Proof of Theorem 7.32 :

In a previous Lemma we showed that:

$$\bar{A} \subset \mathcal{C}(K) \quad \text{and that } \bar{A} \text{ is an algebra.}$$

Step 1 : Let $f \in \bar{A}$. Then $|f| \in \bar{A}$.

Let
$$a = \sup |f(x)|, \ x \in K \qquad (1).$$

Let $\varepsilon > 0$.

We now use the Corollary of Stone-Weirstrass theorem proven earlier: $\exists c_1, \ldots, c_n \in \mathbb{R}$ such that:

$$\Big| \underbrace{\sum_{i=1}^{n} c_i y^i}_{P_n(y)} - |y| \Big| < \varepsilon, \qquad y \in [-a, a] \qquad (2)$$

$$P_n(y) \to |y| \text{ uniformly on } [-a, a]$$
$$P_n(0) = 0$$

Since $\bar{A}$ is an algebra we have:

$$g := \sum_{i=1}^{n} c_i f^i \in \bar{A}.$$

From (1) and (2).

$$\left| \sum_{i=1}^{n} c_i [f(x)]^i - |f(x)| \right| < \varepsilon, \quad \forall x \in K$$
(Since $f(x) \in [-a, a]$)

$$\Rightarrow \left| g(x) - |f(x)| \right| < \varepsilon, \quad \forall x \in K.$$

$$\Rightarrow \| g - |f| \| < \varepsilon \Rightarrow d(g, |f|) < \varepsilon.$$

This proves that $|f|$ is a limit point of $\overline{A}$ since $N_\varepsilon(|f|) \cap \overline{A}$ contains $g \in \overline{A}$, $g \neq |f|$. Since $\overline{A}$ is closed then $|f| \in \overline{A}$.

Step 2 : If $f \in \overline{A}$ and $g \in \overline{A}$ then

$\max(f, g) \in \overline{A}$ and $\min(f, g) \in \overline{A}$.

In order to show this recall that if $h = \max(f, g)$ then:

$$h(x) = \begin{cases} f(x), & \text{if } f(x) \geq g(x) \\ g(x), & \text{if } f(x) < g(x). \end{cases}$$

We also have the identities;

$$\max(f, g) = \frac{f+g}{2} + \frac{|f-g|}{2},$$

$$\min(f, g) = \frac{f+g}{2} - \frac{|f-g|}{2}$$

By step 1:

$\max(f, g) \in \overline{A}$ and $\min(f, g) \in \overline{A}$.

- By iteration, it is also true that:

$$\text{if } f_1, f_2, \ldots, f_n \in \overline{A} \Longrightarrow \begin{array}{l} \max(f_1, \ldots, f_n) \in \overline{A} \\ \min(f_1, \ldots, f_n) \in \overline{A} \end{array}$$

Step 3: Given $f \in \mathscr{C}(K)$, $x \in K$ and $\varepsilon > 0$, there exists $g_x \in \overline{A}$ such that:

$$\begin{cases} g_x(x) = f(x) \\ g_x(t) > f(t) - \varepsilon, \quad t \in K \end{cases}$$

Proof: Since $A \subset \overline{A}$ and A satisfies (i) separates points and (ii) vanishes at no point of K, then clearly, $\overline{A}$ also satisfies (i) and (ii). Then, Theorem 7.31 implies that for every $y \in K$, we can find $h_y \in \overline{A}$ such that:

$$\boxed{h_y(x) = f(x), \quad h_y(y) = f(y) \qquad (3)}$$

By the continuity of h_y and f, $\exists J_y$, $y \in J_y$, J_y open neighborhood such that:

$$|h_y(t) - f(y)| < \frac{\varepsilon}{2} \text{ and } |f(t) - f(y)| < \frac{\varepsilon}{2} \quad \forall t \in J_y$$

Therefore:

$$\begin{aligned} |h_y(t) - f(t)| &\le |h_y(t) - f(y)| + |f(y) - f(t)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

$$\Rightarrow \quad h_y(t) - f(t) > -\varepsilon$$

$$\Rightarrow \quad h_y(t) > f(t) - \varepsilon, \quad \forall t \in J_y.$$

Hence, we have an open cover of K:

$$K \subset \bigcup_y J_y \quad \text{open cover of } K$$

Since K is compact, there exists a finite subcover:

$$\boxed{K \subset J_{y_1} \cup \cdots \cup J_{y_n}} \quad (4).$$

Define:

$$g_x = \max(h_{y_1}, \ldots, h_{y_n})$$

From Step 2, $g_x \in \bar{A}$.

By (3) and (4) we get:

$$g_x(t) > f(t) - \varepsilon, \quad \forall t \in K$$

(because $g_x(t) = h_{y_i}(t)$ for some $i \in \{1, \ldots, n\}$ and $t \in J_{y_j}$ for some j, $\Rightarrow h_{y_i}(t) \geq h_{y_j}(t) > f(t) - \varepsilon$).

Note also that $g_x(x) = f(x)$.

<u>Step 4</u>: Given $f \in \mathcal{C}(K)$ and $\varepsilon > 0$, there exists $h \in \bar{A}$ such that:

$$|h(x) - f(x)| < \varepsilon, \quad \forall x \in K$$

or equivalently, $\|h - f\| < \varepsilon$, Remark :

We already know that $\overline{\mathcal{A}} \subset \mathcal{C}(K)$. We need to show that $\mathcal{C}(K) \subset \overline{\mathcal{A}}$. Step 4 says that, given any $f \in \mathcal{C}(K)$ and $\varepsilon > 0$, we can find $h \in \overline{\mathcal{A}}$ such that:

$$d(h,f) < \varepsilon .$$

This means that f is a limit point of $\overline{\mathcal{A}}$, but since $\overline{\mathcal{A}}$ is closed we conclude that:

$$f \in \overline{\mathcal{A}} .$$

Therefore,

$$\mathcal{C}(K) \subset \overline{\mathcal{A}} .$$

We conclude:

$$\overline{\mathcal{A}} = \mathcal{C}(K).$$

Proof of Step 4 :

Consider the functions:

$$g_x, \quad x \in K$$

constructed in Step 3.

By the continuity of g_x and f there exists open sets V_x, $x \in V_x$ such that:

$$|g_x(t) - f(x)| < \frac{\varepsilon}{2}, \quad |f(t) - f(x)| < \frac{\varepsilon}{2}, \quad \forall t \in V_x.$$

$$\Rightarrow |g_x(t) - f(t)| \leq |g_x(t) - f(x)| + |f(x) - f(t)|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad \forall t \in V_x$$

Hence,

$$-\varepsilon < g_x(t) - f(t) < \varepsilon, \quad \forall t \in V_x$$

$$\Rightarrow \boxed{g_x(t) < f(t) + \varepsilon, \quad \forall t \in V_x} \quad (5)$$

We have an open cover of K :

$$K \subset \bigcup_x V_x \quad \text{open cover of } K$$

Since K is compact:

$$\boxed{K \subset V_{x_1} \cup \dots \cup V_{x_m}} \quad (6)$$

- Let
$$h = \min(g_{x_1}, \dots, g_{x_m})$$

Step 2 gives that $h \in \bar{A}$

We have, from step 3:

$$g_{x_i}(t) > f(t) - \varepsilon, \ \forall t \in K \qquad (7)$$

On the other hand, from (5) and (6):

$$g_{x_i}(t) < f(t) + \varepsilon, \ \forall t \in V_{x_i} \qquad (8)$$

Let $t \in K$.
Then $h(t) = g_{x_i}(t)$ for some $i \in \{1, \dots, m\}$

Therefore, by (7)

$$\boxed{h(t) > f(t) - \varepsilon} \qquad (*)$$

Now, $t \in V_{x_j}$, for some j because
$$K \subset V_{x_1} \cup \dots \cup V_{x_m}$$

$$\Rightarrow \quad h(t) = g_{x_i}(t) \leq g_{x_j}(t) < f(t) + \varepsilon; \text{ by } (8)$$

$$\Rightarrow \quad \boxed{h(t) < f(t) + \varepsilon} \qquad (**)$$

From $(*)$ and $(**)$:
$$|h(t) - f(t)| < \varepsilon, \quad t \in K. \ \blacksquare$$