, \delta > 0$ s.t.

$$|t - x_0| < \delta, \quad a \le t \le b \Rightarrow |f(t) - f(x_0)| < \varepsilon \qquad (*)$$

Let s, t such that:

$$x_0 - \delta < s \le x_0 \le t < x_0 + \delta \quad \text{and} \quad a \le s < t \le b.$$



Let $\phi(t) = \dfrac{F(t) - F(x_0)}{t - x_0}.$

We will show that $\lim\limits_{t \to x_0} \phi(t) = f(x_0)$.

$$\left| \frac{F(t) - F(x_0)}{t - x_0} - f(x_0) \right| = |\phi(t) - f(x_0)|$$

$$= \left| \frac{1}{t - x_0} \int_{x_0}^{t} f(u)\, du - f(x_0) \right|$$

$$= \left| \frac{1}{t - x_0} \int_{x_0}^{t} f(u)\, du - \frac{1}{t - x_0} \int_{x_0}^{t} f(x_0)\, du \right|$$

$$= \left| \frac{1}{t - x_0} \int_{x_0}^{t} (f(u) - f(x_0))\, du \right|$$

$$\le \frac{1}{t - x_0} \int_{x_0}^{t} |f(u) - f(x_0)|\, du$$

$$< \frac{1}{t - x_0} \cdot \int_{x_0}^{t} \varepsilon\, du = \frac{\varepsilon}{t - x_0} (t - x_0) = \varepsilon.$$

We have shown that

$$\boxed{\lim_{\substack{t \to x_0 \\ t > x_0}} \phi(t) = \phi(x_0+) = f(x_0).} \quad (1)$$

$$\left| \frac{F(s) - F(x_0)}{s - x_0} - f(x_0) \right| = \left| \phi(s) - f(x_0) \right|$$

$$= \left| \frac{1}{s-x_0} \int_a^s f(u)\,du - \frac{1}{s-x_0} \int_a^{x_0} f(u)\,du - f(x_0) \right|$$

$$= \left| -\frac{1}{s-x_0} \int_s^{x_0} f(u)\,du - f(x_0) \right| ; \quad \text{since: } \int_a^{x_0} f(u)\,du = \int_a^s f(u)\,du + \int_s^{x_0} f(u)\,du$$

$$= \left| \frac{1}{x_0 - s} \int_s^{x_0} (f(u) - f(x_0))\,du \right|$$

$$\leq \frac{1}{x_0 - s} \int_s^{x_0} |f(u) - f(x_0)|\,du < \frac{1}{x_0 - s} \int_s^{x_0} \varepsilon\,du$$

$$= \frac{1}{\cancel{x_0 - s}} \cdot \varepsilon\,(\cancel{x_0 - s}) = \varepsilon$$

We have shown that:

$$\boxed{\lim_{\substack{s \to x_0 \\ s < x_0}} \phi(s) = \phi(x_0-) = f(x_0)} \quad (2)$$

From (1) and (2), $\lim_{t \to x_0} \phi(t) = f(x_0)$; i.e, $F'(x_0) = f(x_0)$ ∎

Theorem 6.21. The fundamental theorem of calculus.

Let $f \in \mathcal{R}$ on $[a,b]$. If there is a differentiable function F on $[a,b]$ such that $F' = f$, then:

$$\int_a^b f(x)\,dx = \int_a^b F'(x)\,dx = F(b) - F(a).$$

Proof: Let $\varepsilon > 0$.

Since $f \in \mathcal{R} \Rightarrow \exists P = \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$ s.t.:

$$U(P,f) - L(P,f) < \varepsilon.$$

The mean value theorem $\Longrightarrow \exists t_i \in [x_{i-1}, x_i]$ s.t.:

$$F(x_i) - F(x_{i-1}) = F'(t_i)(x_i - x_{i-1}) = f(t_i)\Delta x_i.$$

$$\Rightarrow \underbrace{\sum_{i=1}^n F(x_i) - F(x_{i-1})}_{= F(b) - F(a)} = \sum_{i=1}^n f(t_i)(x_i - x_{i-1})$$

From Theorem 6.7 (c) we get:

$$\left| \sum_{i=1}^n f(t_i)\Delta x_i - \int_a^b f(x)\,dx \right| < \varepsilon,$$

which gives:

$$\left| F(b) - F(a) - \int_a^b f(x)\,dx \right| < \varepsilon.$$

Since ε is arbitrary, we conclude:

$$\int_a^b f(x)\,dx = F(b) - F(a). \quad \blacksquare$$

Theorem 6.22. Integration by parts.

Let F, G be differentiable on $[a,b]$.

Suppose that $F' = f$, $G' = g$, $f, g \in \mathcal{R}$ on $[a,b]$

Then:

$$\int_a^b F(x)\,g(x)\,dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)\,G(x)\,dx.$$

Proof: Let $H(x) = F(x)G(x)$.

$$H'(x) = FG' + GF' = Fg + Gf.$$

The fundamental theorem of calculus gives:

$$\int_a^b H'(x)\,dx = H(b) - H(a)$$

$$\Rightarrow \int_a^b (Fg + Gf)\,dx = H(b) - H(a)$$

$$\Rightarrow \int_a^b F(x)g(x)\,dx = F(b)G(b) - F(a)G(a) - \int_a^b G(x)f(x)\,dx. \quad \blacksquare$$

Ex:

$$\int_0^1 x\,e^x = [xe^x]_0^1 - \int_0^1 e^x\,dx = e - e + 1 = 1$$

$F(x) = x \quad g(x) = e^x$

$f(x) = 1 \quad G(x) = e^x$

Theorem 6.15. $a < s < b$, f bounded on $[a,b]$,

f continuous at s, and:

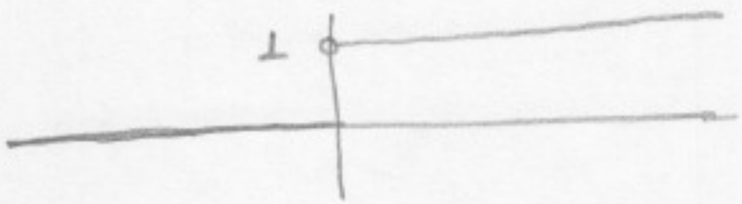
$$\alpha(x) = I(x-s),$$

then:

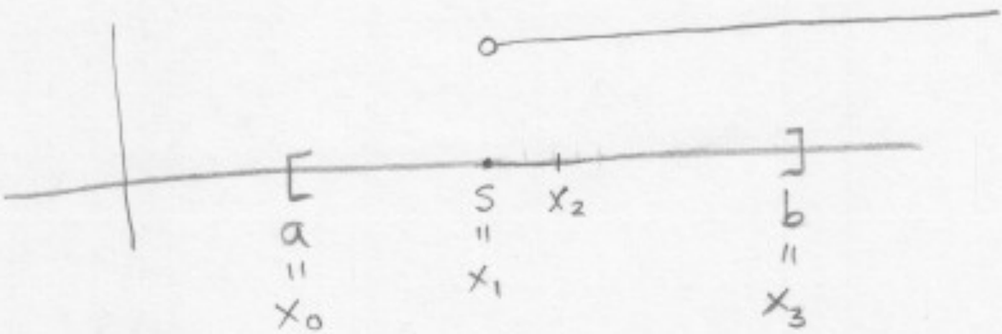
$$\int_a^b f \, d\alpha = f(s)$$

Remark: The unit step function is defined by

$$I(x) = \begin{cases} 0, & x \le 0 \\ 1, & x > 0 \end{cases}$$



Proof:



$$P = \{x_0, x_1, x_2, x_3\}$$

$$U(P, f, \alpha) = M_1(0-0) + M_2(1-0) = M_2$$

$$L(P, f, \alpha) = m_1(0-0) + m_2(1-0) = m_2$$

$$U(P, f, \alpha) - L(P, f, \alpha) = M_2 - m_2$$

- Let $\{\varepsilon_n\}$, $\varepsilon_n \to 0$, $\varepsilon_n > 0$, $n = 1, 2, \ldots$

Since f is continuous at s:

Given ε_n, $\exists\, \delta_n > 0$ s.t:

$$|t-s| < \delta_n \Rightarrow |f(t) - f(s)| < \varepsilon_n \qquad (1)$$

Define:

$$P_n = \{x_0, x_1, x_2, x_3\} = \{a, s, s+\delta_n, b\}$$

$$\Rightarrow U(P_n, f, \alpha) - L(P_n, f, \alpha) = M_{2,n} - m_{2,n}$$

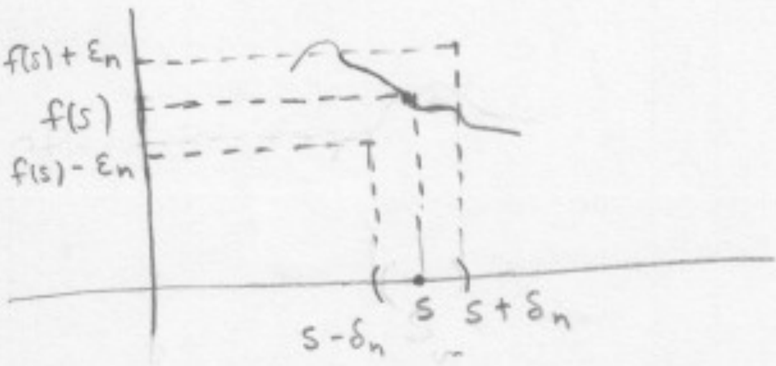
(1) implies that:

$$\boxed{U(P_n, f, \alpha) - L(P_n, f, \alpha) \leq 2\varepsilon_n} \quad (2).$$

Indeed:

$$U(P_n, f, \alpha) = M_{2,n} \leq f(s) + \varepsilon_n$$

$$L(P_n, f, \alpha) = m_{2,n} \geq f(s) - \varepsilon_n$$



$$\therefore\ U(P_n, f, \alpha) - L(P_n, f, \alpha) \leq f(s) + \varepsilon_n - (f(s) - \varepsilon_n)$$
$$= 2\varepsilon_n$$

$$0 \le U(P_n, f, \alpha) - L(P_n, f, \alpha) \le 2\varepsilon_n$$

The squezee theorem yields, since $\varepsilon_n \to 0$,

$$\lim_{n\to\infty} \left(U(P_n, f, \alpha) - L(P_n, f, \alpha)\right) = 0.$$

From homework problem we obtain that $f \in \mathcal{R}(\alpha)$ and:

$$\int_a^b f\, d\alpha = \lim_{n\to\infty} U(P_n, f, \alpha) = \lim_{n\to\infty} L(P_n, f, \alpha).$$

We have:

$$f(s) - \varepsilon_n \le M_{2,n} \le f(s) + \varepsilon_n, \quad n = 1, 2, \dots$$

Letting $\varepsilon_n \to 0$, since $f(s) + \varepsilon_n \to f(s)$ and $f(s) - \varepsilon_n \to f(s)$, the squezee theorem gives:

$$\lim_{n\to\infty} M_{2,n} = f(s)$$

$$\therefore \lim_{n\to\infty} U(P_n, f, \alpha) = f(s)$$

$$\Rightarrow \boxed{\int_a^b f\, d\alpha = f(s)}$$

(We also have $f(s) - \varepsilon_n \le m_{2,n} \le f(s) + \varepsilon_n$, and $\lim_{n\to\infty} m_{2,n} = \lim_{n\to\infty} L(P_n, f, \alpha) = f(s)$).

<u>Theorem 6.16</u>: Suppose $c_n \geq 0$, $\sum_{n=1}^{\infty} c_n$ converges, $\{s_n\}$ is a sequence of distincts points in (a,b), and

$$\alpha(x) = \sum_{n=1}^{\infty} c_n I(x - s_n)$$

Let f be continuous on $[a,b]$. Then:

$$\int_a^b f\,d\alpha = \sum_{n=1}^{\infty} c_n f(s_n)$$

<u>Proof</u>: $\alpha(x)$ exists since $\sum_{n=1}^{\infty} c_n < \infty$, $\alpha(x)$ is monotonic, $\alpha(a) = 0$, $\alpha(b) = \sum_{n=1}^{\infty} c_n$.

f continuous on $[a,b] \Rightarrow f \in \mathcal{R}(\alpha)$.

Let $\varepsilon > 0$, then $\exists N$ s.t:

$$\sum_{n=N+1}^{\infty} c_n < \varepsilon$$

$$\alpha(x) = \sum_{n=1}^{N} c_n I(x - s_n) + \sum_{n=N+1}^{\infty} c_n I(x - s_n)$$

$$= \alpha^1(x) + \alpha^2(x)$$

$$\int_a^b f\,d\alpha^1 = \int_a^b f\,(d\alpha_1^1 + \dots + d\alpha_N^1),$$

$$\alpha_n^1 = c_n I(x - s_n)$$

$$= \sum_{n=1}^{N} \int_a^b f\,d\alpha_n^1$$

$\Rightarrow$ (A) $\boxed{\int_a^b f\,d\alpha^1 = \sum_{n=1}^{N} c_n f(s_n);}$ from Theorem 6.15

f is continuous on $[a,b] \Rightarrow f \in \mathcal{R}(\alpha_2)$.

$$\boxed{\left|\int_a^b f\,d\alpha^2\right| \le \int_a^b |f|\,d\alpha^2 \le M\int_a^b d\alpha^2 \le M\varepsilon} \quad (B)$$

where $M = \sup\{|f(x)| : a \le x \le b\}$, because:

$$\int_a^b d\alpha^2 = \alpha^2(b) - \alpha^2(a)$$
$$= \sum_{n=N+1}^{\infty} c_n I(b-s_n) - \sum_{n=N+1}^{\infty} c_n I(a-s_n)$$
$$= \sum_{n=N+1}^{\infty} c_n - 0 = \sum_{n=N+1}^{\infty} c_n < \varepsilon.$$

Note:

$$\int_a^b f\,d\alpha = \int_a^b f\,d\alpha^1 + \int_a^b f\,d\alpha^2$$

$$\Rightarrow \left|\int_a^b f\,d\alpha - \int_a^b f\,d\alpha^1\right| = \left|\int_a^b f\,d\alpha^2\right| \le M\varepsilon$$

$$\Rightarrow \left|\int_a^b f\,d\alpha - \sum_{n=1}^{N} c_n f(s_n)\right| \le M\varepsilon; \quad \text{by } (A)$$

We have shown that: $\forall \varepsilon > 0, \exists N$ s.t $\sum_{i=N+1}^{\infty} c_i < \varepsilon$.

$\Rightarrow \forall n > N, \ \sum_{i=n}^{\infty} c_i < \varepsilon$.

Let $\beta_n = \sum_{i=1}^{n} c_i f(s_i)$. Therefore:

$$\left|\int_a^b f\,d\alpha - \beta_n\right| \le M\varepsilon, \quad \forall n \ge N.$$

$$\Rightarrow \lim_{n\to\infty} \beta_n = \int_a^b f\,d\alpha. \Rightarrow \boxed{\int_a^b f\,d\alpha = \sum_{i=1}^{\infty} c_i f(s_i)}$$