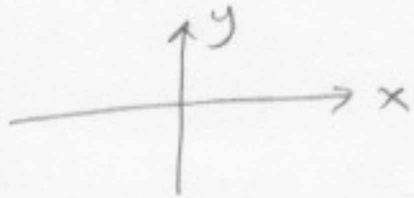
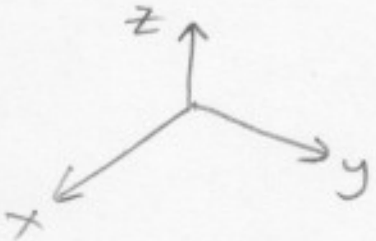
# Euclidean Spaces and Metric Spaces

$\mathbb{R}$

$\mathbb{R}^2$




$\mathbb{R}^3$




Let

$$\mathbb{R}^k = \{\vec{x} = (x_1, \dots, x_k) : x_1, \dots, x_k \in \mathbb{R}\}$$

Define:

$$\vec{x} + \vec{y} = (x_1 + y_1, \dots, x_k + y_k)$$

$$\alpha \vec{x} = (\alpha x_1, \dots, \alpha x_k), \quad \alpha \in \mathbb{R}$$

With these two operations, $\mathbb{R}^k$ is a <u>vector space</u> over $\mathbb{R}$. The inner product (or scalar product) is defined as:

$$\vec{x} \cdot \vec{y} = \sum_{i=1}^{k} x_i y_i$$

$$|\vec{x}| = \sqrt{\vec{x} \cdot \vec{x}} = \left( \sum_{i=1}^{k} x_i^2 \right)^{1/2}$$

- **Theorem**: Let $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^k$, $\alpha \in \mathbb{R}$.

Then:

(a) $|\vec{x}| \geq 0$

(b) $|\vec{x}| = 0$ if and only if $\vec{x} = 0$

(c) $|\alpha \vec{x}| = |\alpha| \, |\vec{x}|$

(d) $|\vec{x} \cdot \vec{y}| \leq |\vec{x}| \, |\vec{y}|$

(e) $|\vec{x} + \vec{y}| \leq |\vec{x}| + |\vec{y}|$

(f) $|\vec{x} - \vec{z}| \leq |\vec{x} - \vec{y}| + |\vec{y} - \vec{z}|$

Proof: (a), (b), (c) are clear.

We now show (d):

Define $A = \sum_{i=1}^{k} x_i^2$, $B = \sum_{i=1}^{k} x_i y_i$ and $C = \sum_{i=1}^{k} y_i^2$. We want to show:

$$B^2 \leq AC \qquad (1)$$

If $C = 0$ the result is clear (since $\vec{y} = \vec{0}$).

Assume $C \neq 0$. Define:

$$\begin{aligned} F(t) &= (x_1 - t y_1)^2 + \cdots + (x_k - t y_k)^2 \\ &= \sum_{i=1}^{k} x_i^2 - 2t \sum_{i=1}^{k} x_i y_i + t^2 \sum_{i=1}^{k} y_i^2 \\ &= A - 2tB + Ct^2 \geq 0 \end{aligned}$$

Since $F(t) \geq 0$ for all $t$ then the quadratic equation $F(t) = 0$ can not have 2 real distinct roots. Then:

In equation:

$$Ct^2 - 2Bt + A = 0$$

we must have:

$$\Delta := (-2B)^2 - 4AC \le 0$$

$$\Rightarrow \quad B^2 \le AC, \text{ which is } (1).$$

If $\vec{x} = s\vec{y}$ then equality holds in (1). On the other hand, if equality holds in (1), then $(-2B)^2 - 4AC = 0$ and hence $F(t) = 0$ has a unique root, say $s$. That is,

$$F(s) = 0$$

$$\Rightarrow \quad x_i - s y_i = 0 \quad , \; i = 1, \dots, k$$

$$\Rightarrow \quad \vec{x} = s\vec{y}.$$

To prove (e) we compute:

$$\begin{aligned} |\vec{x}+\vec{y}|^2 &= (\vec{x}+\vec{y}) \cdot (\vec{x}+\vec{y}) \\ &= \vec{x}\cdot\vec{x} + 2\vec{x}\cdot\vec{y} + \vec{y}\cdot\vec{y} \\ &\le |\vec{x}|^2 + 2|\vec{x}||\vec{y}| + |\vec{y}|^2 \; ; \quad \text{since } \vec{x}\cdot\vec{y} \le |\vec{x}||\vec{y}| \\ &= (|\vec{x}| + |\vec{y}|)^2 \end{aligned}$$

$$\Rightarrow \quad |\vec{x}+\vec{y}| \le |\vec{x}| + |\vec{y}|.$$

For (f) we use (e):

$$|\vec{x} - \vec{z}| = |(\vec{x}-\vec{y}) + (\vec{y}-\vec{z})| \le |\vec{x}-\vec{y}| + |\vec{y}-\vec{z}|. \quad \blacksquare$$

# Metric spaces

Def: A set $X$ is a metric space if there exists a function $d: X \times X \to \mathbb{R}$ such that:

(a) $d(p,q) > 0$ if $p \neq q$, $d(p,p) = 0$

(b) $d(p,q) = d(q,p)$

(c) $d(p,q) \leq d(p,r) + d(r,q)$ for any $r \in X$.

Ex: $\mathbb{R}^k$ is a metric space with a metric $d$ given by the distance function as follows:

$$d(\vec{x}, \vec{y}) = |\vec{x} - \vec{y}| \quad ; \quad \vec{x}, \vec{y} \in \mathbb{R}^k$$

Clearly, (a), (b), (c) are true since:

(a) $|\vec{x} - \vec{y}| > 0$ if $\vec{x} \neq \vec{y}$
$|\vec{x} - \vec{x}| = 0$ ✓

(b) $|\vec{x} - \vec{y}| = |\vec{y} - \vec{x}|$ ✓

(c) We showed the triangle inequality in previous theorem:
$|\vec{x} - \vec{y}| \leq |\vec{x} - \vec{z}| + |\vec{z} - \vec{y}|$, for any $\vec{z} \in \mathbb{R}^k$

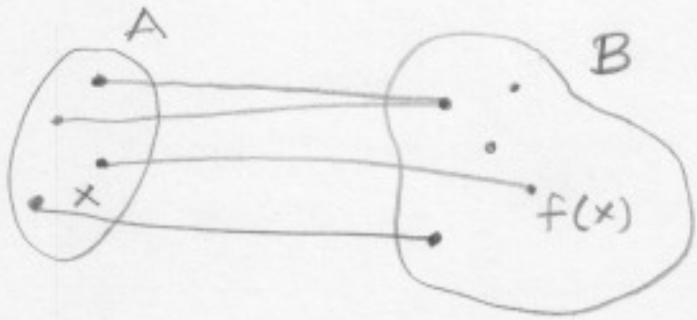
Remark: In this class, when discussing a theorem for $X$, always think of $\mathbb{R}^k$ to put a picture in mind.

# Chapter 2

## Basic topology.

### Finite, Countable and uncountable sets.

<u>Def</u>: A function $f: A \to B$ is a rule that assigns to each $x \in A$ an element of $B$, which is denoted by $f(x)$




$A$ is the domain of $f$. The set of all values of $f$ is the range of $f$.

<u>Def</u>: Let $f: A \to B$

If $E \subset A$, $f(E) := \{f(x) : x \in E\}$. The set $f(E) \subset B$ is called the image of $E$ under $f$. In this notation, $f(A)$ is the range of $f$.

<u>Def</u>: If $f(A) = B$ we say that $f$ maps $A$ <u>onto</u> $B$, or that $f$ is surjective.

If $E \subset B$:

$$f^{-1}(E) = \{x \in A : f(x) \in E\}$$

is the inverse image of $E$ under $f$.

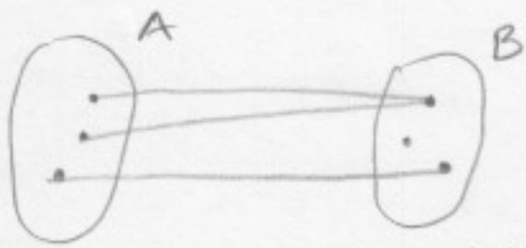
If $y \in B$,

$$f^{-1}(y) = \{x \in A : f(x) = y\}$$

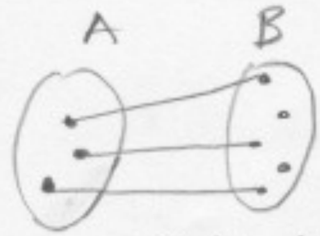
<u>Def</u>: If $f^{-1}(y)$ consists of at most one element of $A$, then $f$ is said to be a 1-1 (one-to-one) mapping of $A$ into $B$. It is equivalent to say that $f$ is a 1-1 mapping (or injective) of $A$ into $B$ if:

$$f(x_1) \neq f(x_2) \quad \text{whenever} \quad x_1 \neq x_2, \quad x_1, x_2 \in A.$$

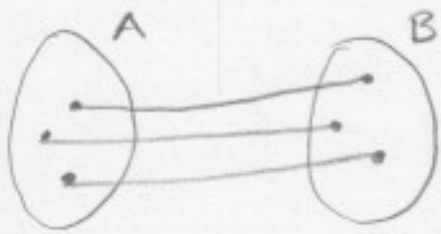
<u>Ex</u>:




$f: A \to B$ is not 1-1




$f: A \to B$ is 1-1
$f$ is not onto.




$f: A \to B$ is 1-1 and onto.

<u>Ex</u>: Let $A = \{x \in \mathbb{R} : x \neq 1\}$ and define $f: A \subset \mathbb{R} \to \mathbb{R}$ as:

$$f(x) = \frac{2x}{x-1}, \quad x \in A.$$

Show that $f$ is injective and determine the range of $A$.

Proof:

Assume that $x_1, x_2 \in A$ are such that
$$f(x_1) = f(x_2).$$
We need to show that $x_1 = x_2$.

We have:
$$\frac{2x_1}{x_1 - 1} = \frac{2x_2}{x_2 - 1}$$
$$\Rightarrow 2x_1(x_2 - 1) = 2x_2(x_1 - 1)$$
$$\Rightarrow \cancel{2x_1x_2} - 2x_1 = \cancel{2x_2x_1} - 2x_2$$
$$\Rightarrow x_1 = x_2 \quad ✓$$

To find the range of $f$, we solve the equation $y = \frac{2x}{x-1}$ for $x$ in terms of $y$:
$$xy - y = 2x \Rightarrow x(y-2) = y$$
$$\Rightarrow x = \frac{y}{y-2},$$
which is defined for $y \neq 2$.

Hence:
$$\text{Range}(f) = f(A) = \{y \in \mathbb{R} : y \neq 2\}.$$

Note that:
$$f : A \to \mathbb{R} \text{ is not on-to.}$$

However, when we consider:
$$f : A \longrightarrow B,$$
with $B = \{y \in \mathbb{R} : y \neq 2\}$ then this function is 1-1 and onto. That is, $f$ is a bijection of $A$ onto $B$.