5.3 Approximation of Measurable functions.

Simple function on X:

$$f = \sum_{i=1}^{k} a_i \chi_{A_i}$$

$\text{Range}(f) = \{a_1, a_2, \ldots, a_k\}$

Only takes a finite number of values.

<u>Thm</u>: Let $f: X \to \overline{\mathbb{R}}$ be an arbitrary (possibly nonmeasurable) function. Then:

(i) $\exists \{f_i\}$, f_i simple function s.t

$f_i(x) \to f(x) \quad \forall x \in X$

(ii) If $f \geq 0$, $\{f_i\}$ can be chosen such that $f_i \uparrow f$.

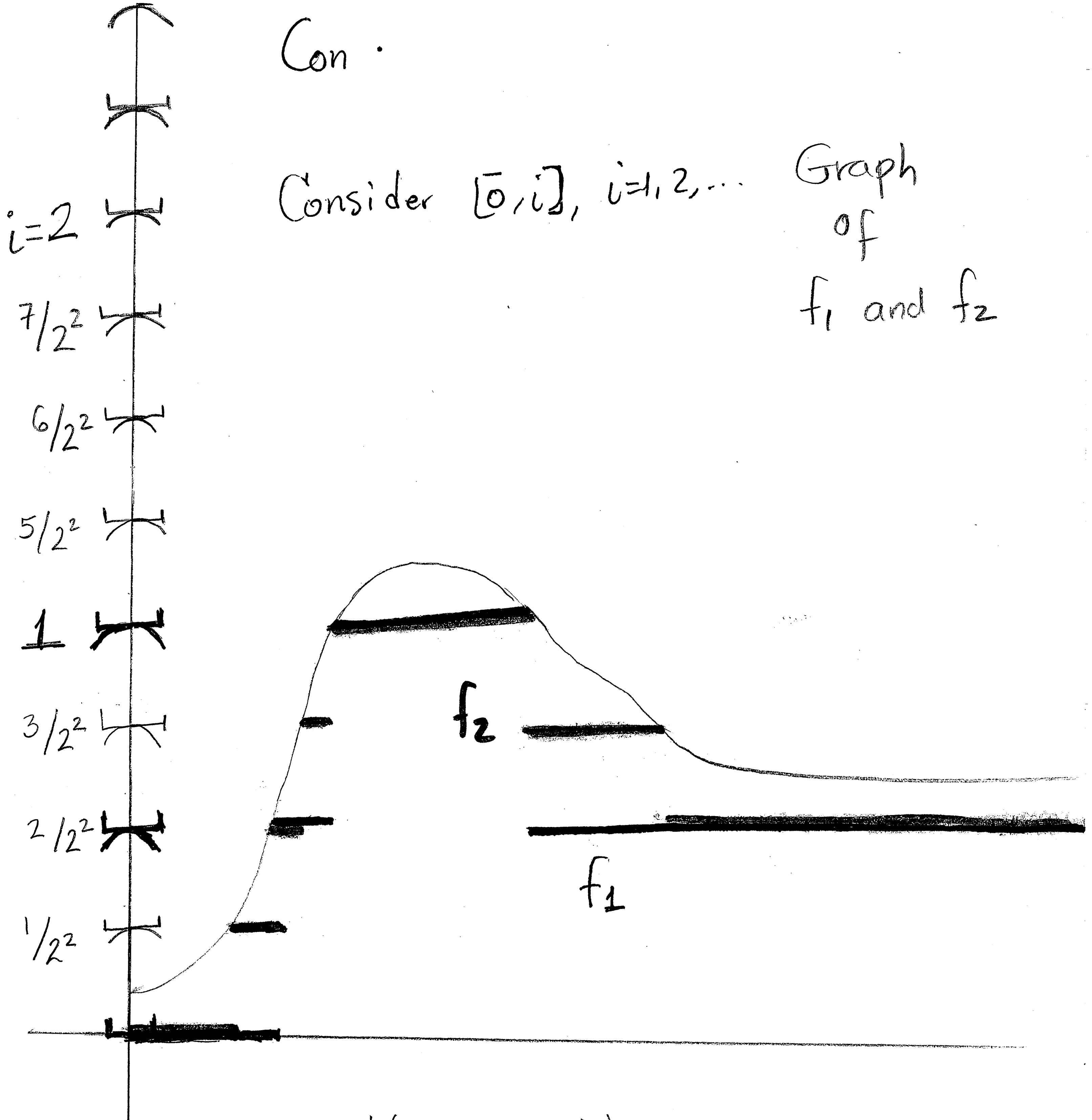
(iii) If f bounded, $\{f_i\}$ can be chosen such that $f_i \to f$ uniformly on X.

(iv) If f is measurable $\Rightarrow \{f_i\}$ can be chosen meas.

Proof: Consider $[0, i]$, $i = 1, 2, \dots$

Con ·

Consider $[0, i]$, $i = 1, 2, \dots$ Graph of f_1 and f_2



$$A_{i,k} = f^{-1}\left(\left[\frac{k-1}{2^i}, \frac{k}{2^i}\right)\right), k = 1, \dots, \frac{i}{1/2^i} \quad A_i = f^{-1}([i, \infty])$$

Define

$$f_i(x) = \begin{cases} \frac{k-1}{2^i}, & x \in f^{-1}\left(\left[\frac{k-1}{2^i}, \frac{k}{2^i}\right)\right) \\ i & x \in f^{-1}([i,\infty]) \end{cases}$$

Note:

$$|f_i(x) - f(x)| < \frac{1}{2^i}, \quad \forall x \in A_i^c$$

and

$$f_1(x) \le f_2(x) \le \cdots \le f(x) \quad \forall x$$

(i) Assume $f(x) \ge 0$.

We need to show

$$f_i(x) \to f(x)$$

Let $x \in f^{-1}(\infty)$. In this case:

$$f_i(x) = i \quad \forall i$$

and

$$f_i(x) = i \to \infty = f(x)$$

Let x s.t. $f(x) < \infty$.

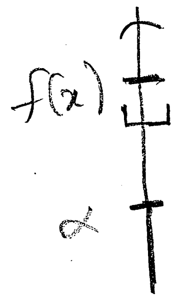
From:

$$f_i(x) \leq f_{i+1}(x) \leq f(x)$$

we obtain.

$$\lim_{i \to \infty} f_i(x) = \sup \{f_i(x)\} = \alpha \leq f(x)$$

If $\alpha < f(x)$,



then $\exists i, k$ s.t $f(x) \in \left[\frac{k-1}{2^i}, \frac{k}{2^i}\right)$, $\alpha < \frac{k-1}{2^i}$

$$\Rightarrow \quad f_i(x) = \frac{k-1}{2^i} > \alpha,$$

which contradicts

$$f_i(x) \leq \sup \{f_i(x)\} = \alpha$$

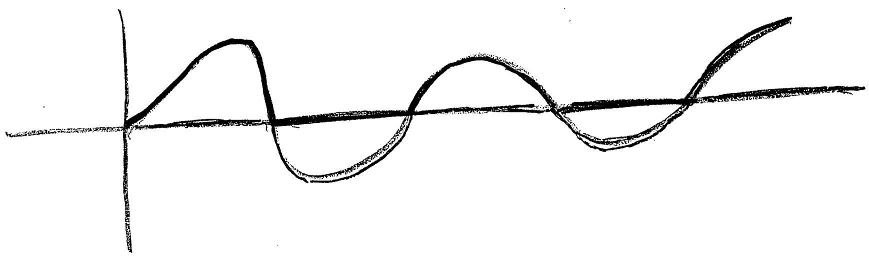
$$\therefore \lim_{i \to \infty} f_i(x) = f(x).$$

In general:

$$f = f^+ - f^-$$

$$f^+(x) = \max(f(x), 0)$$

$$f^-(x) = -\min(f(x, 0))$$



$$f(x) = \lim_{i \to \infty} f_i^+(x) - \lim_{i \to \infty} f_i^-(x)$$

(ii) If $f \geq 0$, then we already showed $f_i \uparrow f$

(iii) Assume now

$$|f(x)| \leq M$$

We first consider the case $f \geq 0$.

Note:

$\exists\, i_0$ s.t $A_i = \emptyset$, $\forall i \geq i_0$

$$\therefore \quad A_i^c = X, \quad \forall i \geq i_0$$

Recall

$$|f_i(x) - f(x)| \le \frac{1}{2^i} \quad \forall x \in A_i^c$$

Let $\varepsilon > 0$

$\exists N, \quad N > i_0$ s.t $\frac{1}{2^N} < \varepsilon$.

Thus $\forall i \ge N, \forall x \in X$.

$$|f_i(x) - f(x)| \le \frac{1}{2^i} < \frac{1}{2^N} < \varepsilon$$

$\therefore \quad |f_i(x) - f(x)| \le \varepsilon \quad \forall i \ge N, \forall x \in X$

$\therefore \quad f_i \to f$ uniformly.

In the general case.

$f_i^+ \to f^+$ unif.

$f_i^- \to f^-$ unif

$\therefore \quad f_i^+ - f_i^- \longrightarrow f^+ - f^-$ unif.

(iv) If f is measurable, then f_i is measurable, since f_i is of the form

$$f_i^+ = \sum_{i=1}^{i2^i} \frac{k-1}{2^i} \chi_{A_{i,k}}$$

$$+ \, i \chi_{A_i}$$

Since each $A_{i,k}$ each measurable, and A_i also is measurable then

f_i^+ is measurable,

f_i^- is measurable

$\Rightarrow$ $f_i = f_i^+ - f_i^-$ is measurable.

Thm (Lusin's Thm).

Suppose $(X, \mathcal{M}, \mu)$ is a measure space where X is a metric space and μ is a finite Borel measure. Let $f: X \to \overline{\mathbb{R}}$ be a measurable function that is finite almost everywhere.

Then:

$\forall \varepsilon > 0 \ \exists F \subset X$, F closed, $\mu(F^c) < \varepsilon$ such that f is continuous on F in the relative topology.

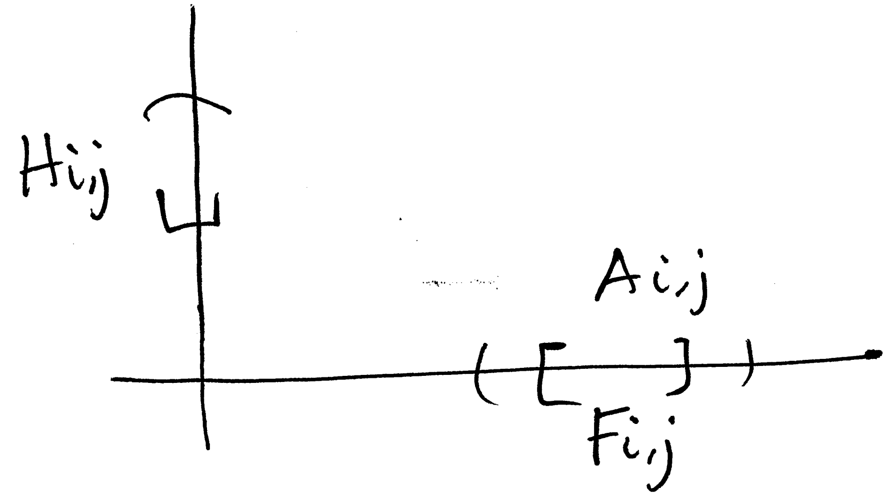
<u>Proof:</u> Let $\varepsilon > 0$.

Fix $i > 0$. Write

$$\mathbb{R} = \bigcup_{j=1}^{\infty} H_{i,j}, \quad H_{ij} \text{ half open intervals whose lengths are } \frac{1}{i}.$$

Def:

$$A_{ij} := f^{-1}(H_{i,j})$$



Thm 115.1 $\Rightarrow$

$\exists\, F_{i,j} \subset A_{i,j}$, $F_{i,j}$ closed such that

$$\mu(A_{i,j} \setminus F_{i,j}) < \frac{\varepsilon}{2^{i+j}}, \quad j = 1, 2, \ldots$$

Define

$$E_k := X \setminus \bigcup_{j=1}^{k} F_{i,j}, \quad k = 1, 2, \ldots$$

$$E_\infty := X \setminus \bigcup_{j=1}^{\infty} F_{i,j}$$

Check ~~that~~ $E_\infty = \bigcap_{k=1}^{\infty} E_k$. (1)

$$\mu(E_\infty) = \mu\left(X \setminus \bigcup_{j=1}^{\infty} F_{i,j}\right) = \sum_{j=1}^{\infty} \mu(A_{i,j} - F_{i,j}) < \frac{\varepsilon}{2^i}$$

From (1); and since $\mu(X) < \infty$

$$\mu(E_\infty) = \lim_{k \to \infty} \mu(E_k), \quad E_1 \supset E_2 \supset E_3 \cdots$$

$$\frac{\varepsilon}{2^i}$$

Thus, $\exists\, J(i)$ such that

$$\boxed{\mu(E_{J(i)}) < \frac{\varepsilon}{2^i}} \quad (2)$$

Select $y_{i,j} \in H_{i,j}$ arbitrary

<u>Def</u>: $B_i = \bigcup_{j=1}^{J(i)} F_{i,j}$.

Define a function g_i on B_i as:

$$g_i(x) = y_{i,j}, \quad \text{whenever } x \in F_{i,j},\ j = 1, \ldots, J(i)$$

The g_i are continuous (relative to B_i) because the closed sets $F_{i,j}$ are disjoints.

<u>Def</u>: $F := \bigcap_{i=1}^{\infty} B_i$

$$
\begin{aligned}
\mu(X \setminus F) &= \mu\left(X \setminus \bigcap_{i=1}^{\infty} B_i\right) \\
&= \mu\left(X \cap \left(\bigcup_{i=1}^{\infty} B_i^c\right)\right) \\
&= \mu\left(\bigcup_{i=1}^{\infty} X \setminus B_i\right) \\
&\le \sum_{i=1}^{\infty} \mu(X \setminus B_i) \\
&< \varepsilon \quad ; \text{ by } (2)
\end{aligned}
$$

So: $\boxed{\mu(X \setminus F) < \varepsilon}$, $\boxed{F \text{ closed}}$

$g_i \to f$ uniformly on F

(because $|f(x) - g_i(x)| < \frac{1}{i}$, $\forall x \in B_i$)

Thus, f is continuous on F (relative to F).