Lesson 35



Thm: Let $\mu$ and $\nu$ be Radon measures on $\mathbb{R}^n$, $\nu$ Borel regular. Then

$$D_\mu \nu(x) = \lim_{r\to 0} \frac{\nu(B(x,r))}{\mu(B(x,r))} \text{ exists and is}$$

finite for $\mu$-a.e. $x$. Moreover $D_\mu \nu$ is $\mu$-measurable.

Proof: For each $x \in \mathbb{R}^n$ define

$$\overline{D}_\mu \nu(x) = \begin{cases} \limsup\limits_{r\to 0} \dfrac{\nu(B(x,r))}{\mu(B(x,r))} & \text{if } \mu(B(x,r))>0 \text{ for all } r>0 \\ +\infty & \text{if } \mu(B(x,r))=0 \text{ for some } r>0 \end{cases}$$

$$\underline{D}_\mu \nu(x) = \begin{cases} \liminf\limits_{r\to 0} \dfrac{\nu(B(x,r))}{\mu(B(x,r))} & \text{if } \mu(B(x,r))>0 \text{ for all } r>0 \\ +\infty & \text{if } \mu(B(x,r))=0 \text{ for some } r>0. \end{cases}$$

We may assume $\nu(\mathbb{R}^n), \mu(\mathbb{R}^n) < \infty$, as we could otherwise consider $\mu$ and $\nu$ restricted to compact subsets of $\mathbb{R}^n$.

Define:

$I := \{x : \overline{D}_\mu \nu(x) = \infty\}$

Define, for all $0 < a < b$,

$R(a,b) = \{x : \underline{D}_\mu \nu(x) < a < b < \overline{D}_\mu \nu(x) < \infty\}$

Note:

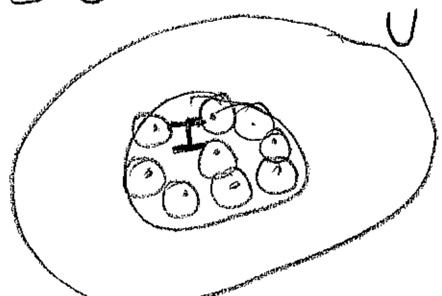
$$\{x : \underline{D}_\mu \nu(x) < \overline{D}_\mu \nu(x) < \infty\} = \bigcup_{\substack{0<a<b \\ a,b \text{ rational}}} R(a,b)$$

Note:

$\forall \alpha > 0$, $I \subset \{x : \overline{D}_\mu \nu(x) \geq \alpha\}$

<u>Claim</u>: $\mu(I) \leq \frac{1}{\alpha} \nu(I)$

Fix $\varepsilon > 0$. Let $U$ be an open set with $I \subset U$




Let $\mathcal{F} = \{ B(x,r),\ x \in I,\ B(x,r) \subset U$, and $\nu(B(x,r)) \geq (\alpha - \varepsilon)\mu(B(x,r)) \}$

A Corollary of Besicovitch's Covering Theorem provide us with a countable collection $\mathcal{G}$ of disjoint balls in $\mathcal{F}$ such that:

$$\mu\Big(I \setminus \bigcup_{B_i \in \mathcal{G}} B_i\Big) = 0$$

Therefore:

$$\mu(I) \leq \sum_{B_i \in \mathcal{G}} \mu(B_i) \leq \frac{1}{\alpha - \varepsilon} \sum_{B_i \in \mathcal{G}} \nu(B_i)$$

$$\leq \frac{1}{\alpha - \varepsilon} \nu(U)$$

$$(*) \therefore \quad \mu(I) \leq \frac{1}{\alpha - \varepsilon} \nu(U) \quad \forall U \text{ open set, } I \subset U$$

Since $\nu$ is Borel regular, one can prove that $\nu(I) = \inf\{\nu(U) : I \subset U,\ U \text{ open}\}$. This and $(*)$ yields:

$$\mu(I) \leq \frac{1}{\alpha - \varepsilon} \nu(I)$$

$\varepsilon$ arbitrary $\Rightarrow$

$\mu(I) \leq \frac{1}{\alpha} \nu(I)$, which proves the claim. Since:

$$\mu(I) \leq \frac{1}{\alpha} \nu(I) \quad \forall \alpha > 0,$$

letting $\alpha \to \infty$ yields $\mu(I) = 0$

Hence

$$\boxed{\overline{D}_\mu \nu(x) < \infty \quad \mu\text{-a.e. } x} \quad (1)$$

In the same way we proved our claim we can show:

$$b\,\mu(R(a,b)) \le \nu(R(a,b)) \le a\,\mu(R(a,b)),$$

and this true for any $0<a<b$.

Therefore, since $b>a$ we obtain,

$$\boxed{\mu(R(a,b))=0} \quad (2)$$

(1) and (2) imply that

$D_\mu \nu(x)$ exists and is finite for $\mu$-a.e. $x$

It can also be proved that $D_\mu \nu$ is $\mu$-measurable.

# Lebesgue Decomposition for Radon measures.

**Theorem**: Suppose $\nu$ is a Radon measure on $\mathbb{R}^n$. Then there exist measures $\mu$ and $\sigma$ such that:

$$\nu = \mu + \sigma, \quad \mu \ll \lambda \quad \text{and} \quad \sigma \perp \lambda$$

**Proof**: We may assume as before that $\nu(\mathbb{R}^n) < \infty$.

Define:

$$\mathcal{E} = \{A \subset \mathbb{R}^n \mid A \text{ is Borel}, \lambda(A^c) = 0\}$$

and choose $B_k \in \mathcal{E}$ such that, for $k = 1, 2, \ldots$

$$\nu(B_k) \leq \inf_{A \in \mathcal{E}} \nu(A) + \frac{1}{k}$$

Write $B = \bigcap_{k=1}^{\infty} B_k$. Since:

$$\lambda(B^c) = \lambda\left(\bigcup_{k=1}^{\infty} B_k^c\right) \leq \sum_{k=1}^{\infty} \lambda(B_k^c) = 0$$

$$\therefore \quad B \in \mathcal{E}, \quad \text{and} \quad \nu(B) = \inf_{A \in \mathcal{E}} \nu(A)$$

**Define**:

$$\mu \equiv \nu \llcorner B$$

$$\sigma = \nu \llcorner B^c$$

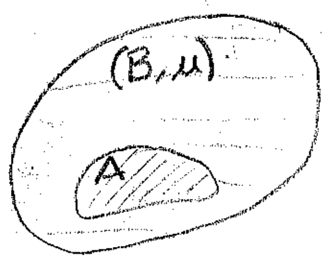
$\mu$ and $\sigma$ are Radon measures. 

Claim: $\mu \ll \lambda$.

Let $A \subset B$, $A$ Borel set such that $\lambda(A) = 0$ and $\mu(A) > 0$. Then

$$B \cap A^c \in \mathcal{E}$$

$(B^c, \sigma)$




$\lambda(B^c) = 0$

and:

$$\nu(B \cap A^c) < \nu(B), \text{ since } \nu(A) > 0$$

But this contradicts:

$$\nu(B) = \inf_{A \in \mathcal{E}} \nu(A)$$

$\therefore$ $\boxed{\mu \ll \lambda}$

Also:
$\sigma(B) = 0 = \lambda(B^c)$ yields that $\sigma \perp \lambda$.

<u>Thm</u> : Suppose $\nu$ is a Radon measure on $\mathbb{R}^n$. Let:

$$\nu = \mu + \sigma$$

be its Lebesgue decomposition, $\mu \ll \lambda$, $\sigma \perp \lambda$.

Let $f = D_\lambda \mu$. Then:

$$\lim_{r\to 0} \frac{\nu(B(x,r))}{\lambda(B(x,r))} = f(x) \text{ for } \lambda\text{-a.e. } x \in \mathbb{R}^n.$$

<u>Proof</u>:

We have shown in a previous theorem that $\sigma \perp \lambda$ implies that:

$$D_\lambda \sigma(x) = 0 \quad \text{for } \lambda\text{-a.e. } x.$$

Thefore:

$$D_\lambda \nu = D_\lambda \mu + D_\lambda \sigma \quad \text{for } \lambda\text{-a.e. } x$$

Hence,

$$D_\lambda \nu(x) = \lim_{r\to 0} \frac{\nu(B(x,r))}{\lambda(B(x,r))}$$

$$= \lim_{r\to 0} \frac{\mu(B(x,r))}{\lambda(B(x,r))} + 0$$

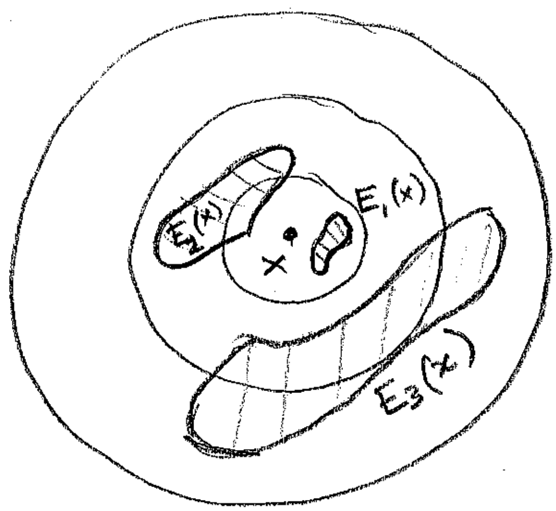
$$= f(x), \quad \text{for } \lambda\text{-a.e. } x$$

**Def**: Let $x \in \mathbb{R}^n$

A sequence of Borel sets $\{E_k(x)\}$ is called a regular differentiation basis at $x$ if $\exists \alpha_x > 0$ such that:

There is a sequence of balls $B(x, r_k)$ with $r_k \longrightarrow 0$ s.t:

$E_k(x) \subset B(x, r_k)$ and $\lambda(E_k(x)) \geq \alpha_x \lambda(B(x, r_k))$




The following theorem shows that we can use the sequence $\{E_k(x)\}$ instead of the sequence of balls $\{B_k(x)\}$

<u>Thm</u>: Let $\nu$ be a Radon measure on $\mathbb{R}^n$. Consider its Lebesgue decomposition:

$$\nu = \mu + \sigma, \quad \mu << \lambda, \quad \sigma \perp \lambda.$$

Let $f = D_\lambda \mu$

Then, for $\lambda$-a.e. $x \in \mathbb{R}^n$,

$$\lim_{k\to\infty} \frac{\sigma(E_k(x))}{\lambda(E_k(x))} = 0 \quad \text{and} \quad \lim_{k\to\infty} \frac{\mu(E_k(x))}{\lambda(E_k(x))} = f(x)$$

for every $\{E_k(x)\}$ regular differentiation basis at $x$.

<u>Proof</u>: For $\lambda$-a.e. $x$:

$$\lim_{k\to\infty} \frac{\sigma(E_k(x))}{\lambda(E_k(x))} = \frac{1}{\alpha_x} \lim_{k\to\infty} \frac{\alpha_x\, \sigma(E_k(x))}{\lambda(E_k(x))}$$

$$\leq \frac{1}{\alpha_k} \lim_{k\to\infty} \frac{\alpha_x\, \sigma(E_k(x))}{\alpha_x\, \lambda(B(x,r_k))}$$

$$\leq \frac{1}{\alpha_k} \lim_{k\to\infty} \frac{\sigma(B(x,r_k))}{\lambda(B(x,r_k))}$$

$= 0$, since $\sigma \perp \lambda \Rightarrow D_\lambda \sigma(x) = 0$ for $\lambda$-a.e. $x$.

Also:

$$\lim_{k\to\infty} \frac{\mu(E_k(x))}{\lambda(E_k(x))} = \lim_{k\to\infty} \frac{\int_{E_k(x)} f(x)\, d\lambda(x)}{\lambda(E_k(x))}$$

$$= \lim_{k\to\infty} \fint_{E_k(x)} f(x)\, d\lambda(x)$$

$$= f(x) \quad \text{for } \lambda\text{-a.e. } x,$$

because

$$\lim_{k\to\infty} \fint_{E_k(x)} |f(y)-f(x)|\, d\lambda(y)$$

$$= \lim_{k\to\infty} \frac{\int_{E_k(x)} |f(y)-f(x)|\, d\lambda(y)}{\lambda(E_k(x))}$$

$$\leq \lim_{k\to\infty} \frac{\int_{B(x,r_k)} |f(y)-f(x)|\, d\lambda(y)}{\alpha_* \lambda(B(x,r_k))}$$

$$= \frac{1}{\alpha_x} \lim_{k\to\infty} \fint_{B(x,r_k)} |f(y)-f(x)|\, d\lambda(y)$$

$= 0$, for $\lambda$-a.e. $x$, since $\lambda$-a.e. $x$ is a Lebesgue point of $f$.

We have proved:

$$\lim_{k\to\infty} \frac{\mu(E_k(x))}{\lambda(E_k(x))} = f(x), \quad \lambda\text{-a.e. } x.$$

Lesson 35



<u>Thm</u>: Let $f$ be a Lebesgue integrable function defined on $[a,b]$. For each $x \in [a,b]$ let:

$$F(x) = \int_a^x f(t)\, d\lambda(t).$$

Then $F'(x) = f(x)$ for $\lambda$-a.e. $x$.

<u>Proof</u>:

For Riemann integrals:

Let $f \in \mathcal{R}$ on $[a,b]$. For $a \le x \le b$, let

$$F(x) = \int_a^x f(t)\, dt$$

Then $F$ is continuous on $[a,b]$, furthermore, if $f$ is continuous at a point $x_0$ of $[a,b]$, then $F$ is differentiable at $x_0$, and

$$F'(x_0) = f(x_0)$$

We have:



$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h \to 0} \frac{\int_a^{x+h} f(t)\, d\lambda(t) - \int_a^{x} f(t)\, d\lambda(t)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\, d\lambda(t) \qquad (*)$$

We define the measure:

$$\mu(E) = \int_E f\, d\lambda, \quad \forall E \text{ Lebesgue measurable, } E \subset [a,b]$$

Note that $\mu$ is a Radon measure and.

$$\mu \ll \lambda.$$

Thus, $f = D_\lambda \mu$ satisfies:

$$f(x) = \lim_{r \to 0} \frac{\mu(B(x,r))}{\lambda(B(x,r))}, \quad \lambda\text{-a.e. } x \in [a,b]$$

Let $\{h_k\}$ be any sequence with $h_k \to 0$.

Define:

$$I_k(x) = [x, x+h_k]$$

Note that $\{I_K(x)\}$ forms a regular differentiation basis at $x$, since for the sequence of open intervals

$$\{B(x, 2h_K)\}$$

we have:

$$I_K(x) \subset B(x, 2h_K) \text{ and } \lambda(I_K(x)) \geq \frac{1}{8} \lambda(B(x, 2h_K))$$

Therefore, the previous theorem yields:

$$\lim_{K \to \infty} \frac{\mu(B(x, 2h_K))}{\lambda(B(x, 2h_K))}$$

$$= \lim_{K \to \infty} \frac{\mu(I_K(x))}{\lambda(I_K(x))} \quad \to (2)$$

$$= f(x), \text{ for } \lambda\text{-a.e. } x.$$

From (2) we obtain:

$$\lim_{K \to \infty} \frac{\int_x^{x+h_K} f(t)\, d\lambda(t)}{h_K} = f(x) \text{ for } \lambda\text{-a.e. } x$$

Since $h_K \to 0$ is arbitrary we conclude $F'(x) = f(x)$ $\lambda$-a.e. $x$.