Lesson 10

Section 4.6

Lebesgue - Stieltjes measure

Def: The Lebesgue-Stieltjes outer measure of an arbitrary set $E \subset \mathbb{R}$ is defined by:

$$\lambda_f^*(E) = \inf \left\{ \sum_{h_k \in \mathcal{F}} \alpha_f(h_k) \right\},$$

where the infimum is taken over all countable collections $\mathcal{F}$ of half-open intervals h_k of the form $(a_k, b_k]$ such that

$$E \subset \bigcup_{h_k \in \mathcal{F}} h_k \quad \text{and} \quad \alpha_f((a_k, b_k]) = f(b_k) - f(a_k)$$

Thm 1: If $f: \mathbb{R} \to \mathbb{R}$ is a nondecreasing function, then λ_f^* is a Carathéodory outer measure on $\mathbb{R}$.

Remark: A Carathéodory outer measure is a Borel outer measure.

Theorem 2: If $f: \mathbb{R} \to \mathbb{R}$ is nondecreasing and right continuous, then

$$\lambda_f((a,b]) = f(b) - f(a)$$

There is an identification between Lebesgue-Stieltjes measures and nondecreasing right continuous functions.

Thm 3: Suppose μ is a finite Borel outer measure on $\mathbb{R}$ and lef

$$f(x) = \mu((-\infty, x])$$

Then the Lebesgue-Stieltjes measure λ_f, agrees with μ on all Borel sets.

Proof:

Let μ be a finite Borel outer measure on $\mathbb{R}$. Define:

$$f(x) = \mu((-\infty, x])$$

Clearly, f is non-decreasing. Fix $x_0 \in \mathbb{R}$ and let $\{x_j\}$ such that:

$$x_j \to x_0, \quad x_j > x_0, \quad x_{j+1} < x_j. \quad (*)$$

It is easy to check that showing $f(x_j) \to f(x_0)$ for each monotonic seq. as in $(*)$ is enough to conclude $f(x_j) \to f(x_0)$ for ANY sequence $x_j \to x_0$, $x_j > x_0$

We have:

$$\bigcap_{j=1}^{\infty} (-\infty, x_j] = (-\infty, x_0]. \quad (*)$$

Indeed, let $x \in (-\infty, x_0]$. Clearly, $x \in (-\infty, x_j]$ $\forall j$ $\Rightarrow$ $x \in \bigcap_{j=1}^{\infty} (-\infty, x_j]$. Now, if $x \notin (-\infty, x_0]$, then $x > x_0$. Thus, $\exists$ x_j such that $x_0 < x_j < x$. Then $x \notin (-\infty, x_j]$ and therefore $x \notin \bigcap_{j=1}^{\infty} (-\infty, x_j]$.

From $(*)$

$$f(x_0) = \mu((-\infty, x_0])$$
$$= \mu\left(\bigcap_{j=1}^{\infty} (-\infty, x_j] \right)$$

$$= \lim_{j\to\infty} \mu((-\infty, x_j])$$

$$= \lim_{j\to\infty} f(x_j)$$

$\therefore$ $f(x_j) \to f(x_0)$

$\therefore$ f is right-continuous.

Let λ_f be the Lebesgue - Stieltjes measure.

$$\mu \longmapsto f \longmapsto \lambda_f.$$

On the one hand:

$$\mu((a,b]) = \mu((-\infty,b] \setminus (-\infty,a])$$
$$= \mu((-\infty,b]) - \mu((-\infty,a])$$
$$= f(b) - f(a)$$

On the other hand, Theorem 2 yields:

$$\lambda_f((a,b]) = f(b) - f(a)$$

$\therefore$ μ and λ_f agree on all half open intervals. Since every open set in $\mathbb{R}$ is a countable union of disjoint half open intervals, then

μ and λ_f agree on all open sets.

Thm 108.1 implies that

μ and λ_f agree on all Borel sets. ∎

Recall

Thm 108.1: φ Borel outer measure. on X

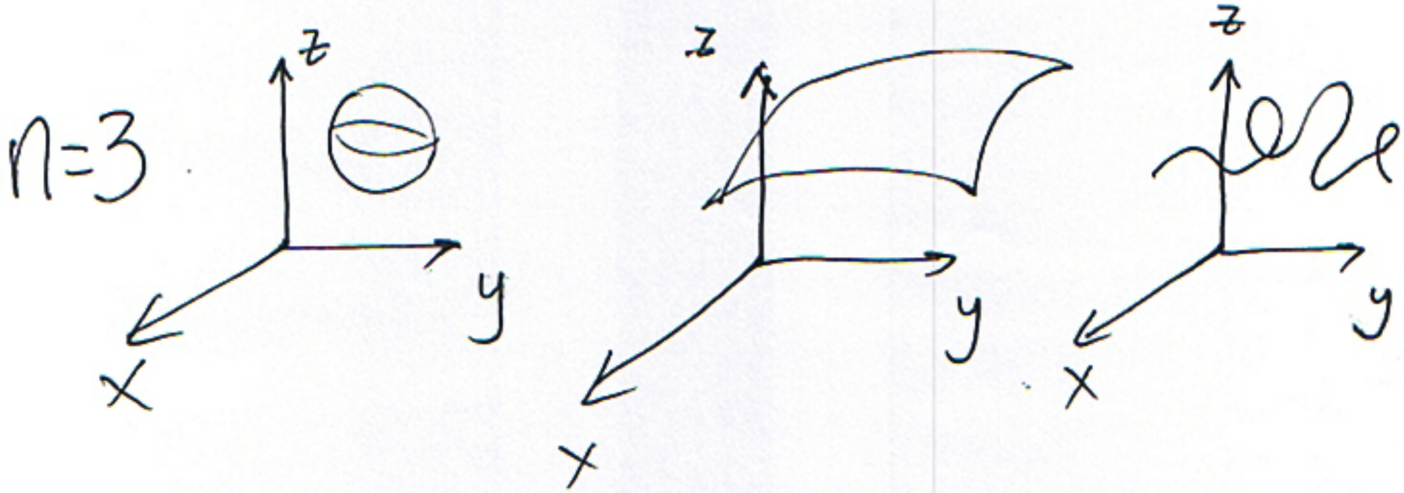
B Borel set

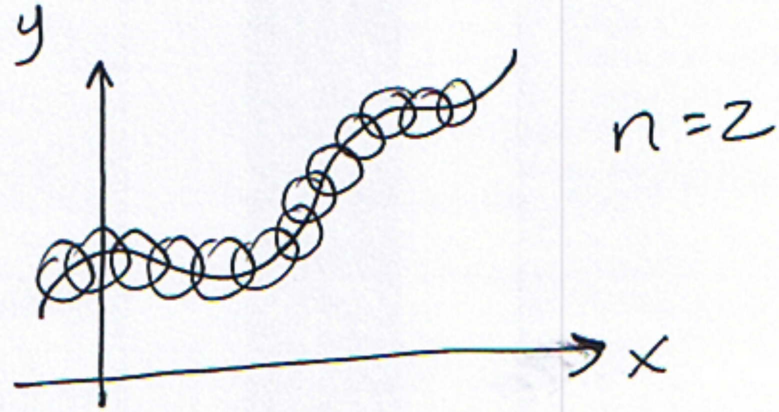
If $B \subset \bigcup_{i=1}^{\infty} V_i$, V_i open, $\varphi(V_i) < \infty$

Then, $\forall \varepsilon > 0$, $\exists\, W \supset B$, W open such that

$$\varphi(W \setminus B) < \varepsilon$$

Hausdorff Measure.





Let $s, \varepsilon > 0$ and $E \subset \mathbb{R}^n$, $0 \leq s \leq n$.

Define:

$$H_\varepsilon^s(E) = \inf\left\{ \sum_{i=1}^{\infty} \alpha(s) 2^{-s} (\operatorname{diam} E_i)^s : E \subset \bigcup_{i=1}^{\infty} E_i, \ \operatorname{diam} E_i < \varepsilon \right\}$$

where,

(10.7)

$$\alpha(s) = \frac{\pi^{s/2}}{\Gamma\left(\frac{s}{2}+1\right)}$$

$$\Gamma(t) = \int_0^\infty e^{-x} x^{t-1}\,dx, \quad 0<t<\infty.$$

Notice that:

$$\varepsilon_1 < \varepsilon_2 \implies H^s_{\varepsilon_1}(E) \geq H^s_{\varepsilon_2}(E)$$

The s-dimensional Hausdorff measure is defined as:

$$H^s(E) := \lim_{\varepsilon \to 0} H^s_\varepsilon(E) = \sup_{\varepsilon > 0} H^s_\varepsilon(E)$$

<u>Note</u>: The previous definition can be made on a metric space X, where,

$$\text{diam } E = \sup_{p,q \in E} \{d(p,q)\}$$

Also, recall that $\text{diam } \bar{E} = \text{diam } E$, so the sets E_i in the covering can be taken to be closed.