Lesson 14

14.1

<u>Def</u>: Suppose $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ are measure spaces. A mapping $f: X \to Y$ is called measurable with respect to $\mathcal{M}$ and $\mathcal{N}$ if:

$$f^{-1}(E) \in \mathcal{M}, \text{ whenever } E \in \mathcal{N}.$$

- If Y is a topological space, then it is always assumed that $\mathcal{N}$ is the σ-algebra of Borel sets. In this case (denoting $\mathcal{N}$ by $\mathcal{B}$):

$$f: (X, \mathcal{M}) \longrightarrow (Y, \mathcal{B}) \text{ is measurable}$$

if: $\quad f^{-1}(E) \in \mathcal{M}$ whenever $E \subset \mathcal{B}$ is Borel.

- Case $Y = \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$

We endow $\overline{\mathbb{R}}$ with the order topology as follows: For each $a \in \mathbb{R}$ let:

$L_a := [-\infty, a)$ and $R_a = (a, \infty]$.

Let $\quad S := \{L_a : a \in \mathbb{R}\} \cup \{R_a : a \in \mathbb{R}\}$

A base for the topology in $\overline{\mathbb{R}}$ is given by:

$$S \cup \{ R_a \cap L_b \quad a, b \in \mathbb{R}, a < b \}$$

<u>Note</u>: the topolgy on $\mathbb{R}$ induced by the order topology on $\overline{\mathbb{R}}$ is precisely the usual topology on $\mathbb{R}$

- Let X be a topological space. Let $f : (X, \mathcal{B}) \longrightarrow (\overline{\mathbb{R}}, \mathcal{B})$ measurable

Then, f is called a <u>Borel measurable function.</u>

- Let $f : (\mathbb{R}^n, \mathcal{M}) \longrightarrow (\overline{\mathbb{R}}, \mathcal{B})$ measurable with $\mathcal{M} = \sigma$- algebra of Lebesgue measurable sets.

Then, f is called a <u>Lebesgue measurable function.</u>

<u>Remark</u>: Let $(X, \mathcal{M}, \mu)$.
Then:

E is measurable $\Longleftrightarrow \chi_E$ is measurable

<u>Remark</u>: Let
$f: (X, \mathcal{M}) \to (Y, \mathcal{B})$ measurable.

Define $\Sigma = \{E : E \subset Y, f^{-1}(E) \in \mathcal{M}\}$

<u>Σ is closed under countable unions:</u>

Let $\{E_i\}$, $E_i \in \Sigma$. Then:

$$f^{-1}\left(\bigcup_{i=1}^{\infty} E_i\right) = \bigcup_{i=1}^{\infty} f^{-1}(E_i)$$

$$E_i \in \Sigma \Rightarrow f^{-1}(E_i) \in \mathcal{M}, \forall i$$

$$\mathcal{M} \ \sigma\text{-algebra} \Rightarrow \bigcup_{i=1}^{\infty} f^{-1}(E_i) \in \mathcal{M}$$

$$\therefore f^{-1}\left(\bigcup_{i=1}^{\infty} E_i\right) \in \mathcal{M}$$

$$\therefore \bigcup_{i=1}^{\infty} E_i \in \Sigma.$$

Σ is closed under complementation. (14.4)

Let $E \in \Sigma$. Then

$$f^{-1}(E^c) = [f^{-1}(E)]^c \in \mathcal{M}$$

$$\therefore \; E^c \in \Sigma.$$

Thus,

$$\boxed{\Sigma \text{ is a } \sigma\text{-algebra}} \quad (*)$$

From $(*)$, the following result follows:

5.1: Let $(X, \mathcal{M}) \xrightarrow{f} (Y, \mathcal{B})$ continuous where $\mathcal{M}$ is a σ-algebra that contains the Borel sets in X. Prove that f is measurable

Thm: Let $f: (X, \mathcal{M}) \to \overline{\mathbb{R}}$. The following are equivalent:

(i) f is measurable

(ii) $\{f > a\} \in \mathcal{M}$ for each $a \in \mathbb{R}$

(vii) $\{f \geq a\} \in \mathcal{M}$ for each $a \in \mathbb{R}$

(iv) $\{f < a\} \in \mathcal{M}$ for each $a \in \mathbb{R}$

(v) $\{f \leq a\} \in \mathcal{M}$ for each $a \in \mathbb{R}$.

<u>Proof</u>:

(i) $\Rightarrow$ (ii)

$$\{f > a\} = f^{-1}(a,\infty) \cup f^{-1}(\infty) \quad \text{--- } (*)$$

Since (a,∞) is open in $\overline{\mathbb{R}}$ (and hence Borel):

$$f^{-1}(a,\infty) \in \mathcal{M}$$

Since $\{\infty\}$ is closed in $\overline{\mathbb{R}}$ (and hence Borel):

$$f^{-1}\{\infty\} \in \mathcal{M}$$

$$\therefore \; f^{-1}(a,\infty) \cup f^{-1}\{\infty\} \in \mathcal{M}$$

From $(*)$:

$$\{f > a\} \in \mathcal{M}.$$

(ii) $\Rightarrow$ (iii)

Note that:

$$\{f \geq a\} = \bigcap_{k=1}^{\infty} \{f > a - \tfrac{1}{k}\}$$

From (ii), $\{f > a - \frac{1}{k}\} \in \mathcal{M}$, $\forall k$.

$$\therefore \; \{f \geq a\} \in \mathcal{M}.$$

(iii) $\Longrightarrow$ (iv)

$$\{f<a\} = \{f \geq a\}^c$$

From (iii), $\{f \geq a\} \in \mathcal{M}$. Since $\mathcal{M}$ is a σ-algebra:

$$\{f \geq a\}^c \in \mathcal{M}.$$

(iv) $\Rightarrow$ (v)

$$\{f \leq a\} = \bigcap_{k=1}^{\infty} \{f < a + \tfrac{1}{k}\} \in \mathcal{M}.$$

(v) $\Rightarrow$ (i)

By Remark (*) with $Y = \overline{\mathbb{R}}$, it is sufficient to show that:

$f^{-1}(U) \in \mathcal{M}$ whenever $U \subset \overline{\mathbb{R}}$ is open.

The open set U can be written as a countable union of elements of the base of the order topology in $\overline{\mathbb{R}}$. Recall that the base of this topology consists of elements of the form:

$$[-\infty, a), \quad (a,b), \quad (a, \infty], \quad a,b \in \mathbb{R}$$

Note:

$$f^{-1}\left(\bigcup_\alpha A_\alpha\right) = \bigcup_\alpha f^{-1}(A_\alpha)$$

(14.7)

since:

$$x \in f^{-1}\left(\bigcup_\alpha A_\alpha\right) \iff f(x) \in \bigcup_\alpha A_\alpha$$

$$\iff f(x) \in A_\alpha, \text{ some } \alpha$$

$$\iff x \in f^{-1}(A_\alpha)$$

$$\iff x \in \bigcup_\alpha A_\alpha$$

Thus, we only need to check:

$$f^{-1}(J) \in \mathcal{M},$$

where J is of the form:

$J_1 = [-\infty, a)$, $J_2 = (a,b)$, or $J_3 = (b,\infty]$,

$a; b \in \mathbb{R}$.

$$\therefore \quad f^{-1}(b,\infty] = \{x : f(x) > b\}$$
$$= \{x : f(x) \leq b\}^c \in \mathcal{M}, \text{ by (v)}$$

$$\therefore \quad f^{-1}(J_3) \in \mathcal{M}.$$

$$\Rightarrow \quad J_1 = \bigcup_{k=1}^{\infty} [-\infty, a_k], \quad a_k < a, \quad a_k \to a$$

$$f^{-1}(J_1) = f^{-1}\left(\bigcup_{k=1}^{\infty}[-\infty, a_k]\right) = \bigcup_{k=1}^{\infty} f^{-1}([-\infty, a_k])$$
$$= \bigcup_{k=1}^{\infty} \{x : f(x) \leq a_k\} \in \mathcal{M}, \text{ by (v)}$$

Finally:

$$
\begin{aligned}
f^{-1}(J_2) &= f^{-1}((a,b)) \\
&= f^{-1}\left([-\infty, b) \cap (a, \infty]\right), \quad a < b \\
&= f^{-1}([-\infty, b)) \cap f^{-1}((a, \infty]) \\
&= f^{-1}(J_1) \cap f^{-1}(J_3) \\
&\in \mathcal{M},
\end{aligned}
$$

where we have used the identity:

$$f^{-1}\left(\bigcap_\alpha A_\alpha\right) = \bigcap_\alpha f^{-1}(A_\alpha)$$

Indeed:

$$
\begin{aligned}
x \in f^{-1}\left(\bigcap_\alpha A_\alpha\right) &\iff f(x) \in \bigcap_\alpha A_\alpha \\
&\iff f(x) \in A_\alpha, \ \forall \alpha \\
&\iff x \in f^{-1}(A_\alpha) \ \forall \alpha \\
&\iff x \in \bigcap_\alpha f^{-1}(A_\alpha) \ \forall \alpha .
\end{aligned}
$$