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A bifurcation phenomenon in a singularly perturbed two-phase free boundary problem of phase transition

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ABSTRACT

In this paper, we prove a bifurcation phenomenon in a two-phase, singularly perturbed, free boundary problem of phase transition. We show that the uniqueness of the solution for the two-phase problem breaks down as the boundary data decreases through a threshold value. For boundary values below the threshold, there are at least three solutions, namely, the harmonic solution which is treated as a trivial solution in the absence of a free boundary, a nontrivial minimizer of the functional under consideration, and a third solution of the mountain-pass type. We classify these solutions according to the stability through evolution. The evolution with initial data near a stable solution, such as the trivial harmonic solution or a minimizer of the functional, converges to the stable solution. On the other hand, the evolution deviates away from a non-minimal solution of the free boundary problem.

1. Introduction

In this paper, we study the functional J_ε defined by

$$J_\varepsilon(u) = \int_\Omega \frac{1}{2}|\nabla u|^2\,\Gamma_\varepsilon(u) + \left(\frac{1}{2}|\nabla u|^2 + u\right)\Theta_\varepsilon(u) + q^2(x)\lambda_\varepsilon^2(u)\,dx \tag{1.1}$$

which is a regularized version of the functional

$$\begin{aligned} J(u) = \int_\Omega &\left(\frac{1}{2}|\nabla u(x)|^2 + q(x)\lambda_1^2\right)\chi_{\{u>0\}}(x) \\ &+ \left(\frac{1}{2}|\nabla u(x)|^2 + u(x) + q(x)\lambda_2^2\right)\chi_{\{u\le 0\}}\,dx. \end{aligned} \tag{1.2}$$

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Here, $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain, $q(x) > 0$ is a weight function, $0 < \lambda_1 < \lambda_2$ are constants, and Γ_ε and $\Theta_\varepsilon \in C_0^\infty(\mathbb{R} \to [0,1])$, and $\lambda_\varepsilon \in C_0^\infty(\mathbb{R} \to [\lambda_1, \lambda_2])$ are smooth functions that satisfy

$$\Gamma_\varepsilon(s) + \Theta_\varepsilon(s) = 1 \tag{1.3}$$

$$\Gamma_\varepsilon(s) = \begin{cases} 0 & \text{if } s \leq 0 \\ 1 & \text{if } s \geq \varepsilon \end{cases} \tag{1.4}$$

$$\lambda_\varepsilon(s) = \begin{cases} \lambda_1 & \text{if } s \leq 0 \\ \lambda_2 & \text{if } s \geq \varepsilon. \end{cases} \tag{1.5}$$

We can readily rewrite the functional J_ε in the form

$$J_\varepsilon(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 + u\Theta_\varepsilon(u) + q^2(x)\lambda_\varepsilon^2(u)\,dx. \tag{1.6}$$

Functionals (1.1) and (1.2) originate in fluid mechanics and thermodynamics. The variational problem of minimizing the slightly different functional

$$J(u) = \int_\Omega \frac{1}{2}|\nabla u(x)|^2 + q(x)\lambda_1^2\chi_{\{u>0\}}(x) + q(x)\lambda_2^2\chi_{\{u\leq 0\}}\,dx,$$

is motivated by applications to the flow of two liquids in the modeling of jets and cavities (see for instance [1,2]), leading to a homogeneous Euler equation. Minimizing the functional (1.2) leads to an inhomogeneous problem that reflects a temperature control through the interior (see [3]). Replacing the free-boundary hypersurface with a thin layer of finite width in the variational problems gives rise to regularized functionals such as (1.1), which are efficient in solving problems such as the description of premixed flames for high activation energy in combustion theory (see [4]).

While a minimizer of (1.2) verifies a free boundary problem

$$\begin{cases} \Delta u = 0 & \text{in } \{u > 0\} \\ \Delta u = 1 & \text{in } \{u \leq 0\}^\circ \\ (u_\nu^+)^2 - (u_\nu^-)^2 = Cq^2(x)\left(\lambda_2^2 - \lambda_1^2\right) & \text{on } \partial\{u > 0\} \cap \Omega \end{cases} \tag{1.7}$$

in a weak sense, the Euler equation for (1.1) is given by

$$-\Delta u + \theta(u)u + \Theta_\varepsilon(u) + 2q^2(x)\lambda_\varepsilon(u)\mu_\varepsilon(u) = 0 \quad \text{in } \Omega \tag{1.8}$$

where $\theta(s) = \Theta_\varepsilon'(s)$, and $\mu_\varepsilon(s) = \lambda_\varepsilon'(s)$. We complete this equation into a boundary value problem by imposing

$$u = \sigma \quad \text{on } \partial\Omega \tag{1.9}$$

for a given positive function σ in the Sobolev space $W^{1,2}(\Omega)$ such that $0 < \varepsilon < \inf_{\partial\Omega} \sigma(x)$ and $J_\varepsilon(\sigma) < \infty$.

In this work, we deal with (1.8)–(1.9), a two-phase free boundary problem of phase transition with 'fattened' free boundary, which may be thought of as a smooth approximation of the two-phase problem (1.7). We always require that the parameter ε verifies the condition $0 < \varepsilon < \min_{\partial\Omega} \sigma(x)$. We are interested in the uniqueness/multiplicity of weak solutions to the boundary value problem (1.8), (1.9). This is a continuation of some authors' previous joint work on a bifurcation (or non-uniqueness) phenomenon for the one-phase free-boundary problem, which can be found in [5–7], and the references therein.

Our main results are Theorem 2.2 and Theorem 5.1. They are proved in Sections 2 and 5, respectively. Theorem 2.2 states the existence of an additional critical point of (1.1) when the boundary data decreases through a threshold value. This is a non-uniqueness result describing a bifurcation phenomenon in the sense that there are at least three solutions for boundary values below the threshold. Namely, the harmonic solution, which is treated as a trivial solution in the absence of a free boundary, a nontrivial minimizer of (1.1), and a third solution of mountain-pass type. We classify these solutions according to the stability through evolution. Theorem 5.1 is a convergence result that describes the evolution of a solution of the corresponding parabolic problem to a solution of the elliptic one. The evolution with initial data near a stable solution (such as the trivial harmonic solution or a minimizer of the functional), converges to the stable solution. On the other hand, the evolution deviates away from a non-minimal solution of the free-boundary problem.

Section 3 is devoted to a parabolic comparison principle which plays a key role in the convergence of the evolution. Theorem 4.1 in Section 4 shows uniform Lipschitz continuity of solutions of the free-boundary problem for small $\varepsilon > 0$. Theorem 4.1 is needed in the proof of the convergence of the evolution, although it is a result of independent interest.

We would like to emphasize that there may be more than three solutions in domains with multiple holes. In particular, in a domain with infinitely many holes, there may be infinitely many solutions as long as the holes keep sufficiently away from one another relative to their size. In this sense, the conclusion in Theorem 5.1 is optimal in general.

2. Existence of multiple solutions

In this section, we prove that the problem (1.8)–(1.9) has three weak solutions, a trivial solution, a minimizer of the functional, and a mountain-pass solution for relatively small boundary data. We denote the trivial solution by u_0. It is given by the harmonic function with boundary data σ, i.e., the solution of

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u = \sigma & \text{on } \partial\Omega. \end{cases}$$

We now show a minimizer u_2 of the functional J_ε exists. We include a short proof for the sake of completeness, although the argument is standard. Notice that, in general, a minimizer of J_ε is not unique.

Theorem 2.1. *There exists a minimizer of the functional J_ε.*

Proof. Define

$$\mathcal{A} = \left\{ v \in W^{1,2}(\Omega) \colon v - \sigma \in W_0^{1,2}(\Omega) \right\}$$

and

$$m = \inf_{v \in \mathcal{A}} J_\varepsilon(v).$$

Let $\{u_k\}$ be a minimizing sequence of J_ε. Then it is a bounded sequence in $W^{1,2}(\Omega)$ as Ω has finite measure, and hence Alaoglu's Theorem implies that, for a subsequence, still denoted by $\{u_k\}$ for convenience, there is a certain $u \in W^{1,2}(\Omega)$ with $u - \sigma \in W_0^{1,2}(\Omega)$ such that

1. $\nabla u_k \rightharpoonup \nabla u$ in $L^2(\Omega)$;
2. $u_k \to u$ a.e. in Ω; and
3. $u_k \Theta(u_k) + q^2 \lambda_\varepsilon^2(u_k) \overset{*}{\rightharpoonup} u\Theta(u) + q^2 \lambda_\varepsilon^2(u)$ in $L^\infty_{loc}(\Omega)$.

Consequently, Fatou's lemma leads to

$$J_\varepsilon(u) \le \liminf_{k\to\infty} J_\varepsilon(u_k),$$

and we are done. □

Besides the two solutions u_0 and u_2 for relatively small boundary data, the problem (1.8)–(1.9) has a third weak solution of mountain-pass type, which we denote u_1. In essence, the mountain-pass lemma is a way to produce a saddle-point solution. In general, u_1 tends to be an unstable solution in contrast to the stable solutions u_0 and u_2.

We devote the rest of the section to the proof of the following existence result.

Theorem 2.2. *If $\varepsilon \ll \sigma_m$ and $J_\varepsilon[u_2] < J_\varepsilon[u_0]$, then there is a third weak solution u_1 of the boundary value problem (1.8)–(1.9). Moreover, $J_\varepsilon[u_1] \geq J_\varepsilon[u_0] + a$ for some $a > 0$ which is independent of ε.*

Let $\sigma_M = max_{\partial\Omega}\sigma(x)$ and $\sigma_m = min_{\partial\Omega}\sigma(x)$. If σ_M is small enough, then $u_0 \neq u_2$. In fact, we may pick $u \in H^1(\Omega)$ so that

$$\begin{cases} u = 0 & \text{in } \Omega_\delta \\ u = \sigma & \text{on } \partial\Omega, \text{ and} \\ -\Delta u = 0 & \text{in } \Omega\backslash\overline{\Omega}_\delta, \end{cases}$$

where $\Omega_\delta = \{x \in \Omega : dist(x, \partial\Omega) > \delta\}$ and $\delta > 0$ is a small constant independent of ε and σ so that $\int_{\Omega_\delta} q^2(x)dx > 0$ is also independent of ε and σ. Without loss of generality, we may assume Ω_δ is smooth, since otherwise we may approximate it with a smooth domain in the argument. Then,

$$J_\varepsilon(u_0) = \int_\Omega \frac{1}{2}|\nabla u_0|^2 + q^2(x)\lambda_2^2\, dx \geq \int_\Omega q^2(x)\lambda_2^2\, dx.$$

As $|\nabla u| \leq C\frac{\sigma_M}{\delta}$ in $\Omega\backslash\overline{\Omega}_\delta$ and $u\Theta_\varepsilon(u) \leq \varepsilon$ for $u \geq 0$, we know that

$$\begin{aligned} J_\varepsilon(u) &= \int_\Omega \frac{1}{2}|\nabla u|^2 + u\Theta_\varepsilon(u) + q^2(x)\lambda_\varepsilon^2(u)\, dx \\ &\leq \int_\Omega \frac{1}{2}|\nabla u|^2 + \varepsilon|\Omega| + \int_{\Omega\backslash\Omega_\delta} q^2(x)\lambda_2^2 + \int_{\Omega_\delta} q^2(x)\lambda_1^2 \\ &\leq C\int_{\Omega\backslash\Omega_\delta} \frac{\sigma_M^2}{\delta^2} + \varepsilon|\Omega| + \int_{\Omega\backslash\Omega_\delta} q^2(x)\lambda_2^2 + \int_{\Omega_\delta} q^2(x)\lambda_1^2. \end{aligned}$$

So, for all small $\varepsilon > 0$,

$$\begin{aligned} J_\varepsilon(u) - J_\varepsilon(u_0) &\leq \int_{\Omega\backslash\Omega_\delta} \frac{1}{2}|\nabla u|^2 + \varepsilon|\Omega| - \int_{\Omega_\delta} q^2(x)\left(\lambda_2^2 - \lambda_1^2\right) dx \\ &\leq C\int_{\Omega\backslash\Omega_\delta} \frac{\sigma_M^2}{\delta^2} + \varepsilon|\Omega| - \int_{\Omega_\delta} q^2(x)\left(\lambda_2^2 - \lambda_1^2\right) dx < 0 \end{aligned}$$

if $\sigma_M \leq \sigma_0$ for some $\sigma_0 = \sigma_0(\delta, \Omega, q)$ small enough. In particular, $J_\varepsilon(u_2) \leq J_\varepsilon(u) < J_\varepsilon(u_0)$, and hence $u_2 \neq u_0$.

Take $H = H_0^1(\Omega)$ as the Hilbert space we are going to deal with. For any $v \in H$, we write $u = v + u_0$ and adopt $\|v\|_H = (\int_\Omega |\nabla v|^2)^{\frac{1}{2}} = (\int_\Omega |\nabla u - \nabla u_0|^2)^{\frac{1}{2}}$ as the norm of v. We define the functional

$$\begin{aligned} I_\varepsilon[v] &= J_\varepsilon(u) - J_\varepsilon(u_0) \\ &= \int_\Omega \frac{1}{2}|\nabla u|^2 + u\left(1 - \Gamma_\varepsilon(u)\right) + q^2(x)\left(\lambda_\varepsilon^2(u) - \lambda_2^2\right) - \int_\Omega \frac{1}{2}|\nabla u_0|^2. \end{aligned}$$

We write $v_2 = u_2 - u_0$. Clearly, $I_\varepsilon[0] = 0$ and $I_\varepsilon[v_2] \leq 0$ by the definition of u_2 as a minimizer of J_ε. In the following, we will apply the Mountain Pass Lemma to show that, as long as $I_\varepsilon[v_2] < 0$, there is a critical point of the functional I_ε which is a weak solution of the problem (1.8)–(1.9).

It is not difficult to see that the Fréchet derivative $I'_\varepsilon[v] \in H^{-1}(\Omega)$ of I_ε is given by

$$I'_\varepsilon[v]\varphi = \int_\Omega \nabla u \cdot \nabla\varphi - u\beta_\varepsilon(u)\varphi + \left(1 - \Gamma_\varepsilon(u)\right)\varphi + 2q^2(x)\lambda_\varepsilon(u)\mu(u)\varphi$$

for $\varphi \in H_0^1(\Omega)$, $\beta_\varepsilon(s) = \Gamma'_\varepsilon(s)$, and $\mu(s) = \lambda'_\varepsilon(s)$. Or equivalently,

$$I'_\varepsilon[v] = -\Delta v - (v + u_0)\beta_\varepsilon(v + u_0) + (1 - \Gamma_\varepsilon(v + u_0)) + 2q^2(x)\lambda_\varepsilon(v + u_0)\mu(v + u_0).$$

In addition, we claim that I_ε' is Lipschitz continuous on H with Lipschitz constant depending on ε, u_0, Γ_ε, β_ε, λ_ε, μ and $\sup q^2$. In fact, for v, w and $\varphi \in H$,

$$\begin{aligned}
&I'[v]\varphi - I'[w]\varphi \\
&= \int_\Omega (\nabla v - \nabla w)\cdot\nabla\varphi - ((v+u_0)\beta_\varepsilon(v+u_0) - (w+u_0)\beta_\varepsilon(w+u_0))\,\varphi \\
&- (\Gamma_\varepsilon(v+u_0) - \Gamma_\varepsilon(w+u_0))\,\varphi \\
&+ 2q^2(x)\,(\lambda_\varepsilon(v+u_0)\mu(v+u_0) - \lambda_\varepsilon(w+u_0)\mu(v+u_0))\,\varphi
\end{aligned}$$

and hence by way of Hölder and Poincaré's inequalities it holds that

$$|I'[v]\varphi - I'[w]\varphi| \le C(u_0, \beta_\varepsilon, \Gamma_\varepsilon, \sup q^2, \lambda_\varepsilon, \mu)\|v-w\|_H\|\varphi\|_H.$$

Next we justify the Palais–Smale condition. Since the mapping

$$v \mapsto -(v+u_0)\beta_\varepsilon(v+u_0) + (1-\Gamma_\varepsilon(v+u_0)) + 2q^2(x)\lambda_\varepsilon(v+u_0)\mu(v+u_0)$$

from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$ is compact due to the Rellich–Kondrachov Compactness Theorem as all the three terms are in $H_0^1(\Omega)$, we are allowed to apply Proposition 2.2, [8], to justify the Palais–Smale condition. As a result, we only need to show that any Palais–Smale sequence of I_ε is bounded in H, i.e. any sequence $\{v_k\}$ in H satisfying

$$|I_\varepsilon[v_k]| \le M, \quad \text{and} \quad I_\varepsilon'[v_k] \to 0 \ \text{ as } k \to \infty$$

must be bounded in norm in H. But this is a straightforward fact, since the assumption $I_\varepsilon[v_k] \le M$ readily implies that

$$\int_\Omega |\nabla u|^2 \le C(|\Omega|)\left(1 + M + \int_\Omega q^2(x)\left(\lambda_2^2 - \lambda_1^2\right) + |\nabla u_0|^2\right)$$

due to the following Hölder–Poincaré-Interpolation manipulation on $\tilde{u} = u\,(1-\Gamma(u)) \in H$,

$$\int_\Omega |\tilde{u}| \le \left(\int_\Omega \tilde{u}^2\right)^{1/2} |\Omega|^{1/2} \le C\left(\int_\Omega |\nabla\tilde{u}|^2\right)^{1/2} |\Omega|^{1/2} \le \delta\int_\Omega |\nabla\tilde{u}|^2 + C(\delta, |\Omega|)$$

which controls the second term in I_ε since $\int_\Omega |\nabla\tilde{u}|^2 \le C\int_\Omega |\nabla u|^2$. The Palais–Smale condition is therefore satisfied by the functional I_ε.

As the last step in verifying the conditions in the Mountain Pass Theorem, we show that there is a closed mountain ridge around the origin of H with the energy I_ε as the elevation function, which is stated as the following lemma.

Lemma 2.3. *For all small $\varepsilon > 0$ such that $C\varepsilon \le \frac{1}{2}\sigma_m$ for a large universal constant C, there exist positive constants δ and a independent of ε, such that, for every v in H with $\|v\|_H = \delta$, the inequality $I_\varepsilon[v] \ge a$ holds.*

Proof. It suffices to prove $I_\varepsilon[v] \ge a > 0$ for every $v \in C_0^\infty(\Omega)$ with $\|v\|_H = \delta$ for δ small enough, as, for fixed ε, $I_\varepsilon[v]$ is continuous in v with respect to the H_0^1-norm, and smooth functions are dense in $H_0^1(\Omega)$.

Let $u = v + u_0$ as before and denote $\Lambda = \Lambda_\varepsilon = \{u \le \varepsilon\}$. We claim that $\Lambda = \emptyset$ if δ is small enough. Let $\mathcal{AC}([a,b], S)$, where $S \subseteq \mathbb{R}^n$, be the set of absolutely continuous functions $\gamma : [a,b] \to S$. For each $\gamma \in \mathcal{AC}([a,b], S)$, we define its length to be $L(\gamma) = \int_a^b |\gamma'(t)|dt$. For $x_0 \in \partial\Omega$, we define the distance from x_0 to Λ to be

$$d(x_0, \Lambda) = \inf\{L(\gamma) : \gamma \in \mathcal{AC}([0,1], \overline{\Omega}), \text{ s.t. } \gamma(0) = x_0, \text{ and } \gamma(1) \in \Lambda\}$$

For a minimizing sequence $\{\gamma_k\}$ of the distance $d(x_0, \Lambda)$, we may require $|\gamma_k(t) - \gamma_k(s)| \le M|t-s|$ for all t, $s \in [0,1]$ and for all k by replacement. In fact, each γ_k consists of parts on $\partial\Omega$ and at most countably many non-overlapping pieces in Ω with endpoints on $\partial\Omega$ with the exception of the last endpoint which is in Λ. Every piece in Ω with endpoints on $\partial\Omega$ or Λ can be replaced by a polygonal with desired tolerance. For each replacement of γ_k, still denoted by γ_k, use the arc-length parameter divided by $L(\gamma_k)$ as the parameter t. So t is approximately the arc-length divided by $d(x_0,\Lambda)$. Then on each part of the replacement either lying totally on $\partial\Omega$ or lying in Ω with endpoints on $\partial\Omega$ or Λ, it holds that $|\gamma_k(t)-\gamma_k(s)|\le M|t-s|$ for some M determined by the smoothness of the boundary $\partial\Omega$. The Arzelà–Ascoli theorem implies there is a minimizing path γ for the distance $d(x_0,\Lambda)$.

If the domain Ω is convex, we prove Λ is empty for δ small enough. For any $x_0\in\partial\Omega$, let γ be a minimizing path of $d(x_0,\Lambda)$ if Λ is nonempty. Then it is clear that γ is a straight line segment and $\gamma(t)\notin\Lambda$ for $t\in[0,1)$. Furthermore, for any two distinct points x_1 and $x_2\in\partial\Omega$, the corresponding minimizing paths do not intersect in $\Omega\backslash\Lambda$. For this reason, we can carry out the following computation. Let $\gamma=\gamma_{x_0}$ be the minimizing path with $\gamma(0)=x_0\in\partial\Omega$ and $\gamma(1)\in\Lambda$. Then $v(x_0)=0$ and $v(\gamma(1))=\varepsilon-u_0(\gamma(1))\le\varepsilon-\sigma_m<0$. So the Fundamental Theorem of Calculus

$$v(\gamma(1))-v(\gamma(0))=\int_0^1\nabla v(\gamma(t))\cdot\gamma'(t)dt$$

implies

$$\sigma_m-\varepsilon\le\int_0^1|\nabla v(\gamma(t))||\gamma'(t)|dt.$$

Hence

$$\begin{aligned}
&(\sigma_m-\varepsilon)H^{n-1}(\partial\Omega)\le\int_{\partial\Omega}\int_0^1|\nabla v(\gamma(t))||\gamma'(t)|dtdH^{n-1}(x_0)\\
&\le\int_{\partial\Omega}(\int_0^1|\gamma'(t)|dt)^{\frac12}(\int_0^1|\nabla v(\gamma(t))|^2|\gamma'(t)|dt)^{\frac12}dH^{n-1}(x_0)\\
&=\int_{\partial\Omega}L(\gamma_{x_0})^{\frac12}(\int_0^1|\nabla v(\gamma(t))|^2|\gamma'(t)|dt)^{\frac12}dH^{n-1}(x_0)\\
&\le(\int_{\partial\Omega}L(\gamma_{x_0})dH^{n-1}(x_0))^{\frac12}(\int_{\partial\Omega}\int_0^1|\nabla v(\gamma(t))|^2|\gamma'(t)|dtdH^{n-1}(x_0))^{\frac12}\\
&\le|\Omega_1|^{\frac12}(\int_\Omega|\nabla v|^2dx)^{\frac12}\\
&\le|\{u>\varepsilon\}|^{\frac12}\delta\le|\{u>0\}|^{\frac12}\delta,
\end{aligned}$$

where the second and third inequalities are due to the application of the Hölder's inequality, and Ω_1 in the fourth inequality is a domain filled with the non-intersecting minimizing paths starting on $\partial\Omega$ and ending in Λ and is a subset of the ε-positive domain $\{u>\varepsilon\}$. If we take δ sufficiently small and independent of ε, the measure $|\{u>0\}|$ of the positive domain would be greater than that of Ω, which is impossible. So Λ must be empty. As a result, $I_\varepsilon[v]=\frac12\delta^2>0$.

In case the domain Ω is not convex, the minimizing paths of $d(x_1,\Lambda)$ and $d(x_2,\Lambda)$ for x_1, $x_2\in\partial\Omega$ may partially coincide. However, if Λ is not empty, we may replace it by a convex set or even a ball on which the value of u is less than $C\varepsilon<\sigma_m$, for ε small. We still denote this new set as Λ. On the other hand, we form the set $\mathcal{DA}(\partial\Omega)$ of the points x_0 on $\partial\Omega$ so that a minimizing path γ of $d(x_0,\Lambda)$ satisfies $\gamma(t)\in\Omega\backslash\Lambda$ for $t\in(0,1)$. We call a point in $\mathcal{DA}(\partial\Omega)$ a **directly accessible** boundary point. Let Ω_1 be the union of these minimizing paths for the directly accessible boundary points. It is not difficult to see that $|\Omega_1|>0$ and hence $H^{n-1}(\mathcal{DA}(\partial\Omega))>0$. Then we may apply the above computation for a convex set to the directly accessible set $\mathcal{DA}(\partial\Omega)$. We will have

$$(\sigma_m-C\varepsilon)H^{n-1}(\mathcal{DA}(\partial\Omega))\le|\Omega\backslash\Lambda|^{\frac12}\delta\le|\Omega|^{\frac12}\delta.$$

For small enough δ, this raises a contradiction $|\Omega| > |\Omega|$. So $u(x) > \varepsilon$ everywhere and hence

$$I_\varepsilon[v] = \int_\Omega \frac{1}{2}|\nabla u|^2 + u\Theta_\varepsilon(u) + q^2(x)\left(\lambda_\varepsilon^2 - \lambda_2^2\right) - \frac{1}{2}|\nabla u_0|^2 = \frac{1}{2}\delta^2 > 0. \quad \square$$

Let

$$\mathcal{G} = \{\gamma \in C([0,1], H) : \gamma(0) = 0 \text{ and } \gamma(1) = v_2\}$$

and

$$c = \inf_{\gamma\in\mathcal{G}} \max_{0\le t\le 1} I_\varepsilon(\gamma(t)).$$

The verified Palais–Smale condition and the preceding lemma allow us to apply the Mountain Pass Theorem as stated, for example, in [8] to conclude that there is a $v_1 \in H$ such that $I_\varepsilon[v_1] = c$, and $I'_\varepsilon[v_1] = 0$ in $H^{-1}(\Omega)$. That is

$$\int_\Omega \nabla u_1 \cdot \nabla\varphi - u_1\beta_\varepsilon(u_1)\varphi + (1 - \Gamma_\varepsilon(u_1))\,\varphi + 2q^2(x)\lambda_\varepsilon(u_1)\mu(u_1)\varphi dx = 0$$

for any $\varphi \in H = H_0^1(\Omega)$, where $u_1 = v_1 + u_0$. So u_1 is a weak solution of the problem (1.8) and (1.9), which concludes the proof of Theorem 2.2.

3. A parabolic comparison principle

In this section we prove a parabolic comparison principle that we will use frequently in the sequel. The proof follows the ideas in [5], where a parabolic comparison principle is proved in a similar setting without the term $\Theta(w)$. We prove our comparison theorem for the following, slightly more general problem,

$$\begin{cases} w_t - \Delta w + \alpha(x,w) = 0 & \text{in } \mathcal{D} = \Omega\times(0,+\infty) \\ w(x,t) = \sigma(x) & \text{on } \partial\Omega\times(0,+\infty) \\ w(x,0) = v_0(x) & \text{for } x\in\overline{\Omega}, \end{cases} \tag{3.1}$$

where α is a smooth function in w satisfying $|\alpha_w(x,w)| \le K$ for all x under consideration.

Theorem 3.1. *Suppose w_1 and w_2 are viscosity sub- and super-solutions of the evolutionary problem (3.1) respectively with $w_1 \le w_2$ on the parabolic boundary $(\overline{\Omega}\times\{0\})\cup(\partial\Omega\times(0,+\infty))$. Then $w_1 \le w_2$ in $\mathcal{D}$.*

Remark 3.2. The comparison principle 3.1 holds also for weak sub- and super-solutions. The proof is in spirit parallel to the following viscosity version. So we omit it.

We adopt the notations $\mathbb{R}_T = [0,T]$ and

$$Hw = w_t - \Delta w + \alpha(x,w). \tag{3.2}$$

Lemma 3.3. *For $T > 0$ small enough, if $Hw_1 \le 0 \le Hw_2$ in $\Omega\times\mathbb{R}_T$ in the viscosity sense and $w_1 < w_2$ on $\partial_p(\Omega\times\mathbb{R}_T)$, then $w_1 \le w_2$ in $\Omega\times\mathbb{R}_T$.*

Proof. For any given small number $\delta > 0$, we define a new function $\tilde{w}_1$ by

$$\tilde{w}_1(x,t) = w_1(x,t) - \frac{\delta}{T-t},$$

where $x \in \overline{\Omega}$ and $0 \le t < T$. In order to prove $w_1 \le w_2$ in $\Omega\times\mathbb{R}_T$, it suffices to prove $\tilde{w}_1 \le w_2$ in $\Omega\times\mathbb{R}_T$ for all small $\delta > 0$. Clearly, $\tilde{w}_1 < w_2$ on $\partial_p(\Omega\times\mathbb{R}_T)$, and $\lim_{t\to T}\tilde{w}_1(x,t) = -\infty$ uniformly on Ω.

Moreover,

$$\begin{aligned}
H\tilde{w}_1 &= w_{1,t} - \frac{\delta}{(T-t)^2} - \Delta w_1 + \alpha(x, w_1 - \frac{\delta}{T-t}) \\
&= Hw_1 - \frac{\delta}{(T-t)^2} + \alpha(x, w_1 - \frac{\delta}{T-t}) - \alpha(x, w_1) \\
&\le Hw_1 - \frac{\delta}{(T-t)^2} + K\frac{\delta}{T-t}, \text{ since } |\alpha_w(x,w)| \le K \\
&\le Hw_1 - \frac{\delta}{(T-t)^2} + \frac{\delta}{2(T-t)^2}, \text{ for } T \le \frac{1}{2K} \text{ so that } K \le \frac{1}{2(T-t)}. \\
&= Hw_1 - \frac{\delta}{2(T-t)^2} \le -\frac{\delta}{2(T-t)^2} \\
&\le -\frac{\delta}{2T^2} < 0.
\end{aligned}$$

The above differential equalities and inequalities are all in the viscosity sense. Every step can be made rigorous in the viscosity sense. We leave the work to the reader. For convenience, we denote $\tilde{w}_1$ by w_1 in the following.

Define, for $j = 1, 2$, $v_j(x,t) = e^{-\lambda t} w_j(x,t)$, where $\lambda > 2K$. So $w_j(x,t) = e^{\lambda t} v_j(x,t)$.

Obviously, $w_1 \le w_2$ in $\Omega \times \mathbb{R}_T$ is equivalent to $v_1 \le v_2$ in $\Omega \times \mathbb{R}_T$. A simple computation shows that in the viscosity sense, $Hw_j = e^{\lambda t}\tilde{H} v_j$, where

$$\tilde{H}v = v_t - \Delta v + e^{-\lambda t}\alpha(x, e^{\lambda t} v) + \lambda v.$$

Then, in the viscosity sense, $\tilde{H}v_1 \le -\frac{\delta}{2T^2}e^{-\lambda t} \le -\frac{\delta}{2T^2}e^{-\lambda T} < 0$ and $\tilde{H}v_2 \ge 0$. Furthermore, $v_1 < v_2$ on $\partial_p(\Omega \times \mathbb{R}_T)$, and $\lim_{t\to T-} v_1(x,t) = -\infty$ uniformly on $\overline{\Omega}$.

Suppose $\sup_{\Omega\times\mathbb{R}_T}(v_1 - v_2) > 0$. Then $\sup_{\Omega\times\mathbb{R}_T}(v_1 - v_2)$ is a maximum and is assumed exclusively in $\Omega \times (0,T)$, due to the last two conditions on v_1 and v_2.

Let

$$M_0 = \sup_{\Omega\times\mathbb{R}_T}(v_1 - v_2) = \max_{\Omega\times\mathbb{R}_T}(v_1 - v_2).$$

For any small $\varepsilon > 0$, we define the penalized function

$$u^\varepsilon(x,y,t) = v_1(x,t) - v_2(y,t) - \frac{1}{2\varepsilon}|x-y|^2, \quad x, y \in \overline{\Omega}, t \in [0,T).$$

We observe first that $\max_{\overline{\Omega}\times\overline{\Omega}\times[0,T)} u^\varepsilon(x,y,t)$ exists as $\lim_{t\to T} v_1(x,t) = -\infty$ uniformly on $\overline{\Omega}$.

Let $M_\varepsilon = u^\varepsilon(x^\varepsilon, y^\varepsilon, t^\varepsilon) = \max_{\overline{\Omega}\times\overline{\Omega}\times[0,T)} u^\varepsilon$, where x^ε, $y^\varepsilon \in \overline{\Omega}$ and $t^\varepsilon \in [0, T') \subset [0,T)$ for some $T' < T$ independent of ε. Clearly, $M_\varepsilon \ge M_0 > 0$. According to Proposition 3.7 in [9], a generalization of Lemma 3.1 in the same manuscript, the conditions $\lim_{\varepsilon\downarrow 0} M_\varepsilon = M_0$ and $\lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon}|x^\varepsilon - y^\varepsilon|^2 = 0$ hold.

We claim that x^ε, $y^\varepsilon \in \Omega$ and $t^\varepsilon > 0$ for all sufficiently small ε.

Suppose not. There exists a sequence $\varepsilon_j \to 0$ such that either $(x^{\varepsilon_j}, t^{\varepsilon_j}) \in \partial_p(\Omega \times \mathbb{R}_T)$ or $(y^{\varepsilon_j}, t^{\varepsilon_j}) \in \partial_p(\Omega \times \mathbb{R}_T)$, and without loss of generality $\{x^{\varepsilon_j}\}$, $\{y^{\varepsilon_j}\}$, $\{t^{\varepsilon_j}\}$ converge. As $\frac{1}{2\varepsilon_j}|x^{\varepsilon_j} - y^{\varepsilon_j}|^2 \to 0$ implies $|x^{\varepsilon_j} - y^{\varepsilon_j}| \to 0$, we may assume $x^{\varepsilon_j} \to x_0$, $y^{\varepsilon_j} \to x_0$, $t^{\varepsilon_j} \to t_0$, where $(x_0, t_0) \in \partial_p(\Omega \times \mathbb{R}_T)$, and $t_0 \le T' < T$. So

$$0 < M_0 \le \limsup_j M_{\varepsilon_j} = v_1(x_0, t_0) - v_2(x_0, t_0) < 0$$

as $(x_0, t_0) \in \partial_p(\Omega \times \mathbb{R}_T)$, which is an obvious contradiction.

For any small $\varepsilon > 0$, Theorem 8.3 in [9] implies that there exist X, $Y \in \mathcal{S}_{n\times n}$, and $b \in \mathbb{R}$ such that $(b, \frac{x^\varepsilon - y^\varepsilon}{\varepsilon}, X) \in \bar{\mathcal{P}}^{2,+} v_1(x^\varepsilon, t^\varepsilon)$, $(b, \frac{x^\varepsilon - y^\varepsilon}{\varepsilon}, Y) \in \bar{\mathcal{P}}^{2,-} v_2(y^\varepsilon, t^\varepsilon)$, and

$$-\frac{3}{\varepsilon}I \le \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \le \frac{3}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

The last inequality implies that $X \leq Y$, while the first two inclusion conditions imply that

$$b - Tr(X) + \lambda v_1(x^\varepsilon, t^\varepsilon) + e^{-\lambda t^\varepsilon} \alpha(x^\varepsilon, e^{\lambda t^\varepsilon} v_1(x^\varepsilon, t^\varepsilon)) \leq -\frac{\delta}{2T^2} e^{-\lambda T} < 0 \tag{3.3}$$

and

$$b - Tr(Y) + \lambda v_2(y^\varepsilon, t^\varepsilon) + e^{-\lambda t^\varepsilon} \alpha(y^\varepsilon, e^{\lambda t^\varepsilon} v_2(y^\varepsilon, t^\varepsilon)) \geq 0. \tag{3.4}$$

Consequently, it follows from the inequalities (3.3), (3.4) and the fact $Tr(X) \leq Tr(Y)$ that

$$\begin{aligned} 0 &> -\frac{\delta}{2T^2} e^{-\lambda T} \\ &\geq \lambda(v_1(x^\varepsilon, t^\varepsilon) - v_2(y^\varepsilon, t^\varepsilon)) + e^{-\lambda t^\varepsilon} \left\{ \alpha(x^\varepsilon, e^{\lambda t^\varepsilon} v_1(x^\varepsilon, t^\varepsilon)) - \alpha(y^\varepsilon, e^{\lambda t^\varepsilon} v_2(y^\varepsilon, t^\varepsilon)) \right\} \\ &\geq \lambda(v_1(x^\varepsilon, t^\varepsilon) - v_2(y^\varepsilon, t^\varepsilon)) - K|v_1(x^\varepsilon, t^\varepsilon) - v_2(y^\varepsilon, t^\varepsilon)|, \text{ as } |\alpha_w(x, w)| \leq K \\ &\geq \lambda(v_1(x^\varepsilon, t^\varepsilon) - v_2(y^\varepsilon, t^\varepsilon)) - \frac{\lambda}{2}|v_1(x^\varepsilon, t^\varepsilon) - v_2(y^\varepsilon, t^\varepsilon)|, \text{ as } \lambda > 2K. \end{aligned}$$

On account of the reasons that justify the preceding claim, we know that there exists a sequence $\varepsilon_j \to 0$ such that $x^{\varepsilon_j} \to x_0$, $y^{\varepsilon_j} \to x_0$, $t^{\varepsilon_j} \to t_0$, and $x_0 \in \Omega$, $0 < t_0 \leq T' < T$. In addition, Proposition 3.7 in [9] implies $v_1(x_0, t_0) - v_2(x_0, t_0) = M_0$. Taking limits in $0 \geq \lambda(v_1(x^{\varepsilon_j}, t^{\varepsilon_j}) - v_2(y^{\varepsilon_j}, t^{\varepsilon_j})) - \frac{\lambda}{2}|v_1(x^{\varepsilon_j}, t^{\varepsilon_j}) - v_2(y^{\varepsilon_j}, t^{\varepsilon_j})|$, we obtain, since $v_1(x_0, t_0) - v_2(x_0, t_0) = M_0 > 0$, that

$$0 \geq \frac{\lambda}{2}(v_1(x_0, t_0) - v_2(x_0, t_0)) > 0,$$

which is an obvious contradiction. We are done. $\square$

The strict inequality restriction on the boundary condition can be loosened to a non-strict one. More precisely,

Lemma 3.4. *For $T > 0$ sufficiently small, if $Hw_1 \leq 0 \leq Hw_2$ in $\Omega \times \mathbb{R}_T$ in the viscosity sense and $w_1 \leq w_2$ on $\partial_p(\Omega \times \mathbb{R}_T)$, then $w_1 \leq w_2$ on $\overline{\Omega \times \mathbb{R}_T}$.*

Proof. For any $\delta > 0$, let $w = w_1 - \delta t - \tilde{\delta}$, where the value of $\tilde{\delta} > 0$ will be taken in the following. Then $w < w_1 \leq w_2$ on $\partial_p(\Omega \times \mathbb{R}_T)$, and

$$\begin{aligned} Hw &= Hw_1 - \delta - \alpha(x, w_1) + \alpha(x, w_1 - \delta t - \tilde{\delta}) \\ &\leq -\delta + K\left(\delta t + \tilde{\delta}\right) \leq -\delta + K\left(\delta T + \tilde{\delta}\right) \\ &< -\delta + \frac{1}{2}\delta + \frac{1}{4}\delta, \quad \text{for } T \text{ small and } \tilde{\delta} \leq \frac{\delta}{4K}, \\ &= -\frac{1}{4}\delta < 0. \end{aligned}$$

Again, the above differential equality and inequalities are in the viscosity sense and can be made rigorous.

The preceding lemma implies $w \leq w_2$ on $\overline{\Omega \times \mathbb{R}_T}$ for small T, for any small $\delta > 0$ and $\tilde{\delta} > 0$. Therefore $w_1 \leq w_2$ on $\overline{\Omega \times \mathbb{R}_T}$. $\square$

Now the parabolic comparison principle, Theorem 3.1, follows from the preceding lemma quite easily as shown by the following argument: Let $T_0 > 0$ be any small value of T in the preceding lemma so that the conclusion of the preceding lemma holds. Then $w_1 \leq w_2$ on $\overline{\Omega \times (0, T_0)}$. In particular, $w_1 \leq w_2$ on $\partial_p(\Omega \times (T_0, 2T_0))$. The preceding lemma may be applied again to conclude that $w_1 \leq w_2$ on $\overline{\Omega \times (T_0, 2T_0)}$. And so on. In the end, we see that $w_1 \leq w_2$ on $\overline{\Omega \times \mathbb{R}_T}$.

4. Uniform Lipschitz continuity

We will need the following result of uniform Lipschitz continuity of solutions of the parabolic equation in the proof of the convergence of the evolution process. On the other hand, this uniform Lipschitz regularity can be applied to the study of the parabolic two-phase free boundary problem beyond its use here.

Theorem 4.1. *Let u be a weak solution in $\Gamma_1 = B_1 \times (-1, 0)$ of*

$$u_t - \Delta u + \theta(u)u + \Theta_\varepsilon(u) + 2q^2(x)\lambda_\varepsilon(u)\mu_\varepsilon(u) = 0 \quad ((x,t) \in \Omega \times (t_0, t_1)) . \tag{4.1}$$

Suppose also that

$$\|u\|_{L^\infty(\Gamma_1)} \leq C$$

for a universal constant C.

Then, there exists a constant $\varepsilon_0 = \varepsilon_0(C, n)$ such that the following uniform gradient bound holds

$$\|\nabla u\|_{L^\infty(\Gamma_{\frac{1}{2}})} \leq \overline{C} \tag{4.2}$$

for a universal constant $\overline{C}$ that is independent of ε in $(0, \varepsilon_0)$.

Proof. Step 1. We prove $\|\nabla u\|_{L^\infty} \leq \overline{C}$ on $\{u \leq 0\} \cap \Gamma_{1/2}$.

For any $\lambda < 0$, let $u_1 = (u - \lambda)^+$ and $u_2 = (u - \lambda)^-$. Then u_1 and u_2 are smooth in their respective support near $\{u = \lambda\}$ and satisfy the hypotheses in Theorems 2 and 3 in [10] if one notices that

$$\begin{aligned} &(\Delta - \partial_t)\, u^- = -1 \text{ in } \{u < 0\}\,; \\ &(\Delta - \partial_t)\, u \geq -2 \text{ in } \{0 \leq u < \varepsilon\}\,; \text{ and} \\ &(\Delta - \partial_t)\, u = 0 \text{ in } \{u \geq \varepsilon\}\,. \end{aligned}$$

Therefore on $\Gamma_{3/4} \cap \{u < 0\}$ and continuous onto $\Gamma_{3/4} \cap \{u \leq 0\}$, $\|\nabla u^-\|_{L^\infty} \leq \overline{C}$ for some universal constant $\overline{C}$ that is independent of ε according to the theorems and the fact that the monotonicity function $\Phi(t) \to C(n)\left|\nabla u^-\right|^4$ as $t \to 0-$ in Theorem 3, [10].

Take $\varepsilon_0 = \frac{\overline{C}}{20}$.

Step 2. We prove $\|\nabla u\|_{L^\infty} \leq \tilde{C}$ on $\{0 < u \leq 2\varepsilon\} \cap \Gamma_{1/2}$ for a universal constant $\tilde{C}$ which is independent of ε.

Pick any (x_0, t_0) in $\{0 < u \leq 2\varepsilon\}$. We define the λ-cylinder $\Gamma_\lambda(x_0, t_0)$ with top center (x_0, t_0) by

$$\Gamma_\lambda(x_0, t_0) = \left\{(x,t)\colon |x - x_0| < \lambda, t_0 - \lambda^2 < t < t_0\right\} . \tag{4.3}$$

We further introduce the new variables $y = \frac{1}{\varepsilon}(x - x_0)$ and $s = \frac{1}{\varepsilon^2}(t - t_0)$. Then

$$(x,t) \in \Gamma_\varepsilon(x_0, t_0) \text{ if and only if } (y,s) \in \Gamma_1 .$$

Define

$$w(y,s) = \frac{1}{\varepsilon} u(x_0 + \varepsilon y, t_0 + \varepsilon^2 s),\ (y,s) \in \Gamma_1 .$$

We can verify that

$$\Delta w - w_t = \varepsilon\,(\Delta u - u_t) = \varepsilon\left(\theta(u)u + \Theta(u) + 2q^2(x)\lambda_\varepsilon(u)\mu_\varepsilon(u)\right)$$

in Γ_1.

In addition, we know in $\{w < 0\}$, $|\nabla w| = |\nabla u| \leq \overline{C}$ from step 1.

Suppose $\overline{B}_{3/4} \cap \{(x,t) \colon w(x,t) \geq 0\} \neq \emptyset$ for some t in $-1 \leq t \leq 0$. Then $w(x,t) \geq -\frac{3}{2}\overline{C}$ for all $x \in \overline{B}_{3/4}$ due to the computation

$$w(x,t) \geq w(x_0,t) - \overline{C}|x-x_0| \geq -\overline{C}|x-x_0| \geq -\frac{3}{2}\overline{C}$$

for some x_0 with $w(x_0,t) = 0$.

Define

$$\mathcal{J} = \left\{t \in [-1,0] \colon \overline{B}_{3/4} \cap \{(x,t) \colon w(x,t) \geq 0\} = \emptyset\right\}. \tag{4.4}$$

The set $\mathcal{J}$ is an open set as w is uniformly continuous. Let (t_α, t_β) be a component of $\mathcal{J}$.

Take $\delta = \frac{1}{18n}$. Let m denote the least integer so that $m\delta \geq 1$. For $1 \leq j \leq m$, we define the interval $I_j = (t_\beta - j\delta, t_\beta - (j-1)\delta) \cap (t_\alpha, t_\beta)$.

If $I_j \neq \emptyset$, we claim that, for $t \in \overline{I}_j$,

$$\inf_x w(x,t) \geq -\frac{3}{2}K^j\overline{C},$$

where K is any universal constant verifying $K > \frac{1}{65}(81 + 288n)$.

We prove the claim by induction. Firstly we notice that $\frac{1}{192n}(65K - 81) > \frac{3}{2}$.

For $j = 1$, suppose for some $t_0 \in I_1$, $\inf_x w(x,t_0) = -M < -\frac{3}{2}K\overline{C}$. Recall that for all t in (t_α, t_β), $w(x,t) < 0$ on $\overline{B}_{3/4}$, and hence $|\nabla w| = |\nabla u| \leq \overline{C}$ there. In particular, we get

$$w(x,t_0) \leq -M + \frac{3}{2}\overline{C} < -\frac{3}{2}(K-1)\overline{C} =: -\overline{M} \text{ for all } x \in \overline{B}_{3/4}.$$

We introduce the super-caloric function $\phi(x,t) = \frac{\overline{M}}{2n}\left(|x|^2 - \left(\frac{3}{4}\right)^2\right) + \left(\overline{M} + 2\varepsilon\right)(t - t_0)$. We see that

$$(\Delta - \partial_t)\,\phi = -2\varepsilon$$

and

$$(\Delta - \partial_t)\,w = \varepsilon\,(\Delta - \partial_t)\,u \geq -2\varepsilon$$

in $B_{3/4} \times (-1,0)$.

We claim that for $(x,t) \in \overline{B}_{3/4} \times [t_0, t_\beta]$, $w(x,t) \leq \phi(x,t)$. In fact,

$$\phi(x,t) = \left(\overline{M} + 2\varepsilon\right)(t - t_0) \geq 0 > w(x,t)$$

on the lateral side $|x| = \frac{3}{4}$. At the bottom where $t = t_0$,

$$w(x,t_0) \leq -\overline{M} < -\frac{\overline{M}}{2n} \leq \phi(x,t_0).$$

In all, $w(x,t) \leq \phi(x,t)$ on the parabolic boundary $\partial_p\left(B_{3/4} \times [t_0, t_\beta]\right)$. Then, the parabolic comparison principle implies that $w(x,t) \leq \phi(x,t)$ on $\overline{B}_{3/4} \times [t_0, t_\beta]$.

Consequently,

$$\begin{aligned}
w(0,t_\beta) \leq \phi(0,t_\beta) &= -\frac{9}{32n}\overline{M} + \left(\overline{M} + 2\varepsilon\right)(t_\beta - t_0) \leq -\frac{9}{32n}\overline{M} + \left(\overline{M} + \frac{\overline{C}}{10}\right)\delta \\
&< -\frac{9}{32n}\overline{M} + \frac{3}{2}K\overline{C}\delta, \quad \text{as } \overline{M} + \frac{\overline{C}}{10} = \frac{3}{2}(K-1)\overline{C} + \frac{\overline{C}}{10} < \frac{3}{2}K\overline{C} \\
&= -\frac{1}{192n}(65K - 81)\,\overline{C} < -\frac{3}{2}\overline{C},
\end{aligned}$$

since $\frac{1}{192n}(65K - 81) > \frac{3}{2}$, which contradicts the estimate $w(x,t_\beta) \geq -\frac{3}{2}\overline{C}$ for all $x \in B_{3/4}$ due to the fact that $\overline{B}_{3/4} \cap \{(x,t_\beta) \colon w > 0\} \neq \emptyset$.

Next, we consider the case $j > 1$. Suppose for some $t_j \in I_j$, $\inf_x w(x,t_j) = -M_j < -\frac{3}{2}K^j\overline{C}$. We observe that

$$w(x,t_j) \le -M_j + \frac{3}{2}\overline{C} < -\frac{3}{2}(K^j - 1)\overline{C} =: -\overline{M}_j \text{ for all } x \in \overline{B}_{3/4}.$$

We introduce the super-caloric function $\phi_j(x,t) = \frac{\overline{M}_j}{2n}\left(|x|^2 - \left(\frac{3}{4}\right)^2\right) + \left(\overline{M}_j + 2\varepsilon\right)(t - t_j)$. We see that

$$(\Delta - \partial_t)\,\phi_j = -2\varepsilon$$

and hence $w - \phi_j$ is sub-caloric in $B_{3/4} \times (t_j, t_\beta - (j-1)\delta)$. We claim that for $(x,t) \in \overline{B}_{3/4} \times [t_j, t_\beta - (j-1)\delta]$, $w(x,t) \le \phi_j(x,t)$. In fact,

$$\phi_j(x,t) = \left(\overline{M}_j + 2\varepsilon\right)(t - t_j) \ge 0 > w(x,t)$$

on the lateral side $|x| = \frac{3}{4}$. At the bottom where $t = t_j$,

$$w(x,t_j) \le -\overline{M}_j < -\frac{\overline{M}_j}{2n} \le \phi_j(x,t_j).$$

In all, $w(x,t) \le \phi_j(x,t)$ on the parabolic boundary $\partial_p\left(B_{3/4} \times [t_j, t_\beta - (j-1)\delta]\right)$. Then the parabolic comparison principle implies that $w(x,t) \le \phi_j(x,t)$ on $\overline{B}_{3/4} \times [t_j, t_\beta - (j-1)\delta]$.

Consequently,

$$\begin{aligned}
w(0, t_\beta - (j-1)\delta) \le \phi_j(0, t_\beta - (j-1)\delta) &= -\frac{9}{32n}\overline{M}_j + \left(\overline{M}_j + 2\varepsilon\right)\delta \\
&\le -\frac{9}{32n}\overline{M}_j + \left(\overline{M}_j + \frac{\overline{C}}{10}\right)\delta \\
&< -\frac{9}{32n}\overline{M}_j + \frac{3}{2}K^j\overline{C}\delta, \quad \text{as } \overline{M}_j + \frac{\overline{C}}{10} < \frac{3}{2}K^j\overline{C} \\
&= -\frac{1}{192n}\left(65K^j - 81\right)\overline{C} < -\frac{1}{192n}\left(65K - 81\right)K^{j-1}\overline{C} \\
&< -\frac{3}{2}K^{j-1}\overline{C},
\end{aligned}$$

since $\frac{1}{192n}(65K - 81) > \frac{3}{2}$, which contradicts the induction hypothesis $w(x, t_\beta - (j-1)\delta) \ge -\frac{3}{2}K^{j-1}\overline{C}$ for all $x \in B_{3/4}$.

The claim is proved.

Therefore,

$$\inf_{x \in \overline{B}_{3/4}} w(x,t) \ge -K^m C$$

for all $t \in (t_\alpha, t_\beta)$ and the universal constants m and K. Moreover

$$(\Delta - \partial_t)\,w = \varepsilon\,(\Delta - \partial_t)\,u \in [-\varepsilon_0 C, C],$$

for some universal constant C independent of ε, or equivalently $(\Delta - \partial_t)\,w$ is also universally bounded, and $w(0,0) = \frac{1}{\varepsilon}u(x_0,t_0) \le \frac{1}{\varepsilon}2\varepsilon = 2$. The classical Harnack's inequality and the standard a priori gradient estimate give that

$$|\nabla u(x_0,t_0)| = |\nabla w(0,0)| \le \tilde{C} \tag{4.5}$$

for a universal constant $\tilde{C}$ that is independent of ε.

Step 3. We finally prove $\|\nabla u\|_{L^\infty} \le C\overline{C}$ on $\Omega_\varepsilon := \{u > \varepsilon\} \cap \Gamma_{1/2}$ by applying the Bernstein technique.

Let $\varphi(x,t)$ be a smooth cut-off function such that $\varphi = 1$ on $\Gamma_{\frac{1}{2}}$ and $\varphi = 0$ in $\Gamma_{3/4}^C$. We consider the function

$$z(x,t) := \varphi^2(x)|\nabla u(x,t)|^2 + \lambda u^2(x,t)$$

in $\Omega_\varepsilon \cap \Gamma_{3/4}$, where $\lambda > 0$ is a large constant that is independent of ε. The parabolic boundary of $\Omega_\varepsilon \cap \Gamma_{3/4}$ consists of two parts, the one that is contained in the parabolic boundary of $\Gamma_{3/4}$ where $\varphi = 0$ and the other that is contained in $\{u < 2\varepsilon\}$ where $|\nabla u| \leq \overline{C}$. In short, $z(x,t)$ is universally bounded on the parabolic boundary $\partial_p\left(\Omega_\varepsilon \cap \Gamma_{3/4}\right)$. On the other hand, in $\{u > \varepsilon\}$, u is caloric and smooth, and

$$\Delta u - \partial_t u = 0$$

and

$$\Delta u_{x_i} - \partial_t u_{x_i} = 0 \ (i = 1, \ldots, n)$$

hold.

Consequently

$$\begin{aligned}
&(\Delta - \partial_t)\, z = (\Delta - \partial_t)\left(\varphi^2 |\nabla u|^2 + \lambda u^2\right) \\
&= \left((\Delta - \partial_t)\,\varphi^2\right)|\nabla u|^2 + 8\varphi\varphi_{x_i} u_{x_i x_j} u_{x_j} + 2\varphi^2\left(\nabla \cdot \left(D^2 u \nabla u\right) - \nabla u \cdot \nabla u_t\right) \\
&\quad + 2\lambda |\nabla u|^2 + 2\lambda u\, (\Delta - \partial_t)\, u \\
&= \left((\Delta - \partial_t)\,\varphi^2\right)|\nabla u|^2 + 8\varphi\varphi_{x_i} u_{x_i x_j} u_{x_j} + 2\varphi^2\left|D^2 u\right|^2 + 2\lambda |\nabla u|^2 \\
&\geq 0
\end{aligned}$$

for sufficiently large universal λ, where we use the identity

$$\nabla \cdot \left(D^2 u \nabla u\right) - \nabla u \cdot \nabla u_t = |D^2 u|^2 + \sum_j u_{x_j}\left(\Delta u_{x_j} - \partial_t u_{x_j}\right) = |D^2 u|^2$$

in the last equation and interpolation in the last inequality. We then conclude the proof of Step 3 by invoking the parabolic maximum principle for $z(x,t)$ in $\Omega_\varepsilon \cap \Gamma_{3/4}$. □

5. Convergence of evolution

Let $\mathcal{D} = \Omega \times (0, +\infty)$, where, as before, $\Omega \subset \mathbb{R}^n$ is a bounded domain. For a function $w : \mathcal{D} \to \mathbb{R}$, we define the positive domain $\mathcal{D}^+(w) = \{(x,t) : w(x,t) > 0\}$ and the free boundary $\mathcal{F}(w) = \partial\mathcal{D}^+(w) \cap \mathcal{D}$ associated with w.

In this section, we consider the convergence of the evolution of the two-phase regularized problem defined below

$$\begin{cases} w_t - \Delta w + \theta(w)w + \Theta_\varepsilon(w) + 2q^2(x)\lambda_\varepsilon(w)\mu_\varepsilon(w) = 0 & \text{in } \mathcal{D} \\ w(x,t) = \sigma(x) & \text{on } \partial\Omega \times (0, +\infty) \\ w(x,0) = v_0(x) & \text{for } x \in \overline{\Omega}. \end{cases} \tag{5.1}$$

Here v_0 is a continuous function on $\overline{\Omega}$ such that $v_0 = \sigma$ on $\partial\Omega$, and $\min_{\partial\Omega} \sigma \gg \varepsilon > 0$. For simplicity in writing, we assume $q(x) \equiv 1$ in the following proofs. We adopt a notation

$$H^\varepsilon w = w_t - \Delta w + \theta(w)w + \Theta_\varepsilon(w) + 2\lambda_\varepsilon(w)\mu_\varepsilon(w)$$

for the nonlinear ε-heat operator.

Define $\mathfrak{S}$ to be the set of viscosity solutions of the stationary problem (1.8)–(1.9). The harmonic function u_0 is the maximum element in $\mathfrak{S}$ as the supremum of all viscosity sub-solutions of (1.8)–(1.9). On the other hand, in many a case, the least element of $\mathfrak{S}$, namely the infimum of all viscosity super-solutions of (1.8)–(1.9), turns out to be a minimizer of the functional J_ε. So it is excusable to abuse the notation a little by using u_2 to denote this least solution.

In addition, one calls u a **non-minimal solution** of the problem (1.8)–(1.9) if it is a viscosity solution of (1.8)–(1.9) but not a local minimizer in the sense that for any $\delta > 0$, there exists v in the admissible set of the functional J_ε with $v = \sigma$ on $\partial\Omega$ such that $\|v - u\|_{L^\infty} < \delta$, and $J_\varepsilon[v] < J_\varepsilon[u]$.

The main result of the convergence of the evolution is stated below.

Theorem 5.1. *Let w be a solution of (5.1). If the initial data v_0 falls into any of the categories specified below, the corresponding conclusion holds.*

1. *If $v_0 \le u_2$ on $\overline{\Omega}$, then $\lim_{t\to+\infty} w(x,t) = u_2(x)$ locally uniformly for $x \in \overline{\Omega}$;*
2. *Define*
$$\bar{u}_2(x) = \inf_{u\in\mathfrak{S}, u\ge u_2, u\ne u_2} u(x), \; x \in \overline{\Omega}.$$
If $\bar{u}_2 \ne u_2$, then for v_0 such that $u_2 \le v_0 \le \bar{u}_2$ but $v_0 \ne \bar{u}_2$, $\lim_{t\to+\infty} w(x,t) = u_2(x)$ locally uniformly for $x \in \overline{\Omega}$;
3. *Define $\bar{u}_0(x) = \sup_{u\in\mathfrak{S}, u\le u_0, u\ne u_0} u(x)$, $x \in \overline{\Omega}$. If $\bar{u}_0 < u_0$, then for v_0 such that $\bar{u}_0 < v_0 < u_0$, $\lim_{t\to+\infty} w(x,t) = u_0(x)$ locally uniformly for $x \in \overline{\Omega}$;*
4. *If $v_0 \ge u_0$ in $\overline{\Omega}$, then $\lim_{t\to+\infty} w(x,t) = u_0(x)$ uniformly for $x \in \overline{\Omega}$;*
5. *Suppose u_1 is a non-minimal solution of (1.8)(1.9). For any small $\delta > 0$, there exists v_0 such that $\|v_0 - u_1\|_{L^\infty(\Omega)} < \delta$ and the corresponding solution w of the problem (5.1) does not satisfy*
$$\lim_{t\to\infty} w(x,t) = u_1(x) \;\; in\ \Omega.$$

Proof. Case (4): The reader can find the proof in [5].

The proof of locally uniform convergence, instead of uniform convergence, in cases (1)-(3), is completed by employing the fact that w is locally Lipschitz continuous in x with a uniform constant independent of ε (i. e., Theorem 4.1) and by showing the weak derivative $w_t \ge 0$.

Case (1): Suppose $v_0 \le u_2$ on $\overline{\Omega}$. We may take a very large negative smooth function $\tilde{v}_0$ with $\tilde{v}_0 \le v_0$ and $-\Delta\tilde{v}_0 \ll 0$, and consider the corresponding solution $\tilde{w}$ of (5.1) which satisfies $\tilde{w} \le w \le u_2$ due to the parabolic comparison principle 3.1. Then we only need to prove that $\tilde{w}$ converges to u_2 locally uniformly in Ω as $t \to +\infty$. In fact, all we need to prove is that the weak time derivative satisfies $\tilde{w}_t \ge 0$ in $\Omega \times (0,+\infty)$ and the modulus of the gradient in space $|\nabla\tilde{w}|$ is locally bounded. We will use v_0 and w for $\tilde{v}_0$ and $\tilde{w}$ in the following for simplicity of notations. For any sub-domain $\tilde{\Omega} \subset\subset \Omega$, we know that

$$\|\nabla w\|_{L^\infty(\tilde{\Omega}\times(0,+\infty))} \le C = C(\|\sigma\|_{L^\infty(\partial\Omega)}, \|v_0\|_{L^\infty(\Omega)})$$

by applying again Theorem 4.1.

Let $z(x,t) = w_t(x,t)$, $(x,t) \in \Omega \times [0,+\infty)$. As $w(x,t) = \sigma(x)$, for $x \in \partial\Omega$, which is time-independent, $z(x,t) = 0$ for $x \in \partial\Omega$ and any $t > 0$. On the other hand, for t close to 0, w is very large negative, and hence $\beta_\varepsilon(w) = 0$. So the equation

$$w_t - \Delta w + \theta_\varepsilon(w)w + \Theta_\varepsilon(w) + 2\lambda_\varepsilon(w)\mu_\varepsilon(w) = 0$$

implies that $w_t = \Delta w > 0$ as $-\Delta v_0 < 0$. Therefore, z is a viscosity solution of the problem

$$\begin{cases} z_t - \Delta z + \left(2\theta_\varepsilon(w) + \theta_\varepsilon'(w)w + 2\mu_\varepsilon^2(w) + 2\lambda_\varepsilon(w)\mu_\varepsilon'(w)\right) z = 0 & \text{in } \Omega \times (0,+\infty) \\ z(x,t) = 0 & \text{on } \partial\Omega \times (0,+\infty) \\ z(x,0) > 0 & \text{in } \Omega. \end{cases} \tag{5.2}$$

The first equation is obtained by differentiating the first equation in (5.1) with respect to t. According to the parabolic comparison principle, Theorem 3.1, $z(x,t) \ge 0$ in $\Omega \times (0,+\infty)$ as 0 is a trivial solution of the problem (5.2). Therefore $\lim_{t\to+\infty} w(x,t) = u^\infty(x)$ monotonically and the limit $u^\infty(x) \le u_2(x)$ on $\overline{\Omega}$. As $|\nabla w| \le C$, the convergence is uniform in $\tilde{\Omega}$, which implies u^∞ is a solution of (1.8)–(1.9) on $\overline{\Omega}$. According to the minimality of u_2, we know that $u^\infty = u_2$ on $\overline{\Omega}$.

We take care of the cases (2) and (3) in a similar way.

Case (2): Suppose $u_2 \le v_0 \le \bar{u}_2$ but $v_0 \ne \bar{u}_2$ identically in Ω. As in Case (1), we may replace v_0 by a super-solution between u_2 and $\bar{u}_2$ here. Indeed, u_2 is the infimum of super-solutions of (1.8)–(1.9). Since

$\bar{u}_2 \neq u_2$, there is a super-solution v_0 between the two which is not identically $\bar{u}_2$. The corresponding solution w then satisfies $w_t(x,t) = 0$ on $\partial\Omega \times (0,+\infty)$ and, for t near 0,

$$w_t = \Delta w - \theta_\varepsilon(w)w - \Theta_\varepsilon(w) - 2\lambda_\varepsilon(w)\mu_\varepsilon(w) \le 0.$$

So the time-derivative $z(x,t) = w_t(x,t)$ of w then verifies

$$\begin{cases} z_t - \Delta z + \left(2\theta_\varepsilon(w) + \theta'_\varepsilon(w)w + 2\mu^2_\varepsilon(w) + 2\lambda_\varepsilon(w)\mu'_\varepsilon(w)\right) z = 0 & \text{in } \Omega \times (0,+\infty) \\ z(x,t) = 0 & \text{on } \partial\Omega \times (0,+\infty) \\ z(x,0) \le 0 & \text{in } \Omega. \end{cases}$$

The parabolic comparison principle, Theorem 3.1, implies $z = w_t \le 0$ in $\Omega \times (0,+\infty)$. The boundedness of the gradient ∇w with respect to x on $\tilde{\Omega} \subset\subset \Omega$ and the monotone convergence of $w(x,t)$ as $t \to +\infty$ imply $\lim_{t\to\infty} w(x,t) = u^\infty(x)$ uniformly, which in turn implies u^∞ is a solution of the problem (1.8)–(1.9) between of u_2 and $\bar{u}_2$. In addition, $u^\infty \le v_0 \le \bar{u}_2$ but $v_0 \neq \bar{u}_2$. So $u^\infty = u_2$.

Case (3) is similarly proved as (2) with super-solutions replaced by sub-solutions.

Case (5): There exists a function v_0 such that $\|v_0 - u_1\| < \delta$ and $J_\varepsilon[v_0] < J_\varepsilon[u_1]$. Take v_0 as the initial data of the problem (5.1), and one may take v_0 so that it is not a solution of the problem (1.8)–(1.9). By multiplying w_t to the equation

$$w_t - \Delta w + \theta_\varepsilon(w)w + \Theta_\varepsilon(w) + 2\lambda_\varepsilon(w)\mu_\varepsilon(w) = 0 \ \text{ in } \mathcal{D}$$

and integrating both sides of the equation over $\mathcal{D} \times (0,t)$, one deduces that

$$\int_0^t \int_{\mathcal{D}} w_t^2 - w_t \Delta w + \left(\Theta_\varepsilon(w)w + \lambda^2_\varepsilon(w)\right)_t \, dxdt = 0,$$

or, equivalently,

$$\int_0^t \int_{\mathcal{D}} w_t^2 - \nabla \cdot (w_t \nabla w) + \nabla w_t \cdot \nabla w + \left(\Theta_\varepsilon(w)w + \lambda^2_\varepsilon(w)\right)_t \, dxdt = 0.$$

Applying the Divergence Theorem to the second term and noticing the condition that $w_t = 0$ on $\partial\Omega \times (0,\infty)$, one gets

$$\int_0^t \int_{\mathcal{D}} w_t^2 + (\frac{1}{2}|\nabla w|^2)_t + \left(\Theta_\varepsilon(w)w + \lambda^2_\varepsilon(w)\right)_t \, dxdt = 0,$$

and hence

$$\begin{aligned} &\int_0^t \int_{\mathcal{D}} w_t^2 \, dxdt + \int_{\mathcal{D}} \frac{1}{2}|\nabla w(x,t)|^2 + \Theta_\varepsilon(w)w(x,t) + \lambda^2_\varepsilon(w)\, dx \\ &= \int_{\mathcal{D}} \frac{1}{2}|\nabla w(x,0)|^2 + \Theta_\varepsilon(w)w(x,0) + \lambda^2_\varepsilon(w)\, dx. \end{aligned}$$

That is,

$$\int_0^t \int_{\mathcal{D}} w_t^2 \, dxdt + J_\varepsilon(w(\cdot,t)) = J_\varepsilon(v_0).$$

Since v_0 is not a solution of the corresponding stationary problem, w_t is not identically 0, and therefore

$$J_\varepsilon(w(\cdot,t)) < J_\varepsilon(v_0) < J_\varepsilon(u_1),$$

which clearly implies that $w(x,t)$ does not converge to u_1. □

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