Chapter 6

Integration. Section 6.1

Def. A function $f: X \to \overline{\mathbb{R}}$ is called countably-simple if it assumes only a countable number of values, including possibly $\pm\infty$.

Def. $(X, \mathcal{M}, \mu)$ measure space

$f: X \to \overline{\mathbb{R}}$, $f \geq 0$ measurable countably-simple. Then

$$\int_X f \, d\mu = \sum_{i=1}^{\infty} a_i \, \mu(f^{-1}\{a_i\})$$

Note: $0 \cdot \infty = \infty \cdot 0 = 0$

Recall: $f = f^+ - f^-$

$|f| = f^+ + f^-$

<u>Def</u>: If $\int_X f^+ d\mu < \infty$ or $\int_X f^- d\mu < \infty$, then

$$\int_X f\, d\mu := \int_X f^+ d\mu - \int_X f^- d\mu$$

<u>Def</u>: $f: X \to \overline{\mathbb{R}}$ (not necessarily measurable), we define the upper integral of f by

$$\overline{\int_X} f\, d\mu = \inf \left\{ \int_X g\, d\mu : \ g \text{ meas, count.-simple, } g \geq f \ \mu \text{ a.e} \right\}.$$

and the lower integral by:

$$\underline{\int_X} f\, d\mu := \sup \left\{ \int_X g\, d\mu : \ g \text{ meas., Count-simple, } g \leq f \ \mu\text{-a.e.} \right\}.$$

<u>Def</u>: $f: X \to \overline{\mathbb{R}}$ meas.

If $\overline{\int_X} f\, d\mu = \underline{\int_X} f\, d\mu$, we write

$$\int_X f\,d\mu = \overline{\int_X} f\,d\mu$$

$$= \underline{\int_X} f\,d\mu$$

(In this case we say that the integral exists).

If this value is finite, f is said to be.

integrable, (with respect to μ).

Remark :

- If $f = g$ μ-a.e. then
$$\overline{\int_X} f\,d\mu = \overline{\int_X} g\,d\mu, \quad \underline{\int_X} f\,d\mu = \underline{\int_X} g\,d\mu$$

- If f, g are measurable
  $f = g$ μ-a.e.
  f integrable
  $\Rightarrow g$ integrable and $\int_X f\,d\mu = \int_X g\,d\mu$

- Def: $A \subset X$ (possibly nonmeasurable),
$$\overline{\int_A} f\,d\mu := \overline{\int_X} f\chi_A\,d\mu.$$
$$\underline{\int_A} f\,d\mu := \underline{\int_X} f\chi_A\,d\mu$$

Thm

(i) If f is integrable $\Rightarrow$ f is finite μ-a.e

(ii) f, g integrable, $a, b \in \mathbb{R}$

$\Rightarrow$ $af + bg$ integrable and

$$\int_X (af+bg)\, d\mu = a\int_X f\, d\mu + b\int_X g\, d\mu$$

(iii) f, g integrable

$f \leq g$ μ-a.e

$$\Rightarrow \int_X f\, d\mu \leq \int_X g\, d\mu$$

(iv) f integrable, $E \in \mathcal{M}$ $\Rightarrow$

$f\chi_E$ integrable

(v) f integrable $\Leftrightarrow$ $|f|$ integrable

(vi) f integrable $\Rightarrow$

$$\left|\int_X f\, d\mu\right| \leq \int_X |f|\, d\mu$$

Check first:

Proof: (i) – (vi) are true for countably-simple functions.

(i) If $\int_X f\,d\mu < \infty$ then $\exists g, h$ countably simple functions such that $g(x) \le f(x) \le h(x)$, μ-a.e. x, and:

$$-\infty < \int_X g\,d\mu \le \int_X f\,d\mu \le \int_X h\,d\mu < \infty \quad (1)$$

From (1), $g(x) > -\infty$, $h(x) < \infty$, for μ-a.e. x.

Thus:

$$-\infty < g(x) \le f(x) \le h(x) < \infty \quad \text{for } \mu\text{-a.e. } x.$$

$\therefore$ $-\infty < f(x) < \infty$ for μ-a.e. x

$\therefore$ f is finite μ-a.e.

(ii) Suppose f integrable. Let $c > 0$

Let $g \le f$ μ-a.e., g countably-simple function.

Since $\int_X cg\,d\mu = c\int_X g\,d\mu$ and $cg \le cf$ μ-a.e.

it follows that:

$$\underline{\int}_X cf\,d\mu = c\,\underline{\int}_X f\,d\mu$$

$$\underline{\int}_X cf\,d\mu = \sup\left\{\int_X cg\,d\mu\right\} = c\sup\left\{\int_X g\,d\mu\right\} = c\int_X f\,d\mu$$

Also $\overline{\int}_X cf\,d\mu = c\,\overline{\int}_X f\,d\mu$

If $\underline{c<0}$ we note that:

$-f$ is integrable and $\int_X -f\,d\mu = -\int_X f\,d\mu$.

Indeed:

$$g \le f \le h$$
$$\Rightarrow -h \le -f \le -g$$

$$\therefore \overline{\int_X} -f\,d\mu = \inf\left\{\int_X -g\,d\mu\right\}$$
$$= -\sup\left\{\int_X g\,d\mu\right\}$$
$$= -\int_X f\,d\mu$$

and
$$\underline{\int_X} -f\,d\mu = \sup\left\{\int_X -h\,d\mu\right\}$$
$$= -\inf\left\{\int_X h\,d\mu\right\}$$
$$= -\int_X f\,d\mu$$

$$\therefore \int_X -f\,d\mu = -\int_X f\,d\mu$$

Since $cf = |c|(-f)$ and $|c|(-f)$ is integrable and

$$\int_X cf\,d\mu = |c|\int_X -f\,d\mu = -|c|\int_X f\,d\mu = c\int_X f\,d\mu$$

Suppose now
$$\int_X f\,d\mu < \infty, \quad \int_X g\,d\mu < \infty$$

Let f_1, g_1 be integrable countably-simple such that
$$f_1 \leq f, \quad g_1 \leq g \quad \mu\text{-a.e.}$$

Then:
$$f_1 + g_1 \leq f+g \quad \mu\text{-a.e.}$$

$$\Rightarrow \underline{\int}_X (f+g)\,d\mu \geq \int_X (f_1+g_1)\,d\mu$$
$$= \int_X f_1\,d\mu + \int_X g_1\,d\mu$$

$$\therefore \quad \int_X f_1\,d\mu \leq \underline{\int}_X (f+g)\,d\mu - \int_X g_1\,d\mu$$

Fixing g_1 and taking the sup over all such f_1 we obtain
$$\int_X f\,d\mu \leq \underline{\int}_X (f+g)\,d\mu - \int_X g_1\,d\mu$$

$$\therefore \quad \int_X g_1\,d\mu \leq \underline{\int}_X (f+g)\,d\mu - \int_X f\,d\mu$$

Taking now the sup over all such g_1 we obtain:
$$\int_X g\,d\mu \leq \underline{\int}_X (f+g)\,d\mu - \int_X f\,d\mu$$

Thus: $$\int_X f\,d\mu + \int_X g\,d\mu \leq \underline{\int}_X (f+g)\,d\mu$$

In the same way we prove:

$$\overline{\int_X} (f+g)\,d\mu \le \int_X f\,d\mu + \int_X g\,d\mu .$$

$\therefore$

$$\int_X f\,d\mu + \int_X g\,d\mu \le \underline{\int_X} (f+g)\,d\mu \le \overline{\int_X} (f+g)\,d\mu \le \int_X f\,d\mu + \int_X g\,d\mu$$

$$\therefore \quad \int_X (f+g)\,d\mu = \int_X f\,d\mu + \int_X g\,d\mu$$

(iii) Let f, g integrable
$f \le g$ μ-a.e.

Then (ii) implies:

$g-f$ integrable. Therefore:

$$\int_X (g-f)\,d\mu = \overline{\int_X} (g-f)\,d\mu \ge 0 \,; \quad \text{since } g-f \ge 0 \ \mu\text{-a.e.}$$

Now, by (ii):

$$\int_X g\,d\mu = \int_X f + (g-f)\,d\mu = \int_X f\,d\mu + \int_X (g-f)\,d\mu$$
$$\ge \int_X f\,d\mu$$

$$\therefore \int_X f\,d\mu \le \int_X g\,d\mu .$$

(iv) Let $\int_X f\,d\mu < \infty$.

Let $\varepsilon > 0$.

$\exists$ g, h integrable countably-simple functions such that $g \leq f \leq h$ μ-a.e. and

$$\int_X (h-g)\,d\mu < \varepsilon$$

Since $h-g \geq 0$ μ-a.e. $\Rightarrow$

$$(h-g)\chi_E \leq h-g \quad \mu\text{-a.e.}$$

Since (iv) is true for simple functions, $(h-g)\chi_E$ is integrable.

$$\Rightarrow \int_X (h-g)\chi_E \leq \int_X (h-g) < \varepsilon \,, \quad E \in \mathcal{M}$$

$$\therefore \int_X (h-g)\chi_E\,d\mu < \varepsilon$$

$$\therefore \int_X (h\chi_E - g\chi_E)\,d\mu < \varepsilon$$

Now, $h\chi_E$, $g\chi_E$ are integrable and thus:

$$\therefore \int_X h\chi_E\,d\mu - \int_X g\chi_E\,d\mu < \varepsilon.$$

$$\overline{\int_X} f\chi_E\,d\mu \leq \int_X h\chi_E\,d\mu < \varepsilon + \int_X g\chi_E\,d\mu \qquad (1)$$

$$\leq \varepsilon + \underline{\int_X} \chi_E f\,d\mu$$

(Since $f\chi_E \leq h\chi_E$, $g\chi_E \leq f\chi_E$ μ-a.e.)

(19.6)

Note that

$\underline{\int}_X f\chi_E$ and $\overline{\int}_X f\chi_E$ are finite

Since

$$-\infty < \int_X g\chi_E \leq \underline{\int}_X f\chi_E \,du \leq \overline{\int}_X f\chi_E \,du \leq \int_X h\chi_E \,du < \infty$$

Therefore; from (1):

$$0 \leq \overline{\int}_X f\chi_E \,du - \underline{\int}_X f\chi_E \,du < \varepsilon$$

Since ε is arbitrary we conclude:

$$\overline{\int}_X f\chi_E \,du = \underline{\int}_X f\chi_E \,du < \infty$$

$\therefore$ $f\chi_E$ is integrable.

(v) Let f integrable

By (iv), $f^+ = f\chi_{\{x: f(x)>0\}}$

and $f^- = -f\chi_{\{x: f(x)<0\}}$ are integrable

$\therefore$ $|f| = f^+ + f^-$ is integrable.

Now, if $|f|$ is integrable, then

$f^+ = |f|\chi_{\{x: f(x)>0\}}$ and $f^- = |f|\chi_{\{x: f(x)<0\}}$ are int.

$\therefore$ $f = f^+ - f^-$ is integrable.

(vi) If f is integrable, then by (v), f^+, f^- are integrable and

$$\left|\int_X f\,d\mu\right| = \left|\int_X f^+\,d\mu - \int_X f^-\,d\mu\right|$$

$$\leq \int_X f^+\,d\mu + \int_X f^-\,d\mu = \int_X |f|\,d\mu$$

$$\therefore \left|\int_X f\,d\mu\right| \leq \int_X |f|\,d\mu$$