

Thm: If f is μ -measurable and $f \geq 0$ μ -a.e., then the integral of f exists; i.e.:

$$\int_{-X} f d\mu = \overline{\int}_X f d\mu$$

Proof:

$$\int_{-X} f d\mu = \infty \Rightarrow \overline{\int}_X f d\mu = \infty.$$

Assume $\int_{-X} f d\mu < \infty$ (i.e.

(i.e. $\mu(f^{-1}(\infty)) = 0$).

If $\mu(f^{-1}(\infty)) > 0$, we can define

$$\tilde{g}(x) = \begin{cases} 0 & x \notin f^{-1}(\infty) \\ \infty & x \in f^{-1}(\infty) \end{cases}$$

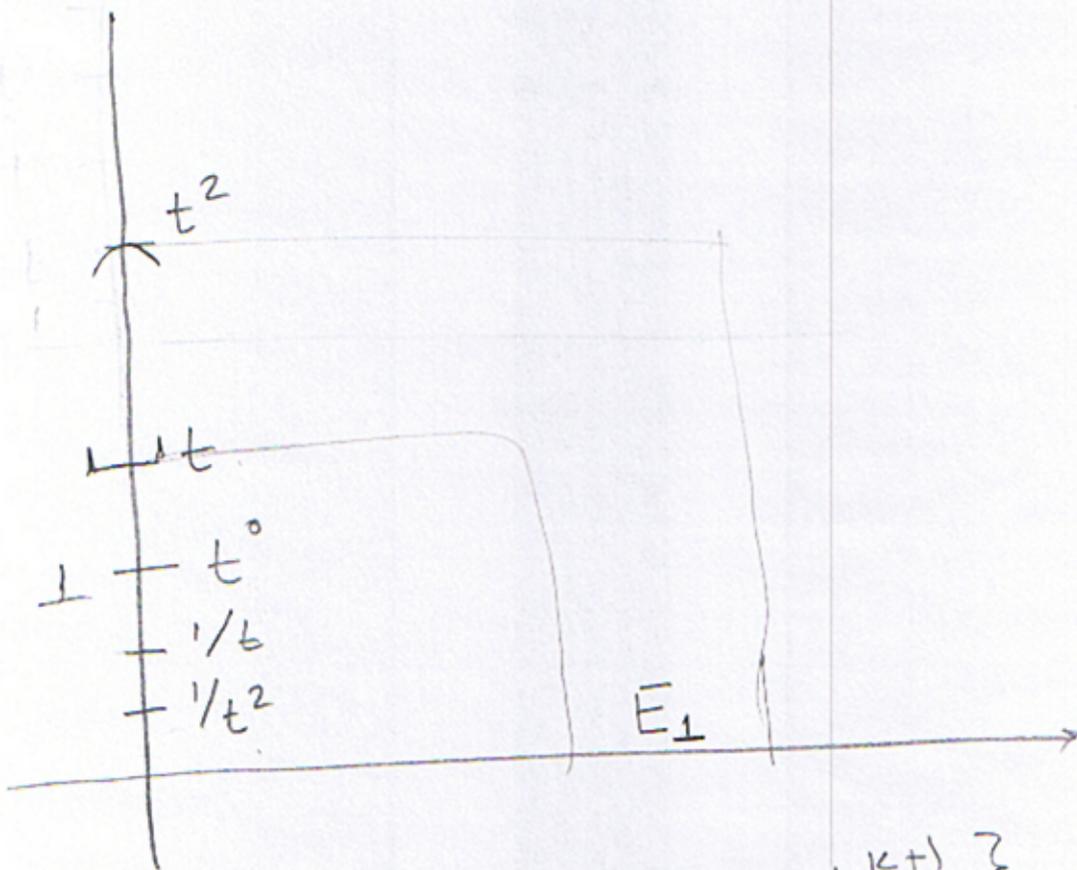
\tilde{g} measurable $\tilde{g} \leq f$ and

hence $\int_{-X} f d\mu = \sup \left\{ \int_X g d\mu : \begin{array}{l} g \text{ meas.} \\ \text{count-simple,} \\ g \leq f \text{ } \mu\text{-a.e.} \end{array} \right\}$

\Rightarrow

$$\int_X f d\mu \geq \int_X \tilde{g} d\mu = \infty$$

$t > 1$



$$E_k = \{x : t^k \leq f(x) < t^{k+1}\}$$

Def :

$$g_t = \sum_{k=-\infty}^{\infty} t^k \chi_{E_k}$$

E_k meas. $\Rightarrow g_t$ meas., countable simple

$$g_t \leq f \leq t g_t \quad \mu\text{-a.e.}$$

$$\begin{aligned} \Rightarrow \int_X g_t d\mu &\leq \int_{-X} f d\mu \leq \int_X f d\mu \\ &\leq \int_X t g_t d\mu \\ &= t \int_X g_t d\mu \\ &\leq t \int_{-X} f d\mu \end{aligned}$$

$$\therefore \int_X f d\mu \leq t \int_{-X} f d\mu$$

$t \rightarrow 1$

$$\int_X f d\mu \leq \int_{-X} f d\mu$$

$$\therefore \int_{-X} f d\mu = \int_X f d\mu$$

Ex: Let $\mu = \lambda$.

$$X = [0, 1]$$

$$f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \notin \mathbb{Q} \end{cases}$$

$$f \geq 0$$

$$\int_{[0,1]} f d\lambda = 1 = \int_{[0,1]} f d\lambda$$

$$\therefore \int_0^1 f d\lambda = 1$$

Riemann integral, f bounded

$$\int_0^1 f dx = \sup L(P, f) = 0$$



$$\int_0^1 f dx = \inf U(P, f) = 1$$

$$U(P, f) = \sum_{i=1}^n M_i (x_i - x_{i-1})$$

$$L(P, f) = \sum_{i=1}^n m_i (x_i - x_{i-1})$$

Upper and lower integral give diff. values

Thm: $f \geq 0$ measurable
 g integrable
 \Rightarrow

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$$\int_X (f+g) d\mu = \int_X (f+g) d\mu = \int_X f d\mu + \int_X g d\mu$$

If f is integrable then Thm 154.5 \Rightarrow
 $f+g$ integrable and

$$\int_X (f+g) d\mu = \int_X f d\mu + \int_X g d\mu$$

Assume then

$$\int_X f d\mu = \infty.$$

Since $\int_X f d\mu = \int_X f d\mu = \infty$ then.

$\exists h_j$ count. simple:

$$0 \leq h_j \leq f$$

$$\int_X h_j d\mu \rightarrow \infty$$

Since g int. $\Rightarrow |g|$ integrable

Let K be countably simple:

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$$K \geq |g|,$$

$$\int_X K \, d\mu \leq \int_X |g| \, d\mu + 1 < \infty$$

Problem 6.1 \Rightarrow

$$\int_X (h_j - K) \, d\mu = \int_X h_j \, d\mu - \int_X K \, d\mu$$

non-neg.
count-
simple

integrable
count-
simple

\Rightarrow

$$\int_X (f+g) \, d\mu \geq \int_X (f - |g|) \, d\mu$$

$$\geq \int_X (h_j - K) \, d\mu$$

$$= \int_X (h_j - K) \, d\mu = \int_X h_j \, d\mu - \int_X K \, d\mu$$

$$\Rightarrow \int_X (f+g) \, d\mu = \infty$$

$$\Rightarrow \int_X (f+g) d\mu = \infty$$

$$\parallel$$

$$\int_X f d\mu + \int_X g d\mu = \infty + \text{finite number.}$$

Corollary: f measurable

f^+ integrable or f^- integrable

\Rightarrow

$$\int_X f d\mu \text{ exists}$$

Proof: Assume f^+ integrable

From previous theorem

$$\int_X (f^- - f^+) d\mu = \int_X (f^- - f^+) d\mu$$

\uparrow non-neg. meas. \uparrow integ.

$$= \int_X f^- - \int_X f^+$$

$\therefore \int_X -f d\mu$ exists. and hence:

$\int_X f d\mu$ exists.

Thm: f μ -measurable
 g integrable
 $|f| \leq |g|$ μ -a.e.
 $\Rightarrow f$ integrable.

Proof: g int. $\Rightarrow |g|$ int.

$|f| \geq 0$ meas.

$$\begin{aligned}\Rightarrow \int_X |f| d\mu &= \overline{\int}_X |f| d\mu \\ &\leq \overline{\int}_X |g| d\mu \\ &= \int_X |g| d\mu < \infty\end{aligned}$$

$\therefore |f|$ integrable.

$\Rightarrow f$ integrable.