Lesson 37

Theorem. Let $f: \mathbb{R} \to \mathbb{R}$ be a non-decreasing function. Then $f'(x)$ exists at λ- a.e. $x \in \mathbb{R}$.

Proof:

Since f is nondecreasing, Theorem 61.2 implies that f is continuous except possibly on a countable set:

$$J = \{x_1, x_2, \dots\}.$$

Let $x_i \in J$. Then

$$f(x_i -) < f(x_i +)$$

Define $g: \mathbb{R} \to \mathbb{R}$ as:

$$g(x) = \begin{cases} f(x), & x \notin J \\ f(x_i +), & x \in J. \end{cases}$$

Then:

g is right continuous, non-decreasing and g agrees with f except possibly on the countable set J.

We now consider the Lebesgue-Stieltjes measure associated to g, as defined in Section 4.6.

Define:

$$\mu := \lambda_g^*\big|_{\mathcal{B}}, \quad \mathcal{B} \text{ Borel sets}.$$

Theorem 93.3 gives that λ_g^* is a Carathéodory outer measure on $\mathbb{R}$, and hence all Borel sets are measurable. Thus,

μ is a Borel measure.

Moreover, since g is right-continuous we obtain (see Theorem 94.1)

$$(1) \quad \mu((a,b]) = g(b) - g(a), \quad a < b.$$

Let $x \in \mathbb{R}$.

Let $\{I_k(x)\}$ be a regular differentiation basis at x given by:

$$I_k(x) = (x, x+h_k], \quad h_k > 0, \quad h_k \to 0.$$

Indeed note that:

$$\lambda(I_k(x)) = \tfrac{1}{2}\lambda\,[(x-h_k, x+h_k)]$$

Then, as proven in previous lectures we have the decomposition:

$$\mu = \tilde{\mu} + \sigma, \quad \tilde{\mu} \ll \lambda, \quad \sigma \perp \lambda$$

and:

$$D_\lambda \mu(x) = D_\lambda \tilde{\mu}(x) + D_\lambda \sigma(x), \quad \lambda\text{-a.e. } x$$

$$= D_\lambda \tilde{\mu}(x) + 0, \quad \lambda\text{-a.e. } x, \text{ since } \sigma \perp \lambda$$

Hence, for λ-almost every x, $\exists\, C_x$ s.t:

$$\lim_{h \to 0} \frac{\mu(B(x,h))}{\lambda(B(x,h))} = \lim_{K \to \infty} \frac{\mu(I_K(x))}{\lambda(I_K(x))}$$

$$= \lim_{K \to \infty} \frac{\mu((x, x+h_K])}{\lambda((x, x+h_K])}$$

$$= C_x$$

Since $h_K \to 0$ is arbitrary we conclude:

$$(2) \quad \lim_{h \to 0} \frac{\mu((x, x+h])}{\lambda((x, x+h])} = C_x, \quad \lambda\text{-a.e } x$$

In a similar way we see that:

$$(3) \quad \lim_{h \to 0} \frac{\mu((x-h, x])}{\lambda((x-h, x])} = C_x, \quad \lambda\text{-a.e. } x.$$

From (1), (2) and (3) we have, for λ-a.e. x;

$$(4) \quad \lim_{h\to 0^+} \frac{g(x+h)-g(x)}{h} = C_x = \lim_{h\to 0^+} \frac{g(x)-g(x-h)}{h}$$

From (4) we conclude:

$\boxed{g'(x) \text{ exists for } \lambda\text{-a.e. } x \text{ and } g'(x) = C_x \text{ for such } x. \quad (5)}$

Let
$G = \{x \in \mathbb{R} ;\ g'(x) \text{ exists and } g(x) = f(x)\}$

Then $\lambda(\mathbb{R} \setminus G) = 0$

We now prove:

<u>Claim</u>: If $x \in G \Rightarrow f'(x)$ exists and $f'(x) = g'(x)$.

Let $h_k \to 0$, $h_k > 0$. For each h_k choose:

$0 < h_k' < h_k < h_k''$ such that

$$\frac{h_k''}{h_k} \to 1, \quad \frac{h_k'}{h_k} \to 1. \qquad (6)$$

Then:

$$\frac{h_k'}{h_k}\,\frac{f(x+h_k')-f(x)}{h_k'} \le \frac{f(x+h_k)-f(x)}{h_k} \le \frac{f(x+h_k'')-f(x)}{h_k''}\cdot\frac{h_k''}{h_k}$$

Note that h_k' and h_k'' can be choosen so that

$$f(x+h_k') = g(x+h_k') \text{ and } f(x+h_k'') = g(x+h_k'').$$

Since $f(x) = g(x)$ for $x \in G$, then:

$$\frac{h_k'}{h_k}\,\frac{g(x+h_k')-g(x)}{h_k'} \le \frac{f(x+h_k)-f(x)}{h_k} \le \frac{g(x+h_k'')-g(x)}{h_k''}\cdot\frac{h_k''}{h_k}$$

Letting $h_k \to 0$, from (5) and (6) we obtain

$$\lim_{h_k\to 0^+} \frac{f(x+h_k)-f(x)}{h_k} = g'(x) = C_x \quad (7)$$

The same argument shows:

$$\lim_{h_k\to 0^+} \frac{f(x)-f(x-h_k)}{h_k} = g'(x) = C_x \,. \quad (8)$$

Since (7), (8) is true for any $\{h_k\}$, $h_k \to 0^+$, then:

$$f'(x) = g'(x) = C_x \quad \forall x \in G.$$

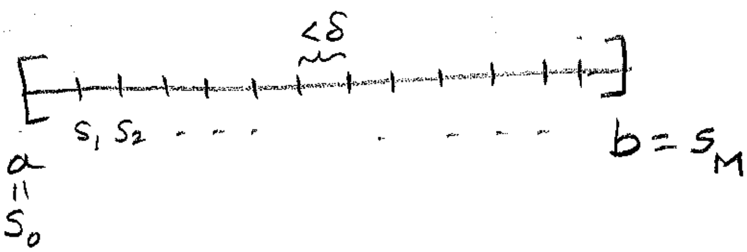
Theorem :

If $f \in AC([a,b])$ then $f \in BV([a,b])$.

Proof : Let $\varepsilon > 0$.

Since f is absolutely continuous, $\exists \delta > 0$ such that $\sum |b_i - a_i| < \delta \Rightarrow \sum |f(b_i) - f(a_i)| < 1$

Subdivide $[a,b]$ in to a finite collection $\mathcal{F}$ of nonoverlapping subintervals $I = [s_{j-1}, s_j]$, $|s_j - s_{j-1}| < \delta$. Let M be the number of those subintervals.



Thus,
$$\sum_{j=1}^{M} |f(s_j) - f(s_{j-1})| < \underbrace{1 + 1 + \cdots + 1}_{M \text{ times}} = M$$

To show that $f \in BV([a,b])$, consider an arbitrary partition

$$a = t_0 < t_1 < \cdots < t_K = b.$$

Since the sum:

$$\sum_{i=1}^{K} |f(t_i) - f(t_{i-1})|$$

is not decreased by adding more points to this partition, we can (if necessary) add more points until each $[t_{i-1}, t_i]$ belongs to one of the subintervals $[s_{j-1}, s_j]$.

For every $j = 1, 2, \dots, M$ let:

$$\alpha_j := \sum_{[t_{i-1}, t_i] \subset [s_{j-1}, s_j]} |f(t_i) - f(t_{i-1})|$$

(the sum is taken over those intervals $[t_{i-1}, t_i]$ that are contained in $[s_{j-1}, s_j]$)

Note that

$$\alpha_j < 1 \quad , \quad \text{since} \quad |s_j - s_{j-1}| < \delta$$

Hence

$$V_f(b) < \sum_{j=1}^{M} \alpha_j < M$$

We conclude $f \in BV([a,b])$.

<u>Theorem</u>: If $f \in AC([a,b])$ with the property that:

$$f' = 0 \quad \text{almost everywhere},$$

then f is constant.

<u>Proof</u>: Let

$$E_f = (a,b) \cap \{x : f'(x) = 0\}.$$

We will use the following:

<u>Lemma</u>: Suppose f is an arbitrary function defined on $[a,b]$. Let

$$E_f := (a,b) \cap \{x : f'(x) \text{ exists and } f'(x) = 0\}$$

Then $\lambda[f(E_f)] = 0$

Note that the hypothesis yields:

$$[a,b] = E_f \cup N, \quad \lambda(N) = 0$$

Therefore:

$$\begin{aligned} \lambda(f([a,b])) &= \lambda(f(E_f \cup N)) \\ &= \lambda(f(E_f) \cup f(N)) \\ &\leq \lambda(f(E_f)) + \lambda(f(N)) \\ &= 0 + \lambda(f(N)); \text{ by Lemma} \\ &= 0 + 0 \quad ; \text{ since } f \text{ satisfies condition } N. \end{aligned}$$

We have shown:

$$\lambda(f[a,b]) = 0$$

Since f is continuous the $f([a,b])$ is an interval and hence $\lambda(f([a,b]))=0$ gives:

$$f([a,b]) = \{c\}, \quad \text{for some constant } c \in \mathbb{R}.$$

$\therefore$ f is constant.