Lesson 9

Existence of Nonmeasurable sets.

(Section 4.5)

<u>Thm</u>. There exits a set $E \subset \mathbb{R}$ that is not Lebesgue measurable.

<u>Proof</u>: Let $x, y \in \mathbb{R}$.

We say

$x \sim y$ if $x - y \in \mathbb{Q}$

$\sim$ is an equivalence relation.

$\Rightarrow$ $\mathbb{R}$ is decomposed is decomposed into disjoint equivalence classes

<u>Def</u>: If $x \in \mathbb{R}$, then E_x denotes the equivalence class that contains x.

- If $x \in \mathbb{Q}$ then $\mathbb{Q} = E_x$
  and $E_x = \{x + r_i : r_i \in \mathbb{Q}\}$
- Each equivalence class is countable and since $\mathbb{R}$ is uncountable, there must be an uncountable number of equivalence classes.

Axiom of Choice implies that:

$\exists S$ such that for each equivalence class E, $S \cap E$ consists precisely of one point

- If $x, y \in S$ then $x - y \notin \mathbb{Q}$, since otherwise they would belong to the same equivalence class contrary to the definition of S

Then
$$D_S := \{x - y : x, y \in S\}$$
is a subset of the irrational numbers and therefore can not contain any interval (*)

Note that

$$\mathbb{R} = \bigcup_{i=1}^{\infty} (S + r_i)$$

where:

$$\mathbb{Q} = \{r_1, r_2, r_3, \dots\}$$

and

$$S + r_i = \{x + r_i : x \in S\}.$$

Indeed,

let $x \in \mathbb{R}$

$\Rightarrow x \in E_x$

$\Rightarrow S \cap E_x = \{z\}$

If $z = x$, then $x \in S + 0$

If $z \neq x$, then

$x - z = r_i$, for some $r_i \in \mathbb{Q}$

$\Rightarrow x = z + r_i$

$\Rightarrow x \in S + r_i$ ∎

We proceed by contradiction and assume S is measurable.

Hence:
$$\lambda\left(\bigcup_{i=1}^{\infty} S+r_i\right) \leq \sum_{i=1}^{\infty} \lambda(S+r_i)$$
$$\lambda\left(\bigcup_{i=1}^{\infty} S+r_i\right) = \lambda(\mathbb{R})$$

Thus, since $\lambda(\mathbb{R}) > 0$ we have:

$$\boxed{\lambda(S) = \lambda(S+r_i) > 0},$$

since otherwise $\lambda(\mathbb{R})$ would be 0.

If $\lambda(S) < \infty$, then Lemma 1 below and (*) give a contradiction. If $\lambda(S) = \infty$, then $0 < \lambda(S \cap I_n) < \infty$ for every $I_n = [-n, n]$ and $S \cap I_n$ is measurable. Since $D_{S\cap I_n} \subset D_S$ does not contain any interval by (*), then Lemma 1 applied to $S \cap I_n$ gives a contradiction.

Lemma 1: If $S \subset \mathbb{R}$ is a Lebesgue measurable set of positive and finite measure, then the set of differences $D_S = \{x-y : x, y \in S\}$ contains an interval.

Proof:

For each $\varepsilon > 0$, there is an open set $U \supset S$ with

$$\lambda(U \setminus S) < \varepsilon \lambda(S); \quad \text{Thm } 91.2$$

i.e $\lambda(U) - \lambda(S) < \varepsilon \lambda(S)$

$$\Rightarrow \quad \lambda(U) < (1+\varepsilon)\lambda(S).$$

Since $U \subset \mathbb{R}$ is open then U is the union of a countable number of disjoint, open intervals,

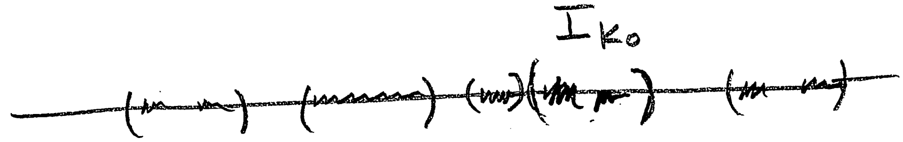
$$U = \bigcup_{k=1}^{\infty} I_k$$

$$\Rightarrow \quad S = \bigcup_{k=1}^{\infty} S \cap I_k \quad \text{and} \quad \lambda(S) = \sum_{k=1}^{\infty} \lambda(S \cap I_k)$$

Since $\lambda(U) < (1+\varepsilon)\lambda(S)$ it follows that, for some k_0,

$$\lambda(I_{k_0}) < (1+\varepsilon)\lambda(S \cap I_{k_0})$$

$$\varepsilon = \frac{1}{3} \quad \Rightarrow \quad \lambda(S \cap I_{k_0}) > \frac{3}{4}\lambda(I_{k_0})$$



Consider

$$S \cap I_{k_0} \quad \text{and} \quad (S \cap I_{k_0}) + t$$

where $0 < |t| < \frac{1}{2}\lambda(I_{k_0})$

$\Rightarrow (S \cap I_{k_0}) \cup ((S \cap I_{k_0}) + t)$ is contained within an interval of length less than $\frac{3}{2}\lambda(I_{k_0})$

If $(S \cap I_{k_0}) \cap [(S \cap I_{k_0}) + t] = \phi$

then

$$\lambda\left[(S \cap I_{k_0}) \cup [(S \cap I_{k_0}) + t]\right] > \frac{3}{4}\lambda(I_{k_0}) + \frac{3}{4}\lambda(I_{k_0}) = \frac{3}{2}\lambda(I_{k_0})$$

which is not possible.

We conclude

$$[S \cap I_{k_0}] \cap [(S \cap I_{k_0}) + t] \neq \phi$$

This means that for each t with

$$|t| < \frac{1}{2}\lambda(I_{k_0})$$

there are points $x, y \in S \cap I_{k_0}$ such that $x - y = t$

Hence:

$$\left(-\frac{1}{2}\lambda(I_{k_0}), \frac{1}{2}\lambda(I_{k_0})\right) \subset \{x-y : x, y \in S \cap I_{k_0}\} \subset D_S .$$