Lesson 15

We work with the measure space $(X, \mathcal{M}, \mu)$.

<u>Thm 1</u>: A function $f: X \to \overline{\mathbb{R}}$ is measurable if and only if:

(i) $f^{-1}\{-\infty\} \in \mathcal{M}$ and $f^{-1}\{\infty\} \in \mathcal{M}$ and

(ii) $f^{-1}(a,b) \in \mathcal{M}$ for all open intervals $(a,b) \subset \mathbb{R}$

<u>Proof</u>: $\Longrightarrow$

If f is measurable, then (i) and (ii) are satisfied since $\{\infty\}$, $\{-\infty\}$ and (a,b) are Borel subsets of $\overline{\mathbb{R}}$

$\Longleftarrow$ Let $E \subset \overline{\mathbb{R}}$ be a Borel set. We need to show that:

$$f^{-1}(E) \in \mathcal{M}.$$

Since the topology on $\mathbb{R}$ induced by the order topology on $\overline{\mathbb{R}}$ is precisely the usual topology on $\mathbb{R}$ it follows that

$E \cap \mathbb{R}$ is a Borel set in $\mathbb{R}$

Also:

$$f^{-1}(E) = f^{-1}(E \cap \mathbb{R}) \cup f^{-1}(E \cap \{\infty, -\infty\})$$

Clearly, by (i):

$$f^{-1}(E \cap \{\infty, -\infty\}) \in \mathcal{M}.$$

We only need to show:

$$f^{-1}(E \cap \mathbb{R}) \in \mathcal{M}$$

Since any open set $U \subset \mathbb{R}$ can be written as a countable union of disjoint open intervals, we have:

$$\left.f^{-1}(U) = f^{-1}\left(\bigcup_{k=1}^{\infty}(a_k, b_k)\right) = \bigcup_{k=1}^{\infty} f^{-1}(a_k, b_k)\right\} \quad (1)$$

and by (ii), $f^{-1}(U) \in \mathcal{M}$.

We define:

$$\Sigma = \{E : E \subset \mathbb{R} \text{ and } f^{-1}(E) \in \mathcal{M}\}$$

By Remark (*), Σ is a σ-algebra and by (1) we obtain that Σ contains all open sets in $\mathbb{R}$. Thus, Σ contains all Borel sets in $\mathbb{R}$. In particular, $f^{-1}(E \cap \mathbb{R}) \in \mathcal{M}$ ■

<u>Lemma</u> : If f and g are measurable functions, then the following sets are measurable:

(i) $X \cap \{x : f(x) > g(x)\}$,

(ii) $X \cap \{x : f(x) \geq g(x)\}$,

(iii) $X \cap \{x : f(x) = g(x)\}$.

<u>Proof</u>:

(i)

$$\underbrace{\{x : f(x) > g(x)\}}_{A} = \overbrace{\bigcup_{r \in \mathbb{Q}} \{f > r\} \cap \{g < r\}}^{B}$$

We now prove that $A = B$.

Let $x \in A$. Then $\exists\, \hat{r} \in \mathbb{Q}$ such that:

$$g(x) < \hat{r} < f(x)$$

$\therefore \quad x \in \{f > \hat{r}\}$ and $x \in \{g < \hat{r}\}$

$\therefore \quad x \in \{f > \hat{r}\} \cap \{g < \hat{r}\}$

$\therefore \quad x \in B$.

Let $x \in B$

$\Rightarrow \exists\ \hat{r}$ s.t $x \in \{f > \hat{r}\} \cap \{g < \hat{r}\}$

$\Rightarrow \quad f(x) > \hat{r}$ and $g(x) < \hat{r}$

$\Rightarrow \quad g(x) < \hat{r} < f(x)$

$\therefore \quad g(x) < f(x)$

$\therefore \quad x \in A$.

We proved earlier that $\{f > r\} \in \mathcal{M}$ & $\{g < r\} \in \mathcal{M}$. We conclude from $A = B$ that $A \in \mathcal{M}$.

(ii) $\{x : f(x) \geq g(x)\} = \{x : f(x) < g(x)\}^c \in \mathcal{M}$,

by (i)

(iii) $\{x : f(x) = g(x)\} =$

$$\{x : f(x) \geq g(x)\} \cap \{x : f(x) \leq g(x)\} \in \mathcal{M},$$

by (ii).

Given, f, g measurable functions we consider:

$$(f+g)(x) = f(x) + g(x)$$
$$(fg)(x) = f(x)\, g(x),$$

where since $\infty - \infty$ is undefined, we let:

$$(f+g)(x) := \alpha,$$

where α is chosen arbitrarily when $x \in [f^{-1}(\infty) \cap g^{-1}(-\infty)] \cup [f^{-1}(-\infty) \cap g^{-1}(\infty)]$

We have the following:

<u>Thm</u>: Let $f, g : X \longrightarrow \overline{\mathbb{R}}$ be measurable functions, then $f+g$ and fg are measurable.

<u>Proof</u>: We consider first the case when f and g have values in $\mathbb{R}$. (The proof is similar in the general case and is left as an exercise (Exercise 5.2).

Define:

$$F : X \to \mathbb{R} \times \mathbb{R} \quad \text{by}$$
$$F(x) = (f(x), g(x))$$

and

$$G: \mathbb{R} \times \mathbb{R} \to \mathbb{R} \quad \text{by}$$

$$G(x,y) = x + y$$

Then

$$(G \circ F)(x) = f(x) + g(x)$$
$$G \circ F : X \to \mathbb{R}$$

From Thm 1, we only need to show that

$$(G \circ F)^{-1}(J) \in \mathcal{M},$$

for every open interval $J = (a,b)$.

<u>Claim</u>: $(G \circ F)^{-1}(J) = F^{-1}(G^{-1}(J))$

$$\begin{aligned} x \in\ & (G \circ F)^{-1}(J) \\ & \iff (G \circ F)(x) \in J \\ & \iff G(F(x)) \in J \\ & \iff F(x) \in G^{-1}(J) \\ & \iff x \in F^{-1}(G^{-1}(J)) \end{aligned}$$

Let $U := G^{-1}(J)$.

U is open in $\mathbb{R}^2$, since G is continuous.

Since $U \subset \mathbb{R}^2$ is open, it can be written as a countable union of closed intervals.

That is,

$$\mathcal{U} = \bigcup_{k=1}^{\infty} I^k,$$

$$I^k = I_1^k \times I_2^k, \quad I_i^k \text{ open intervals on } \mathbb{R}.$$

Thus,

$$F^{-1}\left(\bigcup_{k=1}^{\infty} I^k\right) = \bigcup_{k=1}^{\infty} F^{-1}(I^k)$$

$$= \bigcup_{k=1}^{\infty} F^{-1}(I_1^k \times I_2^k)$$

$$= \bigcup_{k=1}^{\infty} f^{-1}(I_1^k) \cap g^{-1}(I_2^k)$$

$$\in \mathcal{M},$$

since f, g are measurable. $\boxminus$