Suprema and Infima

Definition:

Let S be a subset of $\mathbb{R}$.

(a) An element $u \in \mathbb{R}$ is said to be an upper bound of S if $s \leq u$ for all $s \in S$.

(b) An element $w \in \mathbb{R}$ is said to be a lower bound of S if $w \leq s$ for all $s \in S$.

When a set has an upper bound, we shall say that it is <u>bounded above</u>, and when a set has a lower bound, we shall say that it is <u>bounded below</u>. If a set has both an upper and and a lower bound we shall say that it is <u>bounded</u>. If a set lacks either an upper or a lower bound, we shall say that it is <u>unbounded</u>.

<u>Ex.</u> $\mathbb{R}$ is unbounded

$P = \{x \in \mathbb{R} : x > 0\}$ is unbounded

$P = \{\frac{1}{n} : n = 1, 2, 3, \ldots\}$ is bounded.

Definition:

Let S be a subset of $\mathbb{R}$.

(a) If S is bounded above, then an upper bound of S is said to be a <u>supremum</u> (or a <u>least upper bound</u>) of S if it is less than any other upper bound of S.

(b) If S is bounded below, then a lower bound of S is said to be an <u>infimum</u> (or a <u>greatest lower bound</u>) of S if it is greater than any other lower bound of S.

Expressed differently, a number $u \in \mathbb{R}$ is a supremum of a subset S of $\mathbb{R}$ if it satisfies the two conditions;

(i) $s \leq u$ for all $s \in S$

(ii) If v is any number such that $s \leq v$ for all $s \in S$, then $u \leq v$.

Note that (i) makes u an upper bound of S, and (ii) shows that u is less than any other upper bound of S.

Similarly, a number $w \in \mathbb{R}$ is a infimum of a subset S of $\mathbb{R}$ if:

(i) $w \leq s$ for all $s \in S$

(ii) if p is any othe number such that

$$p \leq s \quad \text{for all } s \in S, \text{ then } \quad p \leq w$$

Actually, there can be only one supremum for a given subset S of $\mathbb{R}$. Indeed, if u_1, u_2 are both suprema of S, then from (ii) above:

$$u_1 \leq u_2 \quad \text{and} \quad u_2 \leq u_1,$$

that is, $u_1 = u_2$

In a similar way, there can be only one infimum for a given set S of $\mathbb{R}$. When these numbers exist, we shall denote them by:

$$\sup S \quad \text{and} \quad \inf S.$$

We have the following:

Theorem (Supremum Property): Every non-empty set of real numbers which has an upper bound has a supremum.

Theorem (Infimum Property): Every non-empty set of real numbers which has a lower bound has an infimum.

These theorems are part of Theorem 1.19, in Rudin's book. We can now show the Archimedean Property of $\mathbb{R}$.

Theorem (Archimedean Property): If $x \in \mathbb{R}$, $y \in \mathbb{R}$ and $x > 0$, then there exists a positive integer n such that $nx > y$.

Proof: We proceed by contradiction and assume that the conclusion is false. We let

$$A = \{nx : n = 1, 2, \cdots\}.$$

Then, y is an upper bound of A. Hence, the Supremum Property implies that $\sup A$ exists. We denote $\sup A$ as α.

Since $x>0$, we have that:

$\alpha - x < \alpha$ and $\alpha - x$ is not an upper bound of A (because α is the least upper bound of A). Therefore,

$$\alpha - x < mx,$$

for some positive integer m. Hence:

$$\alpha < (1+m)x \in A,$$

which contradicts that α is an upper bound of A. $\blacksquare$

<u>Ex</u>: Let $E = \{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{5}, \dots\}$

$= \{\frac{1}{n} : n \text{ is a positive integer}\}$

Then $\inf E = 0$ because

(i) $0 \leq \frac{1}{n}$, $n = 1, 2, \dots$; that is, 0 is a lower bound of E

(ii) 0 is the greatest lower bound of E, for if we could find $\beta > 0$ such that:

$$\beta \leq \frac{1}{n}, \quad n = 1, 2, \dots,$$

then for N large enough (by Archemedean

Property):

$$N\beta > 1,$$

that is, $0 < \frac{1}{N} < \beta$, which contradicts (1) which says that β is a lower bound of E. ∎

<u>Ex</u>: Suppose that A and B are nonempty subsets of $\mathbb{R}$ that satisfy:

$$a \leq b \quad \text{for all } a \in A, \text{ and all } b \in B$$

Then:

$$\sup A \leq \inf B$$

<u>Proof</u>: Fix $b \in B$. We have:

$$a \leq b, \quad \text{for all } a \in A$$

$\Rightarrow$ b is an upper bound of A.

$\Rightarrow \sup A \leq b$, since $\sup A$ is the least upper bound.

Clearly, previous inequality holds for any $b \in B$ (since we took an arbitrary $b \in B$ in the previous argument). Hence, $\sup A$ is a lower bound of B. But, since $\inf B$ is the greatest lower bound of B we get:

$$\sup A \leq \inf B. \quad ∎$$