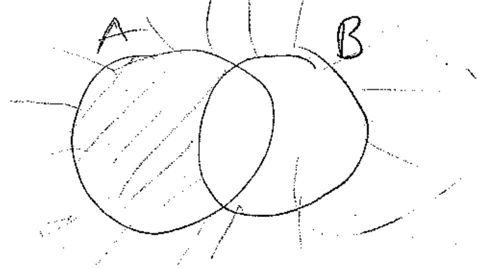
Chapter 11
The Lebesgue Theory.

Def: A family $\mathcal{R}$ of sets is called a ring if $A \in \mathcal{R}$ and $B \in \mathcal{R}$ implies:

$$A \cup B \in \mathcal{R} \;, \quad A - B \in \mathcal{R}.$$

$$(A - B = \{ x \in A : x \notin B \}, \text{ also } A - B = A \cap B^c)$$



Since $A \cap B = A - (A - B)$

$$\begin{aligned} A - (A - B) &= A \cap [A \cap B^c]^c \\ &= A \cap (A^c \cup B) \\ &= (A \cap A^c) \cup (A \cap B) \\ &= (A \cap B) \cup \emptyset = A \cap B \end{aligned}$$

$$\Rightarrow A, B \in \mathcal{R} \Rightarrow A \cap B \in \mathcal{R}$$

A ring $\mathcal{R}$ is called a σ-ring if.

$$A_n \in \mathcal{R}, \; n = 1, 2, \dots \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{R}$$

We have $\displaystyle\bigcap_{n=1}^{\infty} A_n = A_1 - \bigcup_{n=1}^{\infty} (A_1 - A_n)$

$$\begin{aligned} &\parallel \\ A_1 \cap \Big[\bigcup_{n=1}^{\infty} A_1 \setminus A_n \Big]^c &= A_1 \cap \Big[\bigcap_{n=1}^{\infty} (A_1 \setminus A_n)^c \Big] \\ &= A_1 \cap \Big[\bigcap_{n=1}^{\infty} [A_1 \cap A_n^c]^c \Big] \\ &= A_1 \cap \Big[\bigcap_{n=1}^{\infty} (A_1^c \cup A_n) \Big] \end{aligned}$$

$$\Longrightarrow \quad A_n \in \mathcal{R},\ n=1,2,\ldots \quad \Longrightarrow \quad \bigcap_{n=1}^{\infty} A_n \in \mathcal{R}$$

Def: ϕ is a set function defined on $\mathcal{R}$ if ϕ assigns to every $A \in \mathcal{R}$ a number $\phi(A)$ of the extended real number system.

ϕ is additive if:

$$A \cap B = 0 \implies \phi(A \cup B) = \phi(A) + \phi(B) \qquad [0 = \text{empty set}]$$

ϕ is countably additive if:

$$A_i \cap A_j = \phi,\ (i \neq j) \implies \phi\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \phi(A_n)$$

If ϕ is additive:

(a) $\phi(0) = 0$

(b) $\phi(A_1 \cup \cdots \cup A_n) = \phi(A_1) + \cdots + \phi(A_n)$, if $A_i \cap A_j = \phi$, $i \neq j$

(c) $\phi(A_1 \cup A_2) + \phi(A_1 \cap A_2) = \phi(A_1) + \phi(A_2)$

(d) $\phi(A) \geq 0$ $\forall A$ and $A_1 \subset A_2 \implies \phi(A_1) \leq \phi(A_2)$

(e) $\phi(A - B) = \phi(A) - \phi(B)$ if $B \subset A$, and $|\phi(B)| < +\infty$

<u>Thm 11.3</u>: Suppose ϕ is countably additive on a ring $\mathcal{R}$. Suppose $A_n \in \mathcal{R}$, $n = 1, 2, \ldots$, $A_1 \subset A_2 \subset A_3 \subset \cdots$, $A \in \mathcal{R}$, and

$$A = \bigcup_{n=1}^{\infty} A_n$$

Then:

$$\lim_{n \to \infty} \phi(A_n) = \phi(A)$$

<u>Proof</u>: Let

$$B_1 = A_1$$
$$B_2 = A_2 - A_1$$
$$B_3 = A_3 - A_2$$
$$\vdots$$
$$B_n = A_n - A_{n-1}$$
$$\vdots$$

Then

$$B_i \cap B_j = 0 \quad i \neq j$$

$$A_n = B_1 \cup \cdots \cup B_n$$

$$A = \bigcup_{n=1}^{\infty} B_n$$

$$\Rightarrow \quad \phi(A_n) = \sum_{i=1}^{n} \phi(B_i)$$

$$\downarrow \qquad \downarrow$$

and

$$\phi(A) = \sum_{i=1}^{\infty} \phi(B_i)$$

$$\left.\right\} \quad \phi\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} \phi(B_i)$$

If $\sum_{i=1}^{\infty} \phi(B_i)$ converges then $\sum_{i=1}^{n} \phi(B_i) = S_n \longrightarrow \sum_{i=1}^{\infty} \phi(B_i)$

$\Rightarrow \phi(A_n) \to \phi(A)$.

Construction of the Lebesgue measure

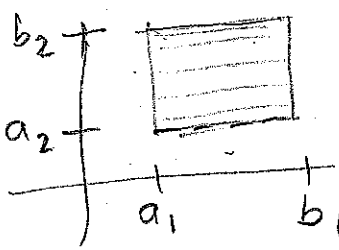
<u>Def</u>: Consider $\mathbb{R}^p$

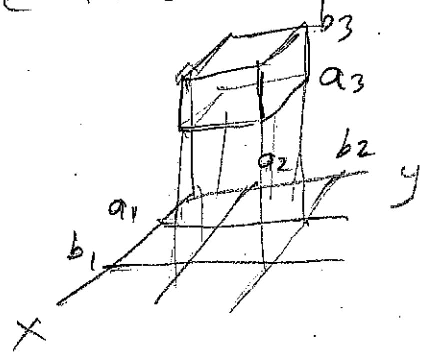
An interval in $\mathbb{R}^p$ is the set of all points $\vec{x} = (x_1, \dots, x_p)$ s.t.:

$$a_i \le x_i \le b_i \qquad i = 1, \dots, p. \qquad (*)$$

One or more of the $\le$ could be replaced by $<$.

$a_i = b_i$ is possible, so the empty set is included among the intervals:





If A is the union of a finite number of intervals, A is said to be an elementary set.

If I is an interval, we define

$$m(I) = \prod_{i=1}^{p} (b_i - a_i)$$

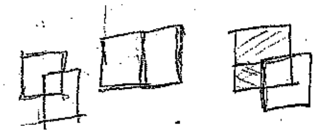
(no matter whether $\le$ or $<$ appears in $*$).

If $A = I_1 \cup \dots \cup I_n$, $I_i \cap I_j = \phi$

then we define:

$$m(A) = m(I_1) + \dots + m(I_n) \qquad (*)$$

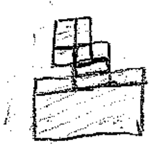
Let $\mathcal{E}$ denote the set of all elementary subsets of $\mathbb{R}^p$. Then:



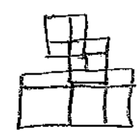
$A = \bigcup_{i=1}^{p} I_n \quad B = \bigcup_{i=1}^{q} I_n \in \mathcal{E}$

(a) $\mathcal{E}$ is a ring $\left\{ \Rightarrow A \cup B \in \mathcal{E} \quad A \setminus B \in \mathcal{E} \right.$

(b) If $A \in \mathcal{E}$, then A is the union of a finite number of disjoint intervals.



(c) If $A \in \mathcal{E}$, $m(A)$ is well defined by $(*)$, that is, two different decompositions of A in to disjoint intervals give the same value $m(A)$ with formula $(*)$.



(d) m is additive on $\mathcal{E}$ $\left\{ \begin{array}{l} A \cap B = 0 , \quad A, B \in \mathcal{E} \\ m(A \cup B) = m(A) \cup m(B) \end{array} \right.$

That is

$$m : \mathcal{E} \longrightarrow \mathbb{R}$$

is an additive set function, defined on the ring $\mathcal{E}$.

<u>Note</u>: If $p = 1, 2, 3$, then m is length, area and volume.

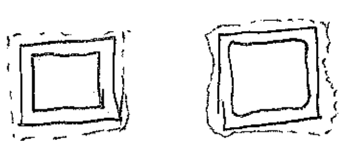
<u>Def</u>: A nonnegative additive set function ϕ defined on $\mathscr{E}$ is said to be <u>regular</u> if the following is true:
To every $A \in \mathscr{E}$ and to every $\varepsilon > 0$ $\exists\, F, G \in \mathscr{E}$ such that F is closed, G is open,

$$F \subset A \subset G, \text{ and}$$

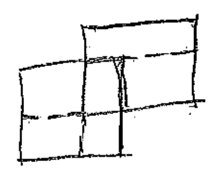
$$\phi(G) - \varepsilon \le \phi(A) \le \phi(F) + \varepsilon$$

$$\phi(F) \le \phi(A) \le \phi(G) \qquad \begin{array}{l} \phi(A) - \phi(F) \le \varepsilon \\ \phi(G) - \phi(A) \le \varepsilon \end{array}$$

<u>Ex</u>: The set function m is regular.

<u>Case 1</u>:


A single interval

<u>Case 2</u>:

$= A$

General case
Use case (1) and write A as the union of disjoint intervals. Then use <u>Case 1</u>.

"The main Theorem in the first part of the chapter is to show that every regular set function on $\mathscr{E}$ can be extended to a countably additive set function on a σ-ring which contains $\mathscr{E}$."

Def: Let μ be additive, regular, nonnegative, and finite on $\mathscr{E}$. Consider countable coverings of any set $E \subset \mathbb{R}^p$ by open elementary sets A_n:

$$E \subset \bigcup_{n=1}^{\infty} A_n \quad , \quad A_n \text{ open} , \ A_n \in \mathscr{E}$$

Define:

$$\mu^*(E) = \inf \sum_{n=1}^{\infty} \mu(A_n)$$

the inf being taken over all countable coverings of E by open elementary sets. $\mu^*(E)$ is called the outer measure of E, corresponding to μ.

Note: $\mu^*(E) \geq 0 \quad \forall E$
$\mu^*(E_1) \leq \mu^*(E_2)$ if $E_1 \subset E_2$.

<u>Thm</u>: (a) For every $A \in \mathcal{E}$,

$$\mu^*(A) = \mu(A)$$

(b) If $E = \bigcup_{n=1}^{\infty} E_n$, then

$$\mu^*(E) \le \sum_{n=1}^{\infty} \mu^*(E_n) \quad ; \ (E, E_n \subset \mathbb{R}^p)$$

<u>Remark</u>: (a) says that μ^* is an extension of μ from $\mathcal{E}$ to the family of all subsets of $\mathbb{R}^p$. Property (b) is called <u>subadditivity</u>.

<u>Proof</u>:

Let $A \in \mathcal{E}$ and $\varepsilon > 0$.

μ regular $\Rightarrow$ $\exists\, G \in \mathcal{E}$, (G open [illegible]) such that:

$$\mu(G) \le \mu(A) + \varepsilon.$$

But $\mu^*(A) \le \mu^*(G) = \mu(G)$ since G is open and by def of μ^*.

$\Rightarrow \mu^*(A) \le \mu(A) + \varepsilon$

ε arbitrary

$\Rightarrow \mu^*(A) \le \mu(A) \rightarrow (1)$

The definition of $\mu^* \Rightarrow \exists \{A_n\}$,

A_n open, $A_n \in \mathcal{E}$, $A \subset \bigcup_{n=1}^{\infty} A_n$

$$\Rightarrow \sum_{n=1}^{\infty} \mu(A_n) \le \mu^*(A) + \varepsilon$$

Since μ is regular $\Rightarrow \exists\ F \subset A$, F closed, $F \in \mathcal{E}$ s.t.

$$\mu(F) \geq \mu(A) - \varepsilon$$

F compact $\Rightarrow$

$$F \subset A_1 \cup \cdots \cup A_N$$

$A_1 \subset A_2 \Rightarrow \phi(A_1) \le \phi(A_2)$

Hence:

$$\mu(A) \le \mu(F) + \varepsilon \le \mu(A_1 \cup \cdots \cup A_N) + \varepsilon$$

$\phi(A_1 \cup A_2) + \phi(A_1 \cap A_2) = \phi(A_1) + \phi(A_2)$

$$\le \sum_{n=1}^{N} \mu(A_n) + \varepsilon$$

$$\le \sum_{n=1}^{\infty} \mu(A_n) + \varepsilon$$

$$\le \mu^*(A) + \varepsilon$$

$$\Rightarrow \mu(A) \le \mu^*(A) \rightarrow (2)$$

From (1) and (2) we obtain $\mu(A) = \mu^*(A)$.

For (b)

Suppose $E = \bigcup_{n=1}^{\infty} E_n$ and assume $\mu^*(E_n) < \infty \ \forall n$.

Given $\varepsilon > 0$, $\exists \ \{A_k^n\}$ $k = 1, 2, \ldots$,

$$E_n \subset \bigcup_{k=1}^{\infty} A_k^n \quad A_k^n \in \mathcal{E}, \text{ and open}$$

Such that

$$\sum_{k=1}^{\infty} \mu(A_k^n) \le \mu^*(E_n) + \frac{\varepsilon}{2^n} \quad (*)$$

$$\Rightarrow \quad \mu^*(E) \le \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \mu(A_k^n) \le \sum_{n=1}^{\infty} \mu^*(E_n) + \varepsilon, \text{ by } (*)$$

ε arbitrary $\Rightarrow$

$$\mu^*(E) \le \sum_{n=1}^{\infty} \mu^*(E_n)$$

If $\mu^*(E_n) = +\infty$ for some n, result trival.

<u>Def</u>: $A, B \in \mathbb{R}^p$, define

$$S(A, B) = (A - B) \cup (B - A)$$

$$d(A, B) = \mu^*(S(A, B))$$

We write $A_n \to A$ if

$$\lim_{n \to \infty} d(A, A_n) = 0$$

If there is a sequence $\{A_n\}$ of elementary sets such that $A_n \to A$, we say that A is:

"finitely μ-measurable"; i.e. $A \in \mathfrak{M}_F(\mu)$

If A is the union of a countable collection of finitely μ-measurable sets, we say that

"A is μ-measurable", i.e. $A \in \mathfrak{M}(\mu)$.

<u>Thm</u>: $\mathfrak{M}(\mu)$ is a σ-ring, and μ^* is countably additive on $\mathfrak{M}(\mu)$.

<u>Idea</u>: If we define

$A \sim B$ if $d(A, B) = 0$.

Then we divide the subsets of $\mathbb{R}^p$ in to equivalent clases, and $d(A, B)$ makes the set of these equivalent clases in to a metric space. $\mathfrak{M}_F(\mu)$ is obtained as the closure of $\mathscr{E}$.

<u>Note</u>: Replacing $\mu^*(A)$ by $\mu(A)$ if $A \in \mathfrak{M}(\mu)$, we have that μ (originarilly defined on $\mathscr{E}$) is extended to a countably additive set function on the σ-ring $\mathfrak{M}(\mu)$.

This extended set function is called a measure.

The special case $\mu = m$ is called:

"The Lebesgue measure on $\mathbb{R}^p$".

Recall

$\mathcal{R}$ ring, σ-ring

$\phi : \mathcal{R} \to \mathbb{R} \cup \{\infty\}$ set function — additive / countably additive

ϕ additive $\quad \Phi(A \cup B) = \phi(A) + \phi(B) \quad A \cap B = 0$

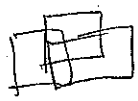
ϕ countably additive $\quad \phi\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \phi(A_n)$

$A_i \cap A_j = 0$

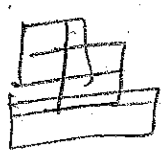
— o —

In $\mathbb{R}^p$

$\mathcal{E} = \{$set of all elementary subsets of $\mathbb{R}^p\}$



$m : \mathcal{E} \to \mathbb{R}$

$A =$  $\quad m(A) = m(I_1) + \dots + m(I_n)$,

m additive set function, $\mathcal{E}$ ring

— o —

Let $\mu : \mathcal{E} \to \mathbb{R}$ be additive, regular, non-negative and finite.

$E \subset \mathbb{R}^p$

Def $\mu^*(E) = \inf \sum_{n=1}^{\infty} \mu(A_n) \qquad E \subset \bigcup_{n=1}^{\infty} A_n$, $A_n \in \mathcal{E}$, A_n open

Then:

Measure spaces.

<u>Def</u>: Let X be a set

X is a measure space if:

(a) $\exists$ a σ-ring $\mathfrak{M}$ of subsets of X (which are called measurable sets) and

(b) $\exists$ a non-negative countably additive set function μ (which is called a measure), defined on $\mathfrak{M}$.

If $X \in \mathfrak{M}$ then X is said to be a measurable space.

<u>Ex</u>: $X = \mathbb{R}^p$ with $\mathfrak{M} = \mathfrak{M}(\mu)$ and μ the Lebesgue measure

<u>Def</u>: Let $f : X \to \tilde{\mathbb{R}}$ $\tilde{\mathbb{R}}$ extended real number system

f is measurable if
$$\{x \,;\, f(x) > a\} \in \mathfrak{M}$$
for every real a.

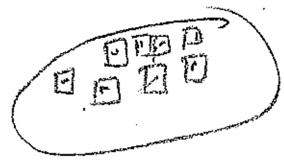
<u>Ex</u>: $X = \mathbb{R}^p$, $\mathcal{M} = \mathcal{M}(\mu)$,

μ Lebesgue measure. If f is continuous then f is measurable, since

$$\{x : f(x) > a\} \text{ is open}$$

$$= f^{-1}((a,\infty))$$

<u>HW 1</u>: $A \subset \mathbb{R}^p$ open $\Rightarrow$ $A \in \mathcal{M}(\mu)$ because A is the union of a contable collection of open intervals



check

Construct a countable base whose members are open intervals.

<u>Thm</u>: The following are equivalent:

(1) $\{x : f(x) > a\} \in \mathcal{M}(\mu)$ $\forall a$

(2) $\{x : f(x) \geq a\} \in \mathcal{M}$ $\forall a$

(3) $\{x : f(x) < a\} \in \mathcal{M}$ $\forall a$

(4) $\{x : f(x) \leq a\} \in \mathcal{M}$ $\forall a$

Sketch of proof: (1) $\Rightarrow$ (2)

$$\underbrace{\{x ; f(x) \geq a\}}_{S_a} = \bigcap_{n=1}^{\infty} \{x : f(x) > a - \tfrac{1}{n}\}.$$

Let $x \in S_a$: If $x \notin \{x : f(x) > a - \frac{1}{N}\}$ $\Rightarrow$

$f(x) \leq a - \frac{1}{N}$ $\Rightarrow$ $f(x) < a$ ∇

Let $x \in \bigcap_{n=1}^{\infty}\{\ \}$, If $f(x) < a$ then $\exists N$ s.t. $f(x) \leq a - \frac{1}{N}$ $\Rightarrow$ $f \notin \{x : f(x) > a - \frac{1}{N}\}$ ∇

Thm: f measurable $\Rightarrow |f|$ measurable

$$\{x : |f(x)| < a\} = \{x : f(x) < a\} \cap \{x : f(x) > -a\} \in \mathcal{M}$$

Thm. Let $\{f_n\}$ be a sequence of measurable functions. For $x \in X$, put

$$g(x) = \sup f_n(x) \quad n = 1, 2, \ldots$$
$$h(x) = \limsup_{n \to \infty} f_n(x)$$

Then g and h are measurable.

Proof:

$$\{x : g(x) > a\} = \bigcup_{n=1}^{\infty} \{x : f_n(x) > a\}$$
$$h(x) = \inf g_m(x)$$

where $g_m(x) = \sup f_n(x) \quad n \geq m$

Corollaries:

(a) If f and g are measurable, then $\max(f, g)$ and $\min(f, g)$ are measurable.

If $f^+ = \max(f, 0)$, $f^- = -\min(f, 0)$.

It follows, in particular, that f^+ and f^- are measurable.

(b) The limit of a convergent sequence of measurable functions is measurable.

Simple function.

<u>Def</u>: Let $s : X \to \mathbb{R}$. If the range of s is finite, we say that s is a <u>simple function</u>.

Let $E \subset X$, and put

$$\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$$

χ_E is called the characteristic function of E.

If range of $s = \{c_1, \dots, c_n\}$, let

$$E_i = \{x : s(x) = c_i\}, \quad i = 1, \dots, n$$

Then $$s = \sum_{i=1}^{n} c_i \chi_{E_i}$$

HW ②: s is measurable if and only if the sets $E_1, \dots, E_n$ are measurable.

Thm 11.20 Let $f : X \to \mathbb{R}$. Then there exists a sequence $\{s_n\}$ of simple functions such that $s_n(x) \to f(x)$ pointwise for every $x \in X$. If f is measurable, $\{s_n\}$ may be chosen to be a sequence of measurable functions. If

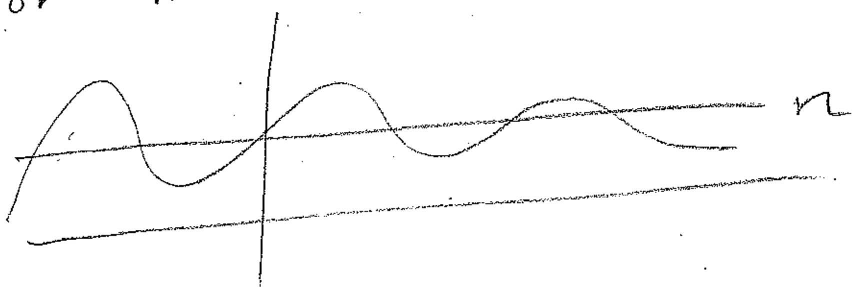
$f \geq 0$, $\{s_n\}$ may be chosen to be a monotonically increasing sequence.

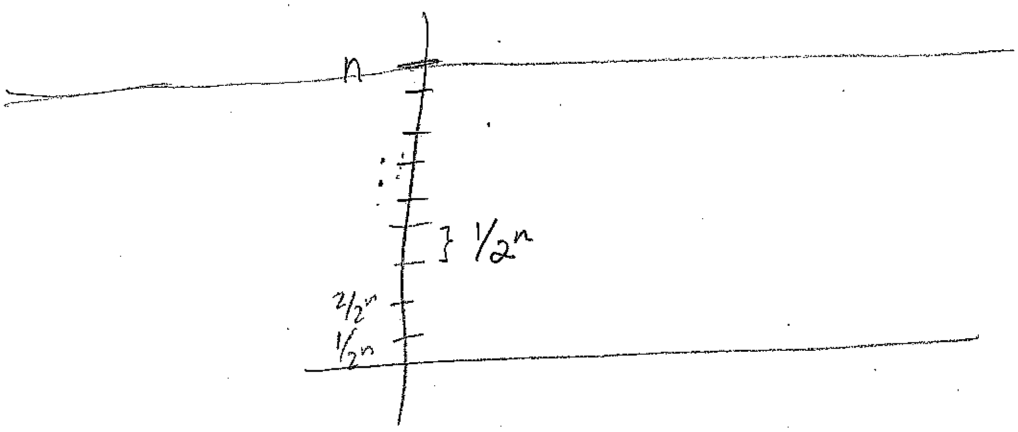
Sketch: If $f \geq 0$ define

$$E_{ni} = \left\{ x : \frac{i-1}{2^n} \leq f(x) < \frac{i}{2^n} \right\},$$

$$F_n = \{ x : f(x) \geq n \}$$

for $n = 1, 2, 3, \ldots$, $i = 1, 2, \ldots, n2^n$





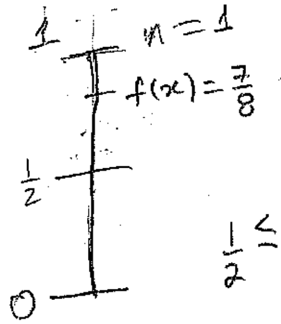
$$\frac{n}{n2^n} = \frac{1}{2^n}$$

Define:

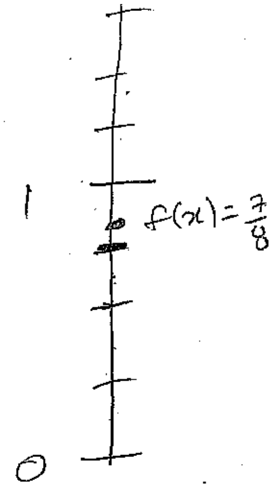
$$S_n = \sum_{i=1}^{n2^n} \frac{i-1}{2^n} \chi_{E_{ni}} + n \chi_{F_n}$$

$$x \in E_{ni} \Rightarrow \frac{i-1}{2^n} \leq f(x) < \frac{i}{2^n} \Rightarrow \frac{2(i-1)}{2^{n+1}} \leq f(x) < \frac{2i}{2^{n+1}}$$

$$\Rightarrow \frac{2i-2}{2^{n+1}} \leq f(x) < \frac{2i}{2^{n+1}} \Rightarrow \frac{j-1}{2^{n+1}} \leq f(x) < \frac{j}{2^{n+1}}, \quad j-1 \geq 2i-2$$



$\frac{1}{2} \le \frac{7}{8} < 1$



$\frac{3}{4} \le \frac{7}{8} < 1$

$n2^n = 1 \cdot 2^1 = 2$

$\frac{n}{n2^n} = \frac{1}{2^n} = \frac{1}{2}$

$n2^n = 2 \cdot 2^2 = 8$

$\frac{n}{n2^n} = \frac{1}{2^n} = \frac{1}{4}$

$S_1(x) = \frac{1}{2}$

$S_2(x) = \frac{3}{4}$

$S_1(x) \le S_2(x)$

$\Rightarrow \quad \underset{\frac{i-1}{2^n}}{\underset{\parallel}{S_1(x)}} \le \underset{\frac{j-1}{2^{n+1}}}{\underset{\parallel}{S_2(x)}}$ and $j-1 \geq 2i-2$

$\Rightarrow \frac{j-1}{2^{n+1}} \geq \frac{2(i-1)}{2^{n+1}} = \frac{i-1}{2^n}$

$x \in F_n \Rightarrow f(x) \geq n \Rightarrow S_n(x) = n$. If $f(x) \geq n+1$ then $S_{n+1}(x) = n+1$ and $S_n(x) \le S_{n+1}(x)$.

If not $n \le f(x) < n+1 \Rightarrow \frac{n2^{n+1}}{2^{n+1}} \le f(x) < \frac{(n+1)2^{n+1}}{2^{n+1}}$

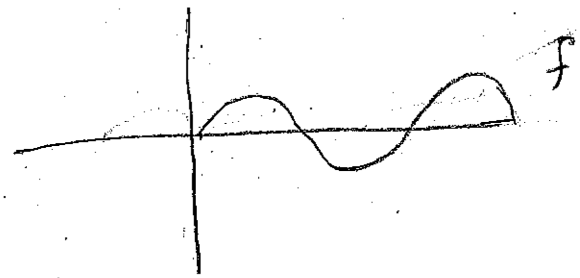
$\Rightarrow \frac{j-1}{2^{n+1}} \le f(x) \le \frac{j}{2^{n+1}}$, $\quad j-1 \geq n2^{n+1}$

$\Rightarrow S_{n+1}(x) = j-1 \geq \frac{n2^{n+1}}{2^{n+1}} = n \quad \Rightarrow S_n(x) \le S_{n+1}(x)$

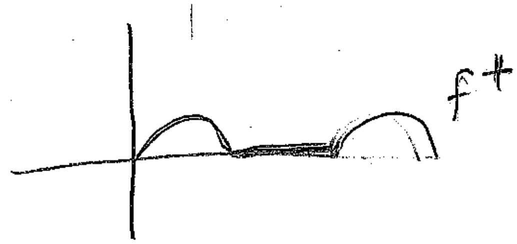
In the general case, we decompose:

$$f = f^+ - f^-$$

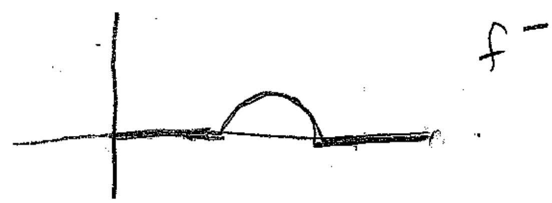
$$f^+ = \max(f, 0)$$
$$f^- = -\min(f, 0)$$



and apply the preceeding construction to f^+ and f^-



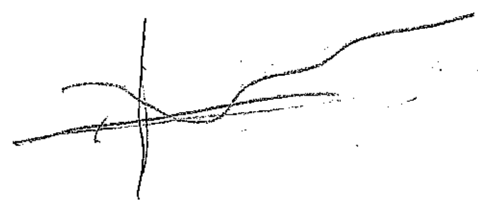
$$\begin{array}{ccccc} s_n(x) = & s_n^+(x) & - & s_n^-(x) \\ \downarrow & \downarrow & & \downarrow \\ f(x) = & f^+(x) & - & f^-(x) \end{array}$$



Integration

X, $\mathcal{M}$ σ-ring of measurable sets, μ a measure

Ex: $X = \mathbb{R}^1$, $\mu = m$.



$$s_n^+(x) \le f^+(x)$$
$$s_n^-(x) \le f^-(x)$$

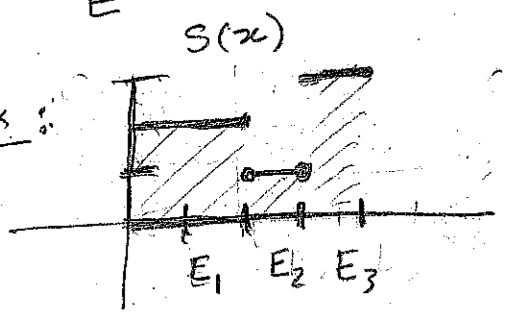
<u>Case 1</u>: $S(x) = \sum_{i=1}^{n} c_i \chi_{E_i}(x)$, $x \in X$, $c_i > 0$

is a measurable simple function.

Let $E \in \mathcal{M}$. We define

$$\int_E s \, d\mu = \sum_{i=1}^{n} c_i \, \mu(E \cap E_i)$$

<u>Ex</u>:



$E = \mathbb{R}$

$\mu = m$

$E_1 = [0, 2]$

$E_2 = (2, 3)$

$E_3 = [3, 4]$

$$\int_{\mathbb{R}} s \, dm = c_1 m(E_1) + c_2 m(E_2) + c_3 m(E_3)$$
$$= 2(2) + 1(1) + 3(1) = 8$$

$$\int_{\mathbb{R}} s \, dm = \int_{\mathbb{R}} s(x) \, dx$$

$\hookrightarrow$ Riemann sense

In Lebesgue sense

Both integrals coincide

<u>Case 2</u>: If $f \geq 0$ is measurable, we define

$$\int_E f \, d\mu = \sup \left\{ \int_E s \, d\mu \right\}$$

where the sup is taken over all measurable simple functions s such that $0 \leq s \leq f$.

$\int_E f\,d\mu$ is called the Lebesgue integral of f, with respect to the measure μ, over the set E. $\int_E f\,d\mu$ could take value $+\infty$.

Case 3: Let f be measurable. Define

$$\int_E f\,d\mu = \int_E f^+\,d\mu - \int_E f^-\,d\mu$$

if at least one the integrals is finite.

Remark: If $\int_E f^+\,d\mu < \infty$, $\int_E f^-\,d\mu < \infty$ then $\int_E f\,d\mu < \infty$ and we say that f is integrable on E in the Lebesgue sense, with respect to μ. We write

$$f \in \mathcal{L}(\mu) \text{ on } E,$$

If $\mu = m$, the standard notation is:

$$f \in \mathcal{L} \text{ on } E.$$

If $\int_E f\,d\mu = +\infty$ (or $-\infty$) then we say f is not integrable.

Thm 11.24: Let $f \in \mathcal{L}(\mu)$ on X. Define

$$\phi(A) = \int_A f\,d\mu, \quad A \in \mathcal{M}$$

Then ϕ is countably additive on $\mathcal{M}$.

<u>Corollary</u>: If $A \in \mathcal{M}$, $B \subset A$ and $\mu(A-B) = 0$, then

$$\int_A f\,d\mu = \int_B f\,d\mu$$

<u>Proof</u>: Since $A = B \cup (A-B)$

$$\Rightarrow \int_A f\,d\mu = \int_{B \cup (A-B)} f\,d\mu = \int_B f\,d\mu + \cancelto{0}{\int_{A-B} f\,d\mu}$$

ϕ additive.

By definition
In case 1:
$E = A-B$
$\mu(E \cap E_i) \le \mu(E) = 0$
$s(x) = \sum c_i \mu(E \cap E_i) = 0$

Remark: A, B $\mu[(A-B) \cup (B-A)] = 0$

$$\Rightarrow \int_A f\,d\mu = \int_B f\,d\mu$$

<u>Def</u>: $f \sim g$ on E if the set

$$\{x : f(x) \neq g(x)\} \cap E$$

has measure zero.

Then
(a) $f \sim f$
(b) $f \sim g \Rightarrow g \sim f$
(c) $f \sim g,\ g \sim h \Rightarrow f \sim h$

If $f \sim g$ on E, clearly

$$\int_A f\,d\mu = \int_A g\,d\mu \qquad \begin{matrix} A \in \mathcal{M} \\ A \subset E \end{matrix}$$

<u>Def</u>: If a property P holds for every $x \in E - A$ and if $\mu(A) = 0$, we say

" P holds for almost all $x \in E$ ".

or

" P holds almost everywhere on E ".

<u>Ex</u>: $f \sim g$ on E means $f = g$ on E almost everywhere

<u>Remark</u>: If $f \in \mathcal{L}(\mu)$ on E then $f(x)$ must be finite almost everywhere on E.

<u>Thm</u>. If $f \in \mathcal{L}(\mu)$ on E, then $|f| \in \mathcal{L}(\mu)$ on E and

$$\left| \int_E f\,d\mu \right| \le \int_E |f|\,d\mu$$

<u>Proof</u>: Write $E = A \cup B$, $f(x) \ge 0$ on A and $f(x) < 0$ on B.

$$\Rightarrow \int_E |f|\,d\mu = \int_A |f|\,d\mu + \int_B |f|\,d\mu = \int_A f^+\,d\mu + \int_B f^-\,d\mu$$

$$\Downarrow \qquad\qquad < +\infty$$

Thm 11.24

$\Rightarrow |f| \in \mathcal{L}(\mu)$. Since $f \le |f|$ and $-f \le |f|$

$$\Rightarrow \int_E f\,d\mu \le \int_E |f|\,d\mu \qquad -\int_E f\,d\mu \le \int_E |f|\,d\mu. \quad \blacksquare$$

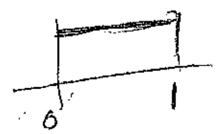
HW #3: $f(x) = \begin{cases} 1 & x \in [0,1] \setminus \mathbb{Q} \\ 0 & x \in \mathbb{Q} \cap [0,1] \end{cases}$

(a) Prove that f is not Riemann integrable, but f is Lebesgue integrable (In this case, $X = \mathbb{R}$, $\mu = m$).

(b) Compute $\int_{[0,1]} f\,dm$

Proof: Let $g = 1$ on $[0,1]$. Then $f = g$ almost everywhere since $m(\{x : f(x) \neq g(x)\} \cap [0,1]) = m(\mathbb{Q} \cap [0,1]) = 0$.

$$\Rightarrow \int_{[0,1]} f\,dm = \int_{[0,1]} g\,dm = 1$$



HW #4: Problem 1, Chapter 11.

Thm 11.28. Lebesgue's monotone convergence theorem.

Let $E \in \mathcal{M}$. Let $\{f_n\}$ be a sequence of measurable functions such that

$$0 \le f_1(x) \le f_2(x) \le \dots \quad (x \in E).$$

Define

$$f(x) := \lim_{n \to \infty} f_n(x) \quad (x \in E)$$

Then

$$\lim_{n \to \infty} \int_E f_n \, d\mu = \int_E f \, d\mu$$

Proof:

$$f_n \le f_{n+1}$$

$$\Rightarrow \int_E f_n \le \int_E f_{n+1}$$

$$\Rightarrow \left\{ \int_E f_n \, d\mu \right\} \text{ is monotone}$$

$$\Rightarrow \int_E f_n \, d\mu \to \alpha, \quad \alpha = \sup \left\{ \int f_n \right\}$$

and since

$$\int f_n \le \int f.$$

we have:

$$\boxed{\alpha \le \int_E f \, d\mu} \quad (1)$$

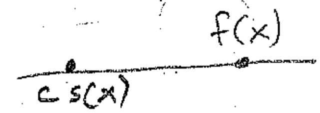
$0 < c < 1$

Let $0 \le s \le f$.

$E_n = \{x : f_n(x) \ge c\,s(x)\}$ $\quad n = 1, 2, 3, \ldots$

$\Rightarrow E_1 \subset E_2 \subset E_3 \subset \cdots \qquad (f_n \le f_{n+1})$

$\Rightarrow \quad E = \bigcup_{n=1}^{\infty} E_n \qquad f(x) = \sup f_n(x)$



$\Rightarrow \int_E f_n\,d\mu \ge \int_{E_n} f_n\,d\mu \qquad (E_n \subset E)$

$\ge c\int_{E_n} s\,d\mu$

$\int_{E_n} s\,d\mu \longrightarrow \int_E s\,d\mu$ $\quad$ HW#5: Read proof of Thm 11.3. Use Thm 11.3 and Thm 11.24 to show $\int_{E_n} s\,d\mu \to \int_E s\,d\mu$.

$\Rightarrow \alpha \ge c\int_E s\,d\mu$

Letting $c \to 1$, we get

$\alpha \ge \int_E s\,d\mu$

$\Rightarrow \boxed{\alpha \ge \int_E f\,d\mu} \to (2) \qquad \left[\text{Since } \int_E f = \sup \int_E s\right]$

From (1) and (2)

$\alpha = \lim \int_E f_n\,d\mu = \int_E f\,d\mu$

<u>Thm 11.29</u>. Suppose $f = f_1 + f_2$, $f_i \in \mathcal{L}(\mu)$ on E $(i = 1, 2)$. Then $f \in \mathcal{L}(\mu)$ on E and

$$\int_E f\,d\mu = \int_E f_1\,d\mu + \int_E f_2\,d\mu$$

<u>Thm 11.30</u>. Suppose $E \in \mathfrak{M}$. Let $\{f_n\}$, $f_n \geq 0$, f_n measurable and

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \qquad (x \in E).$$

then

$$\int_E f\,d\mu = \sum_{n=1}^{\infty} \int_E f_n\,d\mu$$

(Compare with Corollary of Theorem 7.16).

<u>Proof</u>: $g_n(x) = \sum_{i=1}^{n} f_i(x)$ $\qquad g_n(x) \to f(x)$.

$g_n \leq g_{n+1}$

$\Rightarrow \lim_{n \to \infty} \int_E g_n\,d\mu = \int_E f\,d\mu$.

$\Rightarrow \lim_{n \to \infty} \sum_{i=1}^{n} \int_E f_i\,d\mu = \int_E f\,d\mu$

$\Rightarrow \sum_{i=1}^{\infty} \int_E f_i\,d\mu = \int_E f\,d\mu$

Thm 11.31 : Fatou's theorem.

Suppose $E \in \mathcal{M}$. Let $\{f_n\}$, $f_n \geq 0$, f_n measurable and

$$f(x) := \liminf_{n \to \infty} f_n(x), \quad x \in E$$

then

$$\int_E f \, d\mu \leq \liminf_{n \to \infty} \int_E f_n \, d\mu$$

Proof : Define

$$g_n(x) = \inf_{i \geq n} f_i(x)$$

g_n is measurable (Thm 11.17)

$$0 \leq g_1(x) \leq g_2(x) \leq \cdots$$

$$g_n(x) \to f(x)$$

Because: HW #6: Consider the sequence $\{a_n\}$, $a_n \in \mathbb{R}$. Def.

$$c_n = \inf_{i \geq n} a_i$$

Then $\{c_n\}$ is monotone increasing and $\liminf_{n \to \infty} a_n = \lim_{n \to \infty} c_n$

$$\Rightarrow \int_E g_n \, d\mu \to \int_E f \, d\mu$$

$$\Rightarrow \int_E f \, d\mu = \lim_{n \to \infty} \int_E g_n \, d\mu$$

$$= \liminf_{n \to \infty} \int_E g_n \, d\mu \leq \liminf_{n \to \infty} \int_E f_n \, d\mu$$

$\Downarrow$

since $g_n \leq f_n$

Thm 11.32: Lebesgue's dominated convergence theorem.

Suppose $E \in \mathcal{M}$. Let $\{f_n\}$, f_n measurable, such that

$$f_n(x) \to f(x), \quad x \in E \qquad (*)$$

If there exists a function $g \in \mathcal{L}(\mu)$ on E, such that

$$|f_n(x)| \le g(x), \quad \forall n, \forall x \in E$$

then

$$\lim_{n\to\infty} \int_E f_n \, d\mu = \int_E f \, d\mu$$

Remark: The conclusion is the same if $(*)$ is replaced by:

$$f_n(x) \to f(x) \text{ almost everywhere on } E.$$

$$\int_{\tilde{E}} f_n \, d\mu \to \int_{\tilde{E}} f \, d\mu \qquad \mu(E \setminus \tilde{E}) = 0$$

Proof: $f_n \in \mathcal{L}(\mu)$ and $f \in \mathcal{L}(\mu)$ because

$$f_n^+ \le |f_n| \le g \implies \int_E f_n^+ \, d\mu \le \int_E g \, d\mu < \infty$$

Same for f_n^-. Thus $\int_E f_n = \int_E f_n^+ - \int_E f_n^- < \infty$.

$\Rightarrow f_n \in \mathcal{L}(\mu)$. f is measurable by Thm 11.17, Corollary (b). Since $|f(x)| \le g(x)$, same reasoning yields $f \in \mathcal{L}(\mu)$.

Since: $-g(x) \le f_n(x) \le g(x) \quad \forall x \in E$

$$\Rightarrow \quad g - f_n \ge 0, \quad g + f_n \ge 0. \text{ Then}$$

$$\int_E (g-f)\,d\mu \le \liminf_{n\to\infty} \int_E (g-f_n)\,d\mu$$

[by Fatou's Thm since $g-f_n \to g-f$]

So that

$$-\int_E f\,d\mu \le \liminf_{n\to\infty}\left(-\int_E f_n\,d\mu\right)$$

Multiply by -1:

$$\int_E f\,d\mu \geq -\liminf_{n\to\infty}\left(-\int_E f_n\,d\mu\right)$$

$$= \limsup_{n\to\infty} \int_E f_n\,d\mu$$

$\liminf(-a_n) = -\limsup(a_n)$

$$\int_E f\,d\mu \geq \limsup_{n\to\infty} \int_E f_n\,d\mu \longrightarrow (1)$$

Now, $f_n + g \ge 0$, Fatou's thm $\Rightarrow$

$$\int_E (f+g)\,d\mu \le \liminf_{n\to\infty} \int_E (f_n+g)\,d\mu$$

or

$$\int_E f\,d\mu \le \liminf_{n\to\infty} \int_E f_n\,d\mu \longrightarrow (2)$$

$$\Rightarrow \int_E f\,d\mu \le \liminf_{n\to\infty} \int_E f_n\,d\mu \le \limsup_{n\to\infty} \int_E f_n\,d\mu \le \int_E f\,d\mu$$

$$\Rightarrow \int_E f\,d\mu = \lim_{n\to\infty} \int_E f_n\,d\mu.$$

σ-ring versus σ-algebra

X, $\mathcal{M}$ σ-ring (i) $A, B \in \mathcal{M} \Rightarrow A - B \in \mathcal{M}$
(ii) $A_n \in \mathcal{M} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{M}$ $\Bigg\} \Rightarrow \bigcap_{n=1}^{\infty} A_n \in \mathcal{M}$

X, $\mathcal{A}$ σ-algebra (i) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$
(ii) $A_n \in \mathcal{A} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$ $\Bigg\} \Rightarrow \bigcap_{n=1}^{\infty} A_n \in \mathcal{A}$, $A - B \in \mathcal{A}$, $E \cap E^c = \phi \in \mathcal{A}$, $\phi^c = X \in \mathcal{A}$

Note: If $\mathcal{M}$ is a σ-ring and $X \in \mathcal{M}$ then $\mathcal{M}$ is a σ-algebra, because.

$$A \in \mathcal{M} \Rightarrow X - A \in \mathcal{M} \Rightarrow A^c \in \mathcal{M}.$$

For theory of integration as presented in Rudin's book:

$(X, \mathcal{M}$ σ-ring, $\mu) \to X$ is a measure space

<u>If</u> $X \in \mathcal{M} \Rightarrow X$ is a measurable space.

and $\mathcal{M}$ is actually a σ-algebra.

<u>Ex</u>: $(\mathbb{R}^p, \mathcal{M}, \underbrace{m = \mu}_{\text{Lebesgue measure}})$

$\mathbb{R}^p \in \mathcal{M}$ (i.e. $\mathbb{R}^p$ is measurable), and thus $\mathcal{M}$ is also a σ-algebra.

D.C.T.:

$f_n(x) \to f(x)$ on E

$|f_n(x)| \le g(x) \quad x \in E \quad g \in \mathcal{L}(\mu)$

$$\int_E f_n(x)\,d\mu \to \int_E f(x)\,d\mu.$$

Corollary: If $\mu(E) < \infty$, $\{f_n\}$ is uniformly bounded on E, and $f_n(x) \to f(x)$ on E

then
$$\int_E f_n(x)\,d\mu \to \int_E f(x)\,d\mu$$

Proof;
$$|f_n(x)| \le M \qquad n = 1, 2, 3, \dots, \qquad x \in E$$

$g(x) = M, \quad x \in E$

$$\int_E g(x)\,d\mu \le \int_E M\,d\mu = M \int_E d\mu = M \cdot \mu(E) < \infty$$
$$\Rightarrow g \in \mathcal{L}(\mu)$$

Let $X = [a,b]$, $\mu = m$ the Lebesgue measure. Instead of

$$\int_X f\,dm$$

let us write
$$\int_a^b f\,dx.$$

Notation : $\int_a^b f\,dx$ Lebesgue sense

$\mathcal{R}\int_a^b f\,dx$ Riemann sense.

Thm 11.33 :

(a) If $f \in \mathcal{R}$ on $[a,b]$, then $f \in \mathcal{L}$ on $[a,b]$, and

$$\int_a^b f\,dx = \mathcal{R}\int_a^b f\,dx$$

(b) Suppose that f is bounded on $[a,b]$. Then $f \in \mathcal{R}$ on $[a,b]$ if and only if f is continuous almost everywhere on $[a,b]$.

Proof : Suppose f is bounded. Thus there exist a sequence of partitions $P_1, P_2, \ldots, P_k, \ldots$ with

$$P_{k+1} \supset P_k$$

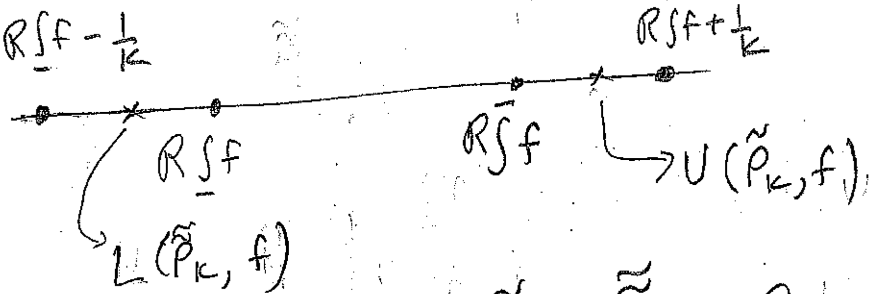
(i.e P_{k+1} is a refinement of P_k)

Such that

$$\boxed{\begin{array}{ll} \lim_{k\to\infty} L(P_k, f) = \mathcal{R}\underline{\int} f\,dx & (1) \\ \lim_{k\to\infty} U(P_k, f) = \mathcal{R}\overline{\int} f\,dx & (2) \end{array}}$$

In order to see (1) and (2) we recall the following:

$$P \subset P^* \Rightarrow \begin{cases} L(P,f) \le L(P^*,f) \\ U(P^*,f) \le U(P,f) \end{cases}$$



If $\quad P_k := \tilde{P}_k \cup \tilde{\tilde{P}}_k \cup P_{k-1}$, $\quad P_0 = \phi$

$0 \le R\underline{\int}f - L(P_k,f) \le \frac{1}{k}$ $\quad 0 \le U(P_k,f) - R\overline{\int}f \le \frac{1}{k}$

Thus P_k is a refinement of P_{k-1}

an $\quad L(P_k,f) \to R\underline{\int}f$, $\quad U(P_k,f) \to R\overline{\int}f$

If $P_k = \{x_0, x_1, \dots, x_n\}$, $x_0 = a$, $x_n = b$, define

$$U_k(a) = L_k(a) = f(a)$$

and $\quad U_k(x) = M_i$, $\quad x_{i-1} < x \le x_i$

$\quad L_k(x) = m_i$, $\quad x_{i-1} < x \le x_i$

$$M_i = \sup_{x_{i-1} \le x \le x_i} f(x) \qquad m_i = \inf_{x_{i-1} \le x \le x_i} f(x)$$

U_k, L_k simple functions.

$$\boxed{L(P_k,f) = \int_a^b L_k \, dx \quad \text{(Lebesgue sense).}}$$

and

$$\boxed{U(P_k, f) = \int_a^b U_k\, dx \quad , \quad \hookrightarrow \text{Lebesgue sense}}$$

$$L_k = \sum m_i \chi_{(x_{i-1}, x_i]} + f(a)\chi_{\{a\}} \qquad U_k = \sum M_i \chi_{(x_{i-1}, x_i]} + f(a)\chi_{\{a\}}$$

$$\int_a^b L_k\, dx = \sum m_i\, m((x_{i-1}, x_i] \cap [a,b]) + \cancel{f(a) m(\{a\})}^{0}$$

$$= \sum m_i \Delta x_i = L(P_k, f)$$

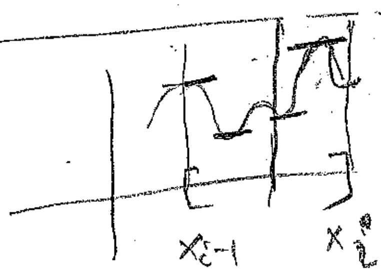
$$\int_a^b M_k\, dx = \sum M_i\, m((x_{i-1}, x_i] \cap [a,b]) + \cancel{f(a) m(\{a\})}^{0}$$

$$= \sum M_i \Delta x_i = U(P_k, f).$$

Also:

$$L_1(x) \le L_2(x) \le \cdots \le f(x) \le \cdots \le U_2(x) \le U_1(x)$$

for all $x \in [a,b]$, since P_{k+1} is a refinement of P_k



Define

$$\left.\begin{aligned} L(x) &= \lim_{k\to\infty} L_k(x) \\ U(x) &= \lim_{k\to\infty} U_k(x) \end{aligned}\right\} \text{defined since they are monotone sequences}$$

- L, U are bounded measurable functions on $[a,b]$
- $L(x) \le f(x) \le U(x) \quad a \le x \le b.$

and

$$\int_a^b L\,dx = \lim_{k\to\infty} \int_a^b L_k = \lim_{k\to\infty} L(P_k, f) = R\underline{\int} f$$

$\Downarrow$

DCT (Corollary)

$$\int_a^b U\,dx = \lim_{k\to\infty} \int_a^b U_k = \lim_{k\to\infty} U(P_k, f) = R\overline{\int} f$$

$\Downarrow$

DCT (Corollary)

$$\Longrightarrow \boxed{\int_a^b L\,dx = R\underline{\int} f \quad , \quad \int_a^b U\,dx = R\overline{\int} f}$$

Now

$$f \in R \iff R\underline{\int} f = R\overline{\int} f \iff \int_a^b L = \int_a^b U \iff \int_a^b (U - L) = 0$$

$\iff U - L = 0$ almost everywhere

$\iff U = L$ a.e

(by HW #4)

Thus

(a) If $f \in \mathcal{R}$ then:

$\Rightarrow L = U$ a.e.

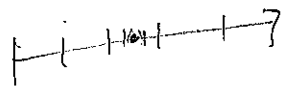
$\Rightarrow f = L = U$ a.e (Since $L(x) \le f(x) \le U(x)$)

$\Downarrow$

f is measurable

$$\Rightarrow f \in \mathcal{L} \text{ and } \int_a^b f = \int_a^b L = \int_a^b U = \mathcal{R}\overline{\int_a^b} f$$
$$= \mathcal{R}\underline{\int_a^b} f$$

(b) Let $x \in [a,b]$ $x \notin \{x_0, \dots, x_n\} = P_k$. Then:



$U(x) = L(x)$ if and only if f is continuous at x.

Let $P = \bigcup_{k=1}^{\infty} P_k$

P is countable $\Rightarrow$ $m(P) \le \sum_{k=1}^{\infty} m(P_k) = 0$

We have then:

f is continuous a.e. on $[a,b]$ $\Leftrightarrow L(x) = U(x)$ a.e

$\Leftrightarrow \mathcal{R}\underline{\int} f = \mathcal{R}\overline{\int} f$

$\Leftrightarrow f \in \mathcal{R}$.

Differentiation and Lebesgue theory.

If $f \in \mathcal{L}$ on $[a,b]$ and we define:

$$F(x) = \int_a^x f(t)\,dt \qquad a \le x \le b$$

then $F'(x) = f(x)$ a.e. on $[a,b]$.

Conversely, if F is differentiable at every point of $[a,b]$ (a.e not enough) and if $F' \in \mathcal{L}$ on $[a,b]$ then

$$F(x) - F(a) = \int_a^x F'(t)\,dt \qquad (a \le x \le b).$$

(Compare with Thm 6.20).

Thm 6.20 : $f \in \mathcal{R}$ on $[a,b]$

$$F(x) = \int_a^x f(t)\,dt$$

Then F is continuous on $[a,b]$. If f is continuous at x_0 then F diff. at x_0 and

$$F'(x_0) = f(x_0)$$

If $f \in \mathcal{R}$ on $[a,b]$ and if $\exists F$ diff. on $[a,b]$ s.t $F' = f$ then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

HW #7: Problem 8 in

HW #8: Problem 16