Lesson 24

<u>Thm</u> : A bounded function $f:[a,b]\to\mathbb{R}$ is Riemann integrable if and only if f is continuous λ-a.e. on $[a,b]$.

<u>Proof</u>:

$\Leftarrow$ Assume that f is continuous λ-a.e. on $[a,b]$. Then $\exists\, N\subset[a,b]$, $\lambda(N)=0$ such that f is continuous at each $x\notin N$.

Let $\{P_k\}$ be any sequence of partitions of $[a,b]$ such that $\lim_{k\to\infty}\|P_k\|=0$.

Let $P=\bigcup_{k=1}^{\infty}P_k$, $P_k=\{x_0^k,x_1^k,\dots x_{m_k}^k\}$. Then

$$\lambda(N\cup P)=0$$

Recall the functions u_k, l_k defined in previous theorem:

$$u_k(x)=\sup_{t\in[x_{i-1}^k,x_i^k]} f(t),\quad x\in[x_{i-1}^k,x_i^k),\ 1\le i\le m_k$$

$$l_k(x)=\inf_{x\in[x_{i-1}^k,x_i^k]} f(t),\quad x\in[x_{i-1}^k,x_i^k),\ 1\le i\le m_k.$$

We now *fix* $x \notin NUP$.

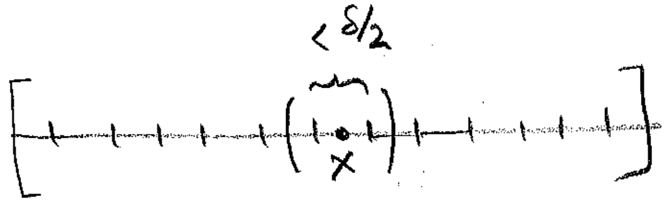
Claim: $\lim\limits_{k \to \infty} \left(u_k(x) - l_k(x) \right) = 0$

Let $\varepsilon > 0$, since f is continuous at x

then $\exists\, \delta > 0$ such that

$$|y - x| < \delta \implies |f(x) - f(y)| < \frac{\varepsilon}{2}, \text{ Le}$$

There exists K_0 such that $\|P_k\| < \frac{\delta}{2}$, $\forall k \geq K_0$.



$K > K_0$

$N_\delta = (x - \delta, x + \delta)$

Let $k > K_0$, then x belongs to some $[x_{i-1}^k, x_i^k)$. Thus:

$$u_k(x) - l_k(x) = \underbrace{\sup_{y \in [x_{i-1}^k, x_i^k]} f(y)}_{= M_i^k} - \underbrace{\inf_{y \in [x_{i-1}^k, x_i^k]} f(y)}_{= m_i^k}$$

$$= M_i^k - m_i^k$$

$$= M_i^k - f(x) + f(x) - m_i^k$$

$$\leq \left| \sup_{y \in N_\delta} f(y) - f(x) \right| + \left| f(x) - \inf_{y \in N_\delta} f(y) \right|$$

Hence

$$u_K(x) - l_K(x) = \left| \lim_{y_i \in N_\delta} f(y_i) - f(x) \right| + \left| f(x) - \lim_{y_j \in N_\delta} f(y_j) \right|$$

$$= \lim_{y_i \in N_\delta} |f(y_i) - f(x)| + \lim_{y_j \in N_\delta} |f(x) - f(y_j)|$$

$$\leq \sup_{y \in N_\delta} |f(y) - f(x)| + \sup_{y \in N_\delta} |f(x) - f(y)|$$

$$= 2 \sup_{y \in N_\delta} |f(y) - f(x)|$$

$$\leq 2\left(\frac{\varepsilon}{2}\right) = \varepsilon. \quad , \quad \forall K \geq K_0$$

thus

$$\lim_{K \to \infty} (u_K(x) - l_K(x)) = 0 \quad , \quad x \notin NUP.$$

By LDCT; since $|u_K - l_K| \leq 2 \sup_{[a,b]} f$,

$$\lim_{K \to \infty} (U(P_K) - L(P_K)) = \lim_{K \to \infty} \int_{[a,b]} (u_K - l_K)\, d\lambda$$

$$= 0$$

$$\therefore \lim_{K \to \infty} (U(P_K) - L(P_K)) = 0$$

$\Rightarrow f$ is Riemann integrable

$\Rightarrow$ Assume now that $f : [a,b] \to \mathbb{R}$ is Riemann integrable.

Let $\{P_K\}$ be a sequence of partitions of $[a,b]$ such that

$$P_K \subset P_{K+1} \quad \text{and}$$

$$\lim_{K \to \infty} \|P_K\| = 0.$$

Let: $$P = \bigcup_{K=1}^{\infty} P_K.$$

Consider the functions u_K, l_K previously defined, and their limits

$$u(x) = \lim_{K \to \infty} u_K(x), \quad l(x) = \lim_{K \to \infty} l_K(x)$$

It was proved in the previous Theorem that $u = l$ λ-a.e. Then, $\exists N$, $\lambda(N) = 0$. Thus $\lambda(N \cup P) = 0$. Fix $x \notin N \cup P$.

<u>Claim</u>: f is continuous at x.

Let $\varepsilon > 0$, then $\exists K$ such that

$$u_K(x) - l_K(x) < \varepsilon.$$

For this K, x belongs to one of the subintervals $[x_{j-1}^K, x_j)$ for some $j \in \{1, 2, \dots, M_K\}$.

Then :

$y \in (x_{j-1}^k, x_j^k)$ implies :

$$f(y) - f(x) \leq \sup_{t \in [x_{j-1}^k, x_j^k]} f(t) - \inf_{t \in [x_{j-1}^k, x_j^k]} f(t)$$

$$= u_k(x) - l_k(x) \; ; \quad \text{since } x \in [x_{j-1}^k, x_j)$$

$$\leq \varepsilon$$

In the same way:

$$f(x) - f(y) \leq u_k(x) - l_k(x) \leq \varepsilon$$

$\therefore \quad |f(x) - f(y)| \leq \varepsilon \quad$ if $\quad |y - x| < \delta$,

where $\quad \delta = \min \{ x - x_{j-1}^k, \; x_j^k - x \}$

$\therefore \quad f$ is continuous at each $x \notin NUP$

$\therefore \quad f$ is continuous at λ-a.e. $x \in [a,b]$

Improper integrals.

<u>Def</u>: $a \in \mathbb{R}$, $f : [a,\infty) \to \mathbb{R}$ be a function that is Riemann integrable on each subinterval of $[a,\infty)$.
The improper integral of f is defined as:

$$(I)\int_a^\infty f(x)\,dx := \lim_{b\to\infty} (R)\int_a^b f(x)\,dx \quad (*)$$

If $(*)$ is finite we say that the improper integral of f exist

We have the theorem:

<u>Thm</u>: Let $f : [a,\infty) \to \mathbb{R}$ be a nonnegative function that is Riemann integrable on any subinterval of $[a,b]$. Then

$$(1)\quad \int_a^\infty f\,d\lambda = \lim_{b\to\infty} (R)\int_a^b f\,dx$$

Thus, f is Lebergue integrable on $[a,\infty)$ if and only if the improper integral $\int_a^\infty f(x)dx$ exists. Moreover, in this case, $\int_a^\infty f(x)\,d\lambda = (I)\int_a^\infty f(x)\,dx$

Proof: Let b_n, $n = 1, 2, 3, \ldots$ by any sequence with $b_n \to \infty$, $b_n > a$. We define

$$f_n = f \chi_{[a,b_n]}, \quad n = 1, 2, 3, \ldots$$

Then the Monotone Convergence Theorem yields:

$$\int_a^\infty f \, d\lambda = \lim_{n\to\infty} \int_a^\infty f_n \, d\lambda \, ; \quad f_n \leq f_{n+1}$$

$$= \lim_{n\to\infty} \int_{[a,b_n]} f \, d\lambda$$

Since f is Riemann integrable on each interval $[a, b_n]$, then Theorem 157.1 yields:

$$\int_{[a,b_n]} f \, d\lambda = (R) \int_a^{b_n} f \, dx$$

and we conclude:

$$\int_a^\infty f \, d\lambda = \lim_{n\to\infty} (R) \int_a^{b_n} f \, dx$$

The second part of the Theorem follows by noticing that both terms in (1) are both finite or ∞ at the same time.

If $f:[a,\infty)\to\mathbb{R}$ takes negative values, then we have:

<u>Thm</u>: Let $f:[a,\infty)\to\mathbb{R}$ be Riemann integrable on every subinterval of $[a,\infty)$. Then f is Lebesgue integrable if and only if the improper integral $(I)\int_a^\infty |f(x)|\,dx$ exists. Moreover, in this case,

$$\int_a^\infty f\,d\lambda = (I)\int_a^\infty f(x)\,dx.$$

<u>Proof</u>: Let $f=f^+-f^-$

f Leb. int on $[a,\infty) \Rightarrow f^+, f^-$ Leb. int. on $[a,\infty)$

Previous Thm $\Rightarrow$

$(I)\int_a^\infty f^+\,dx$ and $(I)\int_a^\infty f^-\,dx$ exist.

Let $\{b_n\}$, $b_n\to\infty$, $b_n>a$. Then; from previous Thm:

(1) $$\int_a^\infty f^+\,d\lambda = (I)\int_a^\infty f^+\,dx = \lim_{n\to\infty}(R)\int_a^{b_n} f^+\,dx$$

(2) $$\int_a^\infty f^-\,d\lambda = (I)\int_a^\infty f^-\,dx = \lim_{n\to\infty}(R)\int_a^{b_n} f^-\,dx.$$

Note:

$$(R)\int_a^{b_n} f\,dx = (R)\int_a^{b_n} f^+ dx - (R)\int_a^{b_n} f^- dx$$

From (1) + (2):

$\lim_{n\to\infty} (R)\int_a^{b_n} f dx$ exists and moreover:

$$\lim_{n\to\infty} (R)\int_a^{b_n} f dx = \lim_{n\to\infty} (R)\int_a^{b_n} f^+ dx - \lim_{n\to\infty} (R)\int_a^{b_n} f^- dx,$$

which implies that the improper integral $\int_a^\infty f dx$ exists and:

$$\int_a^\infty f dx = \int_a^\infty f^+ d\lambda - \int_a^\infty f^- d\lambda = \int_a^\infty f\, d\lambda$$

Analogously,

$$\lim_{n\to\infty} (R)\int_a^{b_n} |f| dx = \lim_{n\to\infty} (R)\int_a^{b_n} f^+ dx + \lim_{n\to\infty} (R)\int_a^{b_n} f^- dx$$

$$< \infty, \quad \text{by (1) and (2)},$$

and thererefore the improper integral $(I)\int_a^\infty |f| dx$ exists and $(I)\int_a^\infty |f| dx = \int_a^\infty f^+ d\lambda + \int_a^\infty f^- d\lambda$

$$= \int_a^\infty |f|\, d\lambda$$

Converse is proved in a similar way.