Section 6.2.

$(X, \mathcal{M}, \mu)$

Lemma (Fatou's Lemma). If $\{f_k\}_{k=1}^{\infty}$ is a sequence of nonnegative μ-measurable functions, then

$$\int_X \liminf_{k\to\infty} f_k \, d\mu \le \liminf_{k\to\infty} \int_X f_k \, d\mu$$

Proof:

$f_k \ge 0$ μ-measurable

$\Rightarrow \liminf_{k\to\infty} f_k$ is μ-measurable

. Let g any measurable countable-simple function s.t.

$$0 \le g \le \liminf f_k \quad \mu\text{-a.e.}$$

Def:

$$g_k(x) = \inf\{f_m(x) : m \ge k\}$$

$$g_k \le g_{k+1}$$

$$\lim_{k\to\infty} g_k = \liminf_{k\to\infty} f_k$$

Let

$$g = \sum_{j=1}^{\infty} a_j \chi_{A_j}, \quad A_j = g^{-1}(a_j)$$

For $0 < t < 1$, let

$$B_{j,k} = A_j \cap \{x : g_k(x) > t a_j\}$$

$$A_j = \bigcup_{k=1}^{\infty} B_{j,k}.$$

(If $x \in A_j \Rightarrow g(x) = a_j$ but $g_k(x)$ is increasing to $\liminf f_k(x) \geq g(x)$; thus $\exists k$ s.t. $g_k(x) > t a_j$)

$$\lim_{k \to \infty} \mu(B_{j,k}) = \mu(A_j), \quad \forall j$$

Note:

$$\sum_{j=1}^{\infty} t a_j \chi_{B_{j,k}} \leq g_k \leq f_m \quad \forall m \geq k$$

In particular

$$\sum_{j=1}^{\infty} t a_j \chi_{B_{j,k}} \leq f_k \quad k = 1, 2, \ldots$$

$\Rightarrow$

$$\int_X \sum t a_j \chi_{B_{j,K}} \le \underline{\int_X} f_K \, d\mu$$

$$\left(\underline{\int_X} f_K \, d\mu = \overline{\int_X} f_K \, d\mu \right)$$

$\therefore$

$$\sum_{j=1}^{\infty} t a_j \mu(B_{j,K}) \le \int_X f_K \, d\mu \quad \forall K$$

Letting $K \to \infty$ in both sides:

$$\lim_{K \to \infty} \sum_{j=1}^{\infty} t a_j \mu(B_{j,K}) \le \liminf_{K \to \infty} \int_X f_K \, d\mu$$

$$\sum_{j=1}^{\infty} t a_j \mu(A_j) \le \liminf_{K \to \infty} \int_X f_K \, d\mu$$

$$\therefore \quad t \int_X g(x) \, d\mu \le \liminf_{K \to \infty} \int_X f_K \, d\mu$$

Let $t \to 1^-$:

$$\int_X g \, d\mu \le \liminf_{k \to \infty} \int_X f_k \, d\mu$$

This is true for every g countable simple, $0 \le g \le \liminf_{k \to \infty} f_k$

Taking sup over all such g:

$$\int_X \underbrace{\liminf_{k \to \infty} f_k}_{\ge 0} \le \liminf_{k \to \infty} \int_X f_k \, d\mu$$

$$\therefore \int_X \liminf_{k \to \infty} f_k \, d\mu \le \liminf_{k \to \infty} \int_X f_k \, d\mu$$

<u>Thm</u>. (Monotone Convergence Theorem).

If $\{f_k\}_{k=1}^{\infty}$ is a sequence of nonnegative μ-measurable functions such that

$$f_k \le f_{k+1}, \quad k = 1, 2, \ldots$$

$\Rightarrow$

$$\lim_{k\to\infty} \int_X f_k \, d\mu = \int_X \lim_{k\to\infty} f_k \, d\mu.$$

<u>Proof</u>: Let

$$f := \lim_{k\to\infty} f_k = \limsup_{k\to\infty} f_k = \sup\{f_k\}$$

$\therefore$ f is measurable.

$$\therefore \int_X f_k \, d\mu \le \int_X f \, d\mu \qquad k = 1, 2, \ldots \quad (*)$$

(If $\underline{\int_X} f \, d\mu = \overline{\int_X} f d\mu = \int_X f d\mu < \infty$ then $(*)$ is clear. If $\int_X f d\mu < \infty$ the f is integrable and hence $f_k \le f$ is also integrable. Hence $(*)$ follows).

If $\int_X f d\mu = \infty$, clearly $(*)$ is true $\forall k$.

$\Rightarrow$

$$\liminf_{k\to\infty} \int_X f_k \, d\mu \leq \int_X f \, d\mu . \qquad (1)$$

$$\int_X \liminf_{k\to\infty} f_k \, d\mu = \int_X f \, d\mu \qquad (2)$$

$$\leq \liminf_{k\to\infty} \int_X f_k \, d\mu$$

From $(1) + (2)$:

$$\boxed{\int_X f \, d\mu = \liminf_{k\to\infty} \int_X f_k \, d\mu}$$

In a similar way:

$$\limsup_{k\to\infty} \int_X f_k \, d\mu \leq \int_X f \, d\mu \qquad (3)$$

$$\int_X f \, d\mu \leq \liminf_{k\to\infty} \int_X f_k \, d\mu \leq \limsup_{k\to\infty} \int_X f_k \, d\mu \qquad (4)$$

From $(3) + (4)$:

$$\boxed{\int_X f \, d\mu = \limsup_{k\to\infty} \int_X f_k \, d\mu}$$

$\therefore$

$$\int_K f\,d\mu = \lim_{k\to\infty} \int_X f_k\,d\mu.$$

$\boxed{\text{Thm}}$: $f_k \geq 0$ μ-measurable.

$$\int_X \sum_{k=1}^{\infty} f_k\,d\mu = \sum_{k=1}^{\infty} \int_X f_k\,d\mu$$

Let

$$g_n = \sum_{k=1}^{n} f_k.$$

$$g_n \leq g_{n+1} \quad g_n \geq 0.$$

$$g_n \longrightarrow \sum_{k=1}^{\infty} f_k \quad \text{pointwise}.$$

$$\lim_{n\to\infty} \int_X g_n\,d\mu = \int_X \lim_{n\to\infty} g_n\,d\mu$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \int_X f_k\,d\mu = \int_X \sum_{k=1}^{\infty} f_k\,d\mu$$

$$\sum_{k=1}^{\infty} \int_X f_k\,d\mu = \int_X \sum_{k=1}^{\infty} f_k\,d\mu$$

$\boxed{\text{Thm}}$: If f is integrable and $\{E_K\}_{K=1}^{\infty}$ is a sequence of disjoints measurable sets s.t.

$$X = \bigcup_{K=1}^{\infty} E_K,$$

then:

$$\int_X f \, d\mu = \sum_{K=1}^{\infty} \int_{E_K} f \, d\mu$$

Proof: Assume first:

$$f \geq 0.$$

Let $f_K = f \chi_{E_K}$.

$$f_K \geq 0, \quad f_K \text{ measurable}.$$

$$\Rightarrow \int_X \sum_{K=1}^{\infty} f_K = \sum_{K=1}^{\infty} \int_X f_K \, d\mu$$

$$\Rightarrow \int_X \sum_{K=1}^{\infty} f \chi_{E_K} = \sum_{K=1}^{\infty} \int_X f \chi_{E_K} \, d\mu$$

$$\Rightarrow \int_X f\,d\mu = \sum_{k=1}^{\infty} \int_{E_k} f\,d\mu.$$

For f integrable, let $f = f^+ - f^-$

<u>Corollary</u>: $f \geq 0$ integrable, $(X, \mathcal{M}, \mu)$

Def. $\nu(E) = \int_E f\,d\mu$, $E \subset X$, E measurable

$\Rightarrow$ ν is a measure in X.

<u>Proof</u>: Check that:

(i) $\nu(\phi) = 0$ ✓

(ii) $\{E_k\}$ disjoint, E_k measurable

$$\Rightarrow \nu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \nu(E_k)$$

(ii) is true because

$$\nu\left(\bigcup_{k=1}^{\infty} E_k\right) = \int_{\bigcup_{k=1}^{\infty} E_k} f\,d\mu = \sum_{k=1}^{\infty} \int_{E_k} f\,d\mu;$$

by previous theorem.