Lecture 23

Lebesgue Dominated Convergence Thm.

Suppose g is integrable, f is measurable, $\{f_k\}_{k=1}^{\infty}$ is a sequence of μ-measurable functions such that

$$|f_k| \le g \quad \mu\text{-a.e.} \quad k=1,2,\ldots$$

and

$$\lim_{k\to\infty} f_k(x) = f(x) \quad \mu\text{-a.e. } x \in X.$$

Then:

$$\lim_{k\to\infty} \int_X |f_k - f| = 0.$$

(Remark: In particular,

$$\lim_{k\to\infty} \int_X f_k \, d\mu = \int_X f \, d\mu$$

because

$$\left| \int_X f_k \, d\mu - \int_X f \, d\mu \right| = \left| \int_X f_k - f \, d\mu \right|$$

$$\le \int_X |f_k - f| \, d\mu$$

$$\longrightarrow 0$$

Proof:

$$|f_k| \le g \quad \mu\text{-a.e.}$$

and hence, letting $k \to \infty$,

$$|f| \le g \quad \mu\text{-a.e.}$$

Since g is integrable $\Rightarrow$ f_k and f are integrable. Define:

$$h_k := 2g - |f_k - f|$$

$$h_k \ge 0. \quad \mu\text{-a.e.}$$

By ~~Fatou's~~ Lemma:

$$\int_X \liminf_{k\to\infty} h_k \le \liminf_{k\to\infty} \int_X h_k \, d\mu.$$

Note:

$$\liminf_{k\to\infty} h_k = \lim_{k\to\infty} h_k$$

$$= 2g - \lim_{k\to\infty} |f_k - f|$$

$$= 2g \quad , \quad \underline{\mu\text{-a.e}}$$

$\therefore$

$$2\int_X g \, d\mu \le \liminf_{k\to\infty} \int_X h_k \, d\mu = \liminf_{k\to\infty} \int_X 2g - |f_k - f| \, d\mu$$

$$= 2\int_X g \, d\mu - \limsup_{k\to\infty} \int_X |f_k - f| \, d\mu$$

$\therefore$ $\limsup_{k\to\infty} \int_X |f_k - f| \, d\mu = 0$, and thus,

$$\lim_{k\to\infty} \int_X |f_k - f| \, d\mu = 0$$

Lecture 23

Riemann and Lebesgue integration. A comparison.

<u>Def</u>: Let $[a,b]$ be a given interval. By a partition P of $[a,b]$ we mean a finite set of points $x_0, x_1, \dots, x_n$

where
$$a = x_0 \le x_1 \le \dots \le x_{n-1} \le x_n = b.$$

We write:
$$\Delta x_i = x_i - x_{i-1}$$

Let $f: [a,b] \to \mathbb{R}$ bounded, and P a partition of $[a,b]$.

Def:
$$M_i = \sup f(x), \quad x_{i-1} \le x \le x_i$$
$$m_i = \inf f(x), \quad x_{i-1} \le x \le x_i$$

$$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i$$

$$L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i$$

and define:
$$\overline{\int_a^b} f\,dx = \inf U(P,f), \quad \underline{\int_a^b} f\,dx = \sup L(P,f)$$

If the upper and lower integrals are equal, we say that f is Riemann-integrable on $[a,b]$ and we denote the common value as

$$(R) \int_a^b f\,dx .$$

<u>Def</u>: We say that the partition Q is a refinement of P if:

$$P \subset Q ;$$

that is, if every point of P is a point of Q. Given two partitions P_1 and P_2, we say that Q is their common refinement if $Q = P_1 \cup P_2$.

<u>Thm</u>: If Q is a refinement of P, then

$$L(P,f) \leq L(Q,f)$$

$$U(Q,f) \leq U(P,f)$$

<u>Def</u>: Given a partition $P = \{a = x_0, x_1, x_2, \ldots, x_m = b\}$ we define:

$$\|P\| = \max\{x_i - x_{i-1} : 1 \le i \le m\}$$

We can prove the following theorem:

<u>Thm 1</u>: Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Then the following are equivalent:

(a) f is Riemann integrable

(b) $\lim_{\|P\| \to 0} \sum_{i=1}^{m} f(x_i^*)(x_i - x_{i-1}) = (R)\int_a^b f(x)\,dx$,

where x_i^* is an arbitrary point of the interval $[x_{i-1}, x_i]$.

(c) $\forall \varepsilon > 0$, there exists a partition P such that

$$U(P,f) - L(P,f) < \varepsilon$$

(d) $\lim_{\|P\| \to 0} (U(P,f) - L(P,f)) = 0$

(e) $\lim_{\|P\| \to 0} U(P,f) = \lim_{\|P\| \to 0} L(P,f) = (R)\int_a^b f(x)\,dx$

Thm: If $f : [a,b] \to \mathbb{R}$ is a bounded Riemann integrable function then f is Lebesgue integrable and:

$$(R) \int_a^b f(x)\,dx = \int_{[a,b]} f \, d\lambda$$

Proof:

Let $\{P_K\}$ be a sequence of partitions such that $P_K \subset P_{K+1}$ and

$$\|P_K\| \to 0 \quad \text{as } K \to \infty.$$

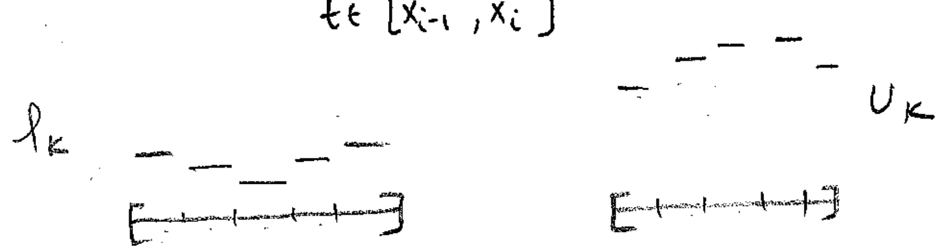
We use the notation:

$$P_K = \{x_0^K, x_1^K, x_2^K, \ldots, x_{m_K}^K\}$$

Define, for each K:

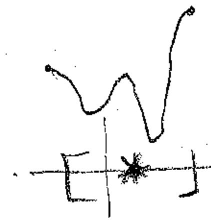
$$l_K(x) = \inf_{t \in [x_{i-1}^K, x_i^K]} f(t), \qquad x \in [x_{i-1}^K, x_i^K)$$

$$u_K(x) = \sup_{t \in [x_{i-1}^K, x_i^K]} f(t), \qquad x \in [x_{i-1}^K, x_i^K)$$



Note:

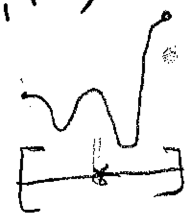
$$\int_{[a,b]} l_k \, d\lambda = L(P_k) \leq U(P_k) = \int_{[a,b]} u_k \, d\lambda \qquad (1)$$



Note that

$$\underline{l_k(x) \uparrow l(x)}, \quad \forall x \in [a,b],$$

for some $l(x)$ since $l_k(x) \leq l_{k+1}(x)$



and:

$$\underline{u_k(x) \downarrow u(x),} \quad \forall x \in [a,b],$$

for some $u(x)$ since $u_k(x) \geq u_{k+1}(x)$

Clearly:

$$l_k(x) \leq f(x) \leq u_k(x) \quad \forall x \in [a,b], \forall k,$$

and thus, letting $k \to \infty$, $l(x) \leq f(x) \leq u(x)$, $\forall x \in [a,b]$

We apply LDCT:

$$\int_{[a,b]} (u - l) \, d\lambda = \lim_{k \to \infty} \int_{[a,b]} (u_k - l_k) \, d\lambda$$

$$= \lim_{k \to \infty} \left(U(P_k) - L(P_k) \right) \text{; by (1) above}$$

$$= 0 \text{ ; by (d) in Thm 1.}$$

$$\therefore \int_{[a,b]} (u - l)\, d\lambda = 0. \qquad (2)$$

From homework Problem # 6.4:

6.4: Suppose f is a nonnegative, integrable function with the property that:

$$\int_X f\, d\mu = 0.$$

Show that $f = 0$ μ-a.e.

Thus, 6.4 and (2) yield:

$$u(x) = l(x) \quad \lambda\text{-a.e. } x$$

But since

$$l(x) \le f(x) \le u(x) \quad \forall x \in [a,b]$$

we conclude:

$$l(x) = f(x) = u(x) \quad \lambda\text{-a.e. on } [a,b].$$

Again using LDCT:

$$\int_{[a,b]} f\, dx = \lim_{k \to \infty} \int_{[a,b]} u_k\, d\lambda = \lim_{k \to \infty} U(P_k)$$

$$= (R) \int_a^b f(x)\, dx.$$