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On the distributional divergence of vector fields vanishing at infinity

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The equation $\operatorname{div} v = F$ has a solution v in the space of continuous vector fields vanishing at infinity if and only if F acts linearly on $\mathrm{BV}_{m/(m-1)}(\mathbb{R}^m)$ (the space of functions in $L^{m/(m-1)}(\mathbb{R}^m)$ whose distributional gradient is a vector-valued measure) and satisfies the following continuity condition: $F(u_j)$ converges to zero for each sequence $\{u_j\}$ such that the measure norms of ∇u_j are uniformly bounded and $u_j \rightharpoonup 0$ weakly in $L^{m/(m-1)}(\mathbb{R}^m)$.

1. Introduction

The equation $\Delta u = f \in L^m(\mathbb{R}^m)$ need not have a solution $u \in C^1(\mathbb{R}^m)$. In this paper we prove that, to each $f \in L^m(\mathbb{R}^m)$, there corresponds a continuous vector field, vanishing at infinity, $v \in C_0(\mathbb{R}^m; \mathbb{R}^m)$ such that $\operatorname{div} v = f$ weakly. In fact, we characterize those distributions F on $\mathbb{R}^m$ such that the equation $\operatorname{div} v = F$ admits a weak solution $v \in C_0(\mathbb{R}^m; \mathbb{R}^m)$. Related results have been obtained in [1–4, 6]. Our first proof, contained in §§ 3–6, follows the same pattern as [2]. A second proof, presented in § 7, is based on the more abstract methods developed in [3].

In this paper $m \geqslant 2$ and $1^* := m/(m-1)$. Let $\mathrm{BV}_{1^*}(\mathbb{R}^m)$ denote the subspace of $L^{1^*}(\mathbb{R}^m)$ consisting of those functions u whose distributional gradient ∇u is a vector-valued measure (of finite total mass). We define a charge vanishing at infinity to be a linear functional $F\colon \mathrm{BV}_{1^*}(\mathbb{R}^m) \to \mathbb{R}$ such that $F(u_j) \to 0$ whenever

$$u_j \rightharpoonup 0 \text{ weakly in } L^{1^*}(\mathbb{R}^m) \quad \text{and} \quad \sup_j \|\nabla u\|_{\mathcal{M}} < \infty. \tag{1.1}$$

We denote by $\mathrm{CH}_0(\mathbb{R}^m)$ the space of charges vanishing at infinity and we note (see proposition 3.2) that it is a closed subspace of the dual of $\mathrm{BV}_{1^*}(\mathbb{R}^m)$ (where the latter is equipped with its norm $\|\nabla u\|_{\mathcal{M}}$). Examples of charges vanishing at infinity include the functions $f \in L^m(\mathbb{R}^m)$ (see proposition 3.4) and the distributional divergence $\operatorname{div} v$ of $v \in C_0(\mathbb{R}^m;\mathbb{R}^m)$ (see proposition 3.5). Our main result thus consists in proving that the operator

$$C_0(\mathbb{R}^m;\mathbb{R}^m) \to \mathrm{CH}_0(\mathbb{R}^m)\colon v \mapsto \operatorname{div} v \tag{1.2}$$

is onto. This is done by applying the closed range theorem. For this purpose we identify $\mathrm{CH}_0(\mathbb{R}^m)^*$ with $\mathrm{BV}_{1^*}(\mathbb{R}^m)$ via the evaluation map (see proposition 5.1). This in turn relies on the fact that $L^m(\mathbb{R}^m)$ is dense in $\mathrm{CH}_0(\mathbb{R}^m)$ (see corollary 4.3, which is obtained by smoothing). Therefore, the adjoint of (1.2) is

$$\mathrm{BV}_{1^*}(\mathbb{R}^m) \to \mathcal{M}(\mathbb{R}^m;\mathbb{R}^m)\colon u \mapsto -\nabla u.$$

The observation that this operator has a closed range follows from compactness in $\mathrm{BV}_{1^*}(\mathbb{R}^m)$ (see proposition 2.6).

Charges vanishing at infinity happen to be the linear functionals on $\mathrm{BV}_{1^*}(\mathbb{R}^m)$ which are continuous with respect to a certain locally convex linear (sequential, non-metrizable, non-barrelled) topology $\mathfrak{T}_{\mathcal{C}}$ on $\mathrm{BV}_{1^*}(\mathbb{R}^m)$. In other words, there exists a locally convex topology $\mathfrak{T}_{\mathcal{C}}$ on $\mathrm{BV}_{1^*}(\mathbb{R}^m)$ such that a sequence $u_j \to 0$ in the sense of $\mathfrak{T}_{\mathcal{C}}$ if and only if the sequence $\{u_j\}$ verifies the conditions of (1.1). Topologies of this type have been studied in [3, § 3]. Referring to the general theory yields a quicker, though very much abstract proof in § 7. In order to appreciate this alternative route, the reader is expected to be familiar with the methods of [3, § 3]. From this perspective the key identification $\mathrm{CH}_0(\mathbb{R}^m)^* \cong \mathrm{BV}_{1^*}(\mathbb{R}^m)$ is simply saying that $\mathrm{BV}_{1^*}(\mathbb{R}^m)[\mathfrak{T}_{\mathcal{C}}]$ is semireflexive; a property which follows from the compactness proposition 2.6.

2. Preliminaries

A continuous vector field $v\colon \mathbb{R}^m \to \mathbb{R}^m$ is said to *vanish at infinity* if, for every $\varepsilon > 0$, there exists a compact set $K \subset \mathbb{R}^m$ such that $|v(x)| \leqslant \varepsilon$ whenever $x \in \mathbb{R}^m \setminus K$. These form a linear space denoted by $C_0(\mathbb{R}^m;\mathbb{R}^m)$, which is complete under the norm $\|v\|_\infty := \sup\{|v(x)|\colon x \in \mathbb{R}^m\}$. The linear subspace $C_c(\mathbb{R}^m;\mathbb{R}^m)$ (respectively, $\mathcal{D}(\mathbb{R}^m;\mathbb{R}^m)$) consisting of those vector fields having compact support (respectively, smooth vector fields having compact support) is dense in $C_0(\mathbb{R}^m;\mathbb{R}^m)$. Thus, each element of the dual, $T \in C_0(\mathbb{R}^m;\mathbb{R}^m)^*$, is uniquely associated with some vector-valued measure $\mu \in \mathcal{M}(\mathbb{R}^m;\mathbb{R}^m)$ as follows:

$$T(v) = \int_{\mathbb{R}^m} \langle v, \mathrm{d}\mu\rangle,$$

according to the Riesz–Markov representation theorem. Furthermore,

$$\|\mu\|_{\mathcal{M}} = \sup\biggl\{\int_{\mathbb{R}^m} \langle v, \mathrm{d}\mu\rangle\colon v \in \mathcal{D}(\mathbb{R}^m;\mathbb{R}^m) \text{ and } \|v\|_\infty \leqslant 1\biggr\}.$$

A vector-valued distribution $T \in \mathcal{D}(\mathbb{R}^m;\mathbb{R}^m)^*$ with the property that

$$\sup\{T(v)\colon v \in \mathcal{D}(\mathbb{R}^m;\mathbb{R}^m) \text{ and } \|v\|_\infty \leqslant 1\} < \infty$$

extends uniquely to an element of $C_0(\mathbb{R}^m;\mathbb{R}^m)$ and is therefore associated with a vector-valued measure as above.

We recall some properties of convolution. Let $1 \leqslant p < \infty$, $u \in L^p(\mathbb{R}^m)$ and $\varphi \in \mathcal{D}(\mathbb{R}^m)$. For each $x \in \mathbb{R}^m$, we define

$$(u * \varphi)(x) = \int_{\mathbb{R}^m} u(y)\varphi(x-y)\,\mathrm{d}y.$$

It follows from Young's inequality that $u * \varphi \in L^p(\mathbb{R}^m)$ and

$$\|u * \varphi\|_{L^p} \leqslant \|u\|_{L^p}\|\varphi\|_{L^1}. \tag{2.1}$$

Furthermore, $u * \varphi \in C^\infty(\mathbb{R}^m)$ and $\nabla(u * \varphi) = u * \nabla\varphi$. In the case when φ is even and $f \in L^q(\mathbb{R}^m)$ with $p^{-1} + q^{-1} = 1$, we have

$$\int_{\mathbb{R}^m} f(u * \varphi) = \int_{\mathbb{R}^m} u(f * \varphi).$$

We fix an *approximate identity on* $\mathbb{R}^m$, $\{\varphi_k\}$ [5, (6.31)], and we infer that

$$\lim_k \|u - u * \varphi_k\|_{L^p} = 0. \tag{2.2}$$

Henceforth we assume that $m \geqslant 2$. We let the Sobolev conjugate exponent of 1 be

$$1^* := \frac{m}{m-1}.$$

Note that $L^{1^*}(\mathbb{R}^m)$ is isometrically isomorphic to $L^m(\mathbb{R}^m)^*$. We will recall the Gagliardo–Nirenberg–Sobolev inequality

$$\|\varphi\|_{L^{1^*}} \leqslant \kappa_m \|\nabla\varphi\|_{L^1}$$

whenever $\varphi \in \mathcal{D}(\mathbb{R}^m)$.

Definition 2.1. We let $\mathrm{BV}_{1^*}(\mathbb{R}^m)$ denote the linear subspace of $L^{1^*}(\mathbb{R}^m)$ consisting of those functions u whose distributional gradient ∇u is a vector-valued measure, i.e.

$$\|\nabla u\|_{\mathcal{M}} = \sup\left\{\int_{\mathbb{R}^m} u \operatorname{div} v \colon v \in \mathcal{D}(\mathbb{R}^m;\mathbb{R}^m) \text{ and } \|v\|_\infty \leqslant 1\right\} < \infty.$$

Readily $|||u||| := \|u\|_{L^{1^*}} + \|\nabla u\|_{\mathcal{M}}$ defines a norm on $\mathrm{BV}_{1^*}(\mathbb{R}^m)$, which makes it a Banach space. In view of proposition 2.5, we will use the equivalent norm $\|u\|_{\mathrm{BV}_{1^*}} := \|\nabla u\|_{\mathcal{M}}$.

Definition 2.2. Given a sequence $\{u_j\}$ in $\mathrm{BV}_{1^*}(\mathbb{R}^m)$, we write $u_j \twoheadrightarrow 0$ whenever

(i) $\sup_j \|\nabla u_j\|_{\mathcal{M}} < \infty$,

(ii) $u_j \rightharpoonup 0$ weakly in $L^{1^*}(\mathbb{R}^m)$.

Proposition 2.3. *Let* $\{u_j\}$ *be a sequence in* $\mathrm{BV}_{1^*}(\mathbb{R}^m)$, $u \in L^{1^*}(\mathbb{R}^m)$, *and assume that* $u_j \rightharpoonup u$ *weakly in* $L^{1^*}(\mathbb{R}^m)$. *It follows that*

$$\|\nabla u\|_{\mathcal{M}} \leqslant \liminf_j \|\nabla u_j\|_{\mathcal{M}}. \tag{2.3}$$

Proof. Let $v \in \mathcal{D}(\mathbb{R}^m;\mathbb{R}^m)$ with $\|v\|_\infty \leqslant 1$. Since $\operatorname{div} v \in L^m(\mathbb{R}^m)$ and $u_j \rightharpoonup u$ weakly in $L^{1^*}(\mathbb{R}^m)$ we have, from definition 2.1,

$$\int_{\mathbb{R}^m} u \operatorname{div} v = \lim_j \int_{\mathbb{R}^m} u_j \operatorname{div} v \leqslant \liminf_j \|\nabla u_j\|_{\mathcal{M}}$$

and, taking the supremum over all such v, we conclude that

$$\|\nabla u\|_{\mathcal{M}} \leqslant \liminf_j \|\nabla u_j\|_{\mathcal{M}}.$$

□

The following density result is basic.

PROPOSITION 2.4. *Let* $u \in \mathrm{BV}_{1^*}(\mathbb{R}^m)$. *The following hold:*

(i) *for every* $\varphi \in \mathcal{D}(\mathbb{R}^m)$, $u * \varphi \in \mathrm{BV}_{1^*}(\mathbb{R}^m)$ *and*

$$\|\nabla(u*\varphi)\|_{L^1} \leqslant \|\nabla u\|_{\mathcal{M}} \|\varphi\|_{L^1};$$

(ii) *if* $\{\varphi_k\}$ *is an approximate identity, then*

$$u - u*\varphi_k \twoheadrightarrow 0 \quad \textit{and} \quad \lim_k \|\nabla(u*\varphi_k)\|_{L^1} = \|\nabla u\|_{\mathcal{M}};$$

(iii) *there exists a sequence* $\{u_j\}$ *in* $\mathcal{D}(\mathbb{R}^m)$ *such that*

$$u - u_j \twoheadrightarrow 0 \quad \textit{as well as} \quad \lim_j \|\nabla u_j\|_{L^1} = \|\nabla u\|_{\mathcal{M}}.$$

Proof. We note that (2.1) yields $u * \varphi \in L^{1^*}$. We have

$$\begin{aligned}
\int_{\mathbb{R}^m} |\nabla(u*\varphi)|(x)\,\mathrm{d}x &= \int_{\mathbb{R}^m} |\varphi * \nabla u|(x)\,\mathrm{d}x \\
&= \int_{\mathbb{R}^m} \left| \int_{\mathbb{R}^m} \varphi(x-y)\,\mathrm{d}\nabla u(y) \right| \mathrm{d}x \\
&\leqslant \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} |\varphi(x-y)|\,\mathrm{d}\|\nabla u\|(y)\,\mathrm{d}x \\
&= \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^m} |\varphi(x-y)|\,\mathrm{d}x \right) \mathrm{d}\|\nabla u\|(y) \\
&= \|\nabla u\|_{\mathcal{M}} \|\varphi\|_{L^1}, \qquad (2.4)
\end{aligned}$$

which shows proposition 2.4(i).

Let $\{\varphi_k\}$ be an approximate identity. From proposition 2.4(i), we obtain

$$\|\nabla(u*\varphi_k)\|_{\mathcal{M}} = \int_{\mathbb{R}^m} |\nabla(u*\varphi_k)|(x)\,\mathrm{d}x \leqslant \|\nabla u\|_{\mathcal{M}} \|\varphi_k\|_{L^1} = \|\nabla u\|_{\mathcal{M}}. \qquad (2.5)$$

Since $u*\varphi_k \to u$ in $L^{1^*}(\mathbb{R}^m)$, then, in particular, $u*\varphi_k \rightharpoonup u$ weakly in $L^{1^*}(\mathbb{R}^m)$; i.e.

$$\int_{\mathbb{R}^m} f[(u*\varphi_k) - u] \to 0 \quad \text{for every } f \in L^m(\mathbb{R}^m). \qquad (2.6)$$

From (2.5) and (2.6) we obtain that $u - u * \varphi_k \twoheadrightarrow 0$. Moreover, from (2.5) and the lower semicontinuity (2.3) we conclude that $\lim_k \|\nabla(u * \varphi_k)\|_{L^1} = \|\nabla u\|_{\mathcal{M}}$, which shows that proposition 2.4(ii) holds.

In order to establish (iii), we choose a sequence $\{\psi_i\}$ in $\mathcal{D}(\mathbb{R}^m)$ such that

$$\mathbf{1}_{B(0,i)} \leqslant \psi_i \leqslant \mathbf{1}_{B(0,2i)} \quad \text{and} \quad \sup_i \|\nabla \psi_i\|_{L^m} < \infty. \tag{2.7}$$

As usual, let $\{\varphi_k\}$ be an approximate identity. Referring to proposition 2.4(ii) we inductively define a strictly increasing sequence of integers $\{k_j\}$ such that

$$\int_{\mathbb{R}^m} |\nabla(u * \varphi_{k_j})| \leqslant \|\nabla u\|_{\mathcal{M}} + \frac{1}{j}.$$

For each j and i, we observe that

$$|\nabla[(u * \varphi_{k_j})\psi_i]| \leqslant |\psi_i \nabla(u * \varphi_{k_j})| + |(u * \varphi_{k_j})\nabla\psi_i|.$$

For fixed j we infer from (2.7) and the relation $|u * \varphi_{k_j}|^{1^*} \in L^1(\mathbb{R}^m)$ that

$$\begin{aligned}\limsup_i \int_{\mathbb{R}^m} |(u * \varphi_{k_j})\nabla\psi_i| &= \limsup_i \int_{B(0,i)^c} |(u * \varphi_{k_j})\nabla\psi_i| \\ &\leqslant \limsup_i \left(\int_{B(0,i)^c} |u * \varphi_{k_j}|^{1^*} \right)^{1/1^*} \|\nabla\psi_i\|_{L^m} \\ &= 0.\end{aligned}$$

According to the three preceding inequalities we can define inductively a strictly increasing sequence of integers $\{i_j\}$ such that

$$\int_{\mathbb{R}^m} |\nabla[(u * \varphi_{k_j})\psi_{i_j}]| \leqslant \int_{\mathbb{R}^m} |\nabla(u * \varphi_{k_j})| + \frac{1}{j} \leqslant \|\nabla u\|_{\mathcal{M}} + \frac{2}{j}.$$

We set $u_j := (u * \varphi_{k_j})\psi_{i_j}$. In view of proposition 2.3, it only remains to show that $u_j \rightharpoonup u$ weakly in $L^{1^*}(\mathbb{R}^m)$. Given $f \in L^m(\mathbb{R}^m)$, we note that

$$\begin{aligned}&\left| \int_{\mathbb{R}^m} f(u - (u * \varphi_{k_j})\psi_{i_j}) \right| \\ &\qquad \leqslant \int_{\mathbb{R}^m} |f||u - (u * \varphi_{k_j})| + \int_{\mathbb{R}^m} |f||u * \varphi_{k_j}||1 - \psi_{i_j}| \\ &\qquad \leqslant \|f\|_{L^m} \|u - (u * \varphi_{k_j})\|_{L^{1^*}} + \left(\int_{B(0,i_j)^c} |f|^m \right)^{1/m} \|u\|_{L^{1^*}} \|\varphi_{k_j}\|_{L^1}.\end{aligned}$$

The latter tends to zero as $j \to \infty$ and the proof is complete. □

Proposition 2.5 (Gagliardo–Nirenberg–Sobolev inequality). *Let* $u \in \mathrm{BV}_{1^*}(\mathbb{R}^m)$. *We have*

$$\|u\|_{L^{1^*}} \leqslant \kappa_m \|\nabla u\|_{\mathcal{M}}.$$

Proof. Since the norm $\|\cdot\|_{L^{1^*}}$ in $L^{1^*}(\mathbb{R}^m)$ is lower semicontinuous with respect to weak convergence, the result is a consequence of proposition 2.4(iii) and the Gagliardo–Nirenberg–Sobolev inequality for functions in $\mathcal{D}(\mathbb{R}^m)$. □

Proposition 2.6 (compactness). *Let* $\{u_j\}$ *be a bounded sequence in* $\mathrm{BV}_{1^*}(\mathbb{R}^m)$, *i.e.* $\sup_j \|\nabla u_j\|_{\mathcal{M}} < \infty$. *Then there exist a subsequence* $\{u_{j_k}\}$ *of* $\{u_j\}$ *and* $u \in \mathrm{BV}_{1^*}(\mathbb{R}^m)$ *such that* $u_{j_k} - u \twoheadrightarrow 0$.

Proof. Since $\{u_j\}$ is bounded in $\mathrm{BV}_{1^*}(\mathbb{R}^m)$, it is also bounded in $L^{1^*}(\mathbb{R}^m)$ according to proposition 2.5. The conclusion thus immediately follows from the fact that $L^{1^*}(\mathbb{R}^m)$ is a reflexive Banach space whose dual is separable, together with proposition 2.3. □

3. Charges vanishing at infinity

Definition 3.1. A *charge vanishing at infinity* is a linear functional

$$F\colon \mathrm{BV}_{1^*}(\mathbb{R}^m) \to \mathbb{R}$$

such that $\langle u_j, F\rangle \to 0$ whenever $u_j \twoheadrightarrow 0$. The collection of these is denoted by $\mathrm{CH}_0(\mathbb{R}^m)$.

We readily see that $\mathrm{CH}_0(\mathbb{R}^m)$ is a linear space. With $F \in \mathrm{CH}_0(\mathbb{R}^m)$ we associate

$$\|F\|_{\mathrm{CH}_0} := \sup\{\langle u, F\rangle \colon u \in \mathrm{BV}_{1^*}(\mathbb{R}^m) \text{ and } \|\nabla u\|_{\mathcal{M}} \leqslant 1\}.$$

We check that $\|F\|_{\mathrm{CH}_0} < \infty$ for each $F \in \mathrm{CH}_0(\mathbb{R}^m)$ according to proposition 2.6; hence $\|\cdot\|_{\mathrm{CH}_0}$ is a norm on $\mathrm{CH}_0(\mathbb{R}^m)$. Note that $\mathrm{CH}_0(\mathbb{R}^m) \subset \mathrm{BV}_{1^*}(\mathbb{R}^m)^*$ and $\|F\|_{\mathrm{CH}_0} = \|F\|_{(\mathrm{BV}_{1^*})^*}$ whenever $F \in \mathrm{CH}_0(\mathbb{R}^m)$.

Proposition 3.2. $\mathrm{CH}_0(\mathbb{R}^m)[\|\cdot\|_{\mathrm{CH}_0}]$ *is a Banach space.*

Proof. Let $\{F_k\}$ be a Cauchy sequence in $\mathrm{CH}_0(\mathbb{R}^m)$. It follows that $\{F_k\}$ converges in $\mathrm{BV}_{1^*}(\mathbb{R}^m)^*$ to some $F \in \mathrm{BV}_{1^*}(\mathbb{R}^m)^*$ and it remains only to check that F is a charge vanishing at infinity. Let $\{u_j\}$ be a sequence in $\mathrm{BV}_{1^*}(\mathbb{R}^m)$ such that $u_j \twoheadrightarrow 0$ and put $\varGamma := \sup_j \|\nabla u_j\|_{\mathcal{M}}$. Given $\varepsilon > 0$, choose an integer k such that $\|F - F_k\|_{\mathrm{BV}_{1^*}^*} \leqslant \varepsilon$. Observe that, for each j,

$$\begin{aligned} |\langle u_j, F\rangle| &\leqslant |\langle u_j, F_k\rangle| + |\langle u_j, F - F_k\rangle| \\ &\leqslant |\langle u_j, F_k\rangle| + \|F - F_k\|_{\mathrm{BV}_{1^*}^*}\varGamma \\ &\leqslant |\langle u_j, F_k\rangle| + \varepsilon\varGamma. \end{aligned}$$

Thus, $\limsup_j |\langle u_j, F\rangle| \leqslant \varepsilon\varGamma$, and since ε is arbitrary the conclusion follows. □

The following is a justification for the terminology 'vanishing at infinity'.

Proposition 3.3. *Let* $F \in \mathrm{CH}_0(\mathbb{R}^m)$ *and* $\varepsilon > 0$. *Then there exists a compact set* $K \subset \mathbb{R}^m$ *such that* $|\langle u, F\rangle| \leqslant \varepsilon\|\nabla u\|_{\mathcal{M}}$ *whenever* $u \in \mathrm{BV}_{1^*}(\mathbb{R}^m)$ *and* $K \cap \operatorname{supp} u = \varnothing$.

Proof. Let $F \in \mathrm{CH}_0(\mathbb{R}^m)$. Assume, if possible, that there exist $\varepsilon > 0$ and a sequence $\{u_j\}$ in $\mathrm{BV}_{1^*}(\mathbb{R}^m)$ such that $\|\nabla u_j\|_{\mathcal{M}} = 1$, $B(0, j) \cap \operatorname{supp} u_j = \varnothing$, and $|\langle u_j, F\rangle| \geqslant \varepsilon$ for every j. We claim that $u_j \twoheadrightarrow 0$. In order to show this, it suffices to establish that $u_j \rightharpoonup 0$ weakly in $L^{1^*}(\mathbb{R}^m)$. Let $f \in L^m(\mathbb{R}^m)$. Given $\eta > 0$, there exists a compact set $K \subset \mathbb{R}^m$ such that

$$\int_{\mathbb{R}^m \setminus K} |f|^m \leqslant \eta^m.$$

If j is sufficiently large for $K \subset B(0, j)$, then

$$\left|\int_{\mathbb{R}^m} f u_j\right| = \left|\int_{\mathbb{R}^m \setminus K} f u_j\right| \leqslant \left(\int_{\mathbb{R}^m \setminus K} |f|^m\right)^{1/m} \|u_j\|_{L^{1^*}} \leqslant \eta \kappa_m.$$

Thus,

$$\limsup_j \left|\int_{\mathbb{R}^m} f u_j\right| \leqslant \eta \kappa_m$$

and, since η is arbitrary, we infer that

$$\int_{\mathbb{R}^m} f u_j \to 0.$$

This establishes our claim and in turn implies that $\lim_j \langle u_j, F\rangle = 0$, which is a contradiction. □

We now turn to giving the two main examples of charges vanishing at infinity. Given $f \in L^m(\mathbb{R}^m)$ (and recalling that $\mathrm{BV}_{1^*}(\mathbb{R}^m) \subset L^{1^*}(\mathbb{R}^m)$), we define

$$\Lambda(f)\colon\ \mathrm{BV}_{1^*}(\mathbb{R}^m) \to \mathbb{R}\colon u \mapsto \int_{\mathbb{R}^m} uf.$$

Proposition 3.4. *Given $f \in L^m(\mathbb{R}^m)$, we have $\Lambda(f) \in \mathrm{CH}_0(\mathbb{R}^m)$ and*

$$\|\Lambda(f)\|_{\mathrm{CH}_0} \leqslant \kappa_m \|f\|_{L^m}.$$

Thus,

$$\Lambda\colon L^m(\mathbb{R}^m) \to \mathrm{CH}_0(\mathbb{R}^m)$$

is a bounded linear operator.

Proof. Let $\{u_j\}$ be a sequence in $\mathrm{BV}_{1^*}(\mathbb{R}^m)$ such that $u_j \twoheadrightarrow 0$. Then $u_j \rightharpoonup 0$ weakly in $L^{1^*}(\mathbb{R}^m)$, whence $\langle u_j, \Lambda(f)\rangle \to 0$, thereby showing that $\Lambda(f) \in \mathrm{CH}_0(\mathbb{R}^m)$. Given $u \in \mathrm{BV}_{1^*}(\mathbb{R}^m)$, we note that

$$|\langle u, \Lambda(f)\rangle| \leqslant \|u\|_{L^{1^*}} \|f\|_{L^m} \leqslant \kappa_m \|\nabla u\|_{\mathcal{M}} \|f\|_{L^m}$$

so that $\|\Lambda(f)\|_{\mathrm{CH}_0} \leqslant \kappa_m \|f\|_{L^m}$. □

Given $v \in C_0(\mathbb{R}^m; \mathbb{R}^m)$ and $u \in \mathrm{BV}_{1^*}(\mathbb{R}^m)$, we note that v is summable with respect to the measure ∇u. Thus, we may define

$$\Phi(v)\colon\ \mathrm{BV}_{1^*}(\mathbb{R}^m) \to \mathbb{R}\colon u \mapsto -\int_{\mathbb{R}^m} \langle v, \mathrm{d}(\nabla u)\rangle.$$

Proposition 3.5. *Given $v \in C_0(\mathbb{R}^m; \mathbb{R}^m)$, we have $\Phi(v) \in \mathrm{CH}_0(\mathbb{R}^m)$ and*

$$\|\Phi(v)\|_{\mathrm{CH}_0} \leqslant \|v\|_\infty.$$

Thus,

$$\Phi\colon C_0(\mathbb{R}^m; \mathbb{R}^m) \to \mathrm{CH}_0(\mathbb{R}^m)$$

is a bounded linear operator.

Proof. Let $v \in C_0(\mathbb{R}^m;\mathbb{R}^m)$ and let $\{u_j\}$ be a sequence in $\mathrm{BV}_{1^*}(\mathbb{R}^m)$ such that $u_j \twoheadrightarrow 0$. Given $\varepsilon > 0$, we choose $w \in \mathcal{D}(\mathbb{R}^m;\mathbb{R}^m)$ such that $\|w - v\|_\infty \leqslant \varepsilon$. Set $\Gamma = \sup_j \|\nabla u_j\|_{\mathcal{M}}$. We note that

$$|\langle u_j, \Phi(v)\rangle| \leqslant \left| \int_{\mathbb{R}^m} \langle v - w, \mathrm{d}(\nabla u_j)\rangle \right| + \left| \int_{\mathbb{R}^m} u_j \operatorname{div} w \right| \leqslant \varepsilon\Gamma + \left| \int_{\mathbb{R}^m} u_j \operatorname{div} w \right|.$$

Since $\operatorname{supp} \operatorname{div} w$ is compact, we infer that $\operatorname{div} w \in L^m(\mathbb{R}^m)$. Hence,

$$\lim_j \int_{\mathbb{R}^m} u_j \operatorname{div} w = 0.$$

Thus, $\limsup_j |\langle u_j, \Phi(v)\rangle| \leqslant \varepsilon\Gamma$ and, from the arbitrariness of ε, we conclude that $\Phi(v) \in \mathrm{CH}_0(\mathbb{R}^m)$.

Finally, if $u \in \mathrm{BV}_{1^*}(\mathbb{R}^m)$, then

$$|\langle u, \Phi(v)\rangle| = \left| \int_{\mathbb{R}^m} \langle v, \mathrm{d}(\nabla u)\rangle \right| \leqslant \|v\|_\infty \|\nabla u\|_{\mathcal{M}},$$

and thus $\|\Phi(v)\|_{\mathrm{CH}_0} \leqslant \|v\|_\infty$. □

4. Approximation

Let $F \in \mathrm{CH}_0(\mathbb{R}^m)$ and $\varphi \in \mathcal{D}(\mathbb{R}^m)$. Our goal is to define a new charge vanishing at infinity, the convolution of F and φ, denoted by $F * \varphi$, to show that it belongs to the range of Λ (see proposition 3.4), and that it approximates F in the norm $\|\cdot\|_{\mathrm{CH}_0}$. We start by observing that if $u \in \mathrm{BV}_{1^*}(\mathbb{R}^m)$, then $u * \varphi \in \mathrm{BV}_{1^*}(\mathbb{R}^m)$ (see proposition 2.4(i)). Therefore,

$$F * \varphi \colon\ \mathrm{BV}_{1^*}(\mathbb{R}^m) \to \mathbb{R} \colon u \mapsto \langle u * \varphi, F\rangle$$

is a well-defined linear functional.

We now show that $F * \varphi$ is indeed a charge vanishing at infinity, in fact, of the special type $\Lambda(f)$ for some $f \in L^m(\mathbb{R}^m)$. We denote by $\mathcal{R}(\Lambda)$ the range of the operator Λ.

Proposition 4.1. *Let $F \in \mathrm{CH}_0(\mathbb{R}^m)$ and $\varphi \in \mathcal{D}(\mathbb{R}^m)$. It follows that $F * \varphi \in \mathrm{CH}_0(\mathbb{R}^m) \cap \mathcal{R}(\Lambda)$.*

Proof. The restriction of F to $\mathcal{D}(\mathbb{R}^m)$ is a distribution, still denoted by F. Thus, the convolution $F * \varphi$ is associated with a smooth function $f \in C^\infty(\mathbb{R}^m)$ as follows:

$$\langle \psi, F * \varphi\rangle = \int_{\mathbb{R}^m} \psi f \tag{4.1}$$

for every $\psi \in \mathcal{D}(\mathbb{R}^m)$ (see, for example, [5, (6.30*b*)]). We claim that $f \in L^m(\mathbb{R}^m)$. Let $\{\psi_j\}$ be a sequence in $\mathcal{D}(\mathbb{R}^m)$ such that $\|\psi_j\|_{L^{1^*}} \to 0$. Note that

$$\sup_j \|\nabla(\psi_j * \varphi)\|_{\mathcal{M}} = \sup_j \|\nabla(\psi_j * \varphi)\|_{L^1} \leqslant \sup_j \|\psi_j\|_{L^{1^*}} \|\nabla\varphi\|_{L^q} < \infty,$$

where $q = m/(m+1)$, according to Young's inequality. For any $g \in L^m(\mathbb{R}^m)$, we have

$$\int_{\mathbb{R}^m} g(\psi_j * \varphi) = \int_{\mathbb{R}^m} \psi_j (g * \varphi) \to 0,$$

since $g*\varphi \in L^m(\mathbb{R}^m)$ and $\psi_j \rightharpoonup 0$ weakly in L^{1^*}. Therefore, $\psi_j*\varphi \twoheadrightarrow 0$ and, in turn, $\langle \psi_j*\varphi, F\rangle = \langle \psi_j, F*\varphi\rangle \to 0$. This shows that $F*\varphi$ is $\|\cdot\|_{L^{1^*}}$-continuous in $\mathcal{D}(\mathbb{R}^m)$. Since $\mathcal{D}(\mathbb{R}^m)$ is dense in $L^{1^*}(\mathbb{R}^m)$, we infer that $F*\varphi$ can be uniquely extended to a continuous linear functional on $L^{1^*}(\mathbb{R}^m)$. Thus, the Riesz representation theorem yields $f \in L^m(\mathbb{R}^m)$ and, therefore, proposition 3.4 gives $\Lambda(f) \in \mathrm{CH}_0(\mathbb{R}^m)$. It only remains to show that $\Lambda(f) = F*\varphi$, which is equivalent to showing that (4.1) actually holds for every $\psi \in \mathrm{BV}_{1^*}(\mathbb{R}^m)$. To see this, we use proposition 2.4(iii) to obtain a sequence $\{\psi_j\} \in \mathcal{D}(\mathbb{R}^m)$ such that $\psi_j \twoheadrightarrow \psi$. Note that equation (4.1) holds for each ψ_j, and the result follows by noting that

$$\int_{\mathbb{R}^m} \psi_j f \to \int_{\mathbb{R}^m} \psi f \quad \text{and} \quad \langle \psi_j, F*\varphi\rangle = \langle \psi_j*\varphi, F\rangle \to \langle \psi*\varphi, F\rangle = \langle \psi, F*\varphi\rangle,$$

since $\psi_j \rightharpoonup \psi$ weakly in $L^{1^*}(\mathbb{R}^m)$ and $\psi_j*\varphi \twoheadrightarrow \psi*\varphi$. □

It remains to show that $F*\varphi$ is a good approximation of F in $\mathrm{CH}_0(\mathbb{R}^m)$ provided that φ is a good approximation of the identity.

PROPOSITION 4.2. *Let $F \in \mathrm{CH}_0(\mathbb{R}^m)$ and let $\{\varphi_k\}$ be an approximate identity such that each φ_k is even. It follows that*

$$\lim_k \|F - F*\varphi_k\|_{\mathrm{CH}_0(\mathbb{R}^m)} = 0.$$

Proof. In order to simplify the notation we put $F_k = F*\varphi_k$.

Since $F \in \mathrm{CH}_0(\mathbb{R}^m)$, the following holds. For every $\varepsilon > 0$, there are $f_1, \dots, f_J \in L^m(\mathbb{R}^m)$ and positive real numbers $\eta_1, \dots, \eta_J$ such that $|\langle u, F\rangle| \leqslant \varepsilon$ whenever $u \in \mathrm{BV}_{1^*}(\mathbb{R}^m)$, $\|\nabla u\|_{\mathcal{M}} \leqslant 2$ and

$$\left| \int_{\mathbb{R}^m} u f_j \right| \leqslant \eta_j$$

for every $j = 1, \dots, J$. We associate an integer k_j with each $j = 1, \dots, J$ such that

$$\|f_j - f_j*\varphi_k\|_{L^m} \leqslant \frac{\eta_j}{\kappa_m}$$

whenever $k \geqslant k_j$. Now, given $u \in \mathrm{BV}_{1^*}(\mathbb{R}^m)$ with $\|\nabla u\|_{\mathcal{M}} \leqslant 1$, and given $k \geqslant \max\{k_1, \dots, k_J\}$, we infer that $\|\nabla(u - u*\varphi_k)\|_{\mathcal{M}} \leqslant 2$ and, for each $j = 1, \dots, J$,

$$\begin{aligned}
\left| \int_{\mathbb{R}^m} (u - u*\varphi_k) f_j \right| &= \left| \int_{\mathbb{R}^m} u f_j - \int_{\mathbb{R}^m} (u*\varphi_k) f_j \right| \\
&= \left| \int_{\mathbb{R}^m} u f_j - \int_{\mathbb{R}^m} u (f_j*\varphi_k) \right| \\
&= \left| \int_{\mathbb{R}^m} u (f_j - f_j*\varphi_k) \right| \\
&\leqslant \|u\|_{L^{1^*}} \|f_j - f_j*\varphi_k\|_{L^m} \\
&\leqslant \eta_j.
\end{aligned}$$

Therefore,

$$|\langle u, F - F_k\rangle| = |\langle u - u*\varphi_k, F\rangle| \leqslant \varepsilon.$$

Taking the supremum over all such u, we obtain

$$\|F - F_k\| \leqslant \varepsilon$$

whenever $k \geqslant \max\{k_1, \dots, k_J\}$, and the proof is complete. □

COROLLARY 4.3. *$\mathcal{R}(\Lambda)$ is dense in* $\mathrm{CH}_0(\mathbb{R}^m)$.

5. Duality

PROPOSITION 5.1. *The evaluation map*

$$\mathrm{ev}\colon \mathrm{BV}_{1^*}(\mathbb{R}^m) \to \mathrm{CH}_0(\mathbb{R}^m)^*$$

is a bijection.

Proof. Since

$$\langle F, \mathrm{ev}(u)\rangle = \langle u, F\rangle,$$

we readily infer that ev is injective. We now turn to proving that ev is surjective. Let $\alpha \in \mathrm{CH}_0(\mathbb{R}^m)^*$. It follows from proposition 3.4 that $\alpha \circ \Lambda \in (L^m(\mathbb{R}^m))^*$. Thus, there exists $u \in L^{1^*}(\mathbb{R}^m)$ such that

$$\langle \Lambda(f), \alpha\rangle = \int_{\mathbb{R}^m} uf \quad \text{for every } f \in L^m(\mathbb{R}^m). \tag{5.1}$$

Given $v \in \mathcal{D}(\mathbb{R}^m; \mathbb{R}^m)$, we note that the charges $\Phi(v)$ (see proposition 3.5) and $\Lambda(\operatorname{div} v)$ (see proposition 3.4) coincide, according to proposition 2.4(iii), because they trivially coincide on $\mathcal{D}(\mathbb{R}^m)$. Thus,

$$\begin{aligned}\int_{\mathbb{R}^m} u \operatorname{div} v &= \langle \Lambda(\operatorname{div} v), \alpha\rangle \\ &= \langle \Phi(v), \alpha\rangle \\ &\leqslant \|\alpha\|_{\mathrm{CH}_0^*} \|\Phi(v)\|_{\mathrm{CH}_0} \\ &\leqslant \|\alpha\|_{\mathrm{CH}_0^*} \|v\|_\infty\end{aligned}$$

according to proposition 3.5. This proves that $u \in \mathrm{BV}_{1^*}(\mathbb{R}^m)$. It then follows from (5.1) that

$$\langle \Lambda(f), \alpha\rangle = \langle \Lambda(f), \mathrm{ev}(u)\rangle$$

for every $f \in L^m(\mathbb{R}^m)$. Since $\mathcal{R}(\Lambda)$ is dense in $\mathrm{CH}_0(\mathbb{R}^m)$ (by corollary 4.3), we conclude that $\alpha = \mathrm{ev}(u)$. □

REMARK 5.2. Note that the evaluation map is in fact an isomorphism of the Banach spaces $\mathrm{BV}_{1^*}(\mathbb{R}^m)[\|\cdot\|_{\mathrm{BV}_{1^*}}]$ and $\mathrm{CH}_0(\mathbb{R}^m)^*$, according to the open mapping theorem.

6. Proof of the main theorem

THEOREM 6.1. *Let F be a distribution in $\mathbb{R}^m$. The following conditions are equivalent:*

(i) *there exists $v \in C_0(\mathbb{R}^m; \mathbb{R}^m)$ such that $\Phi(v) = F$;*

(ii) *F is a charge vanishing at infinity.*

Proof. That (i) implies (ii) is proven by proposition 3.5. In order to prove that (ii) implies (i) we shall first show that $\mathcal{R}(\Phi)$ is dense in $\mathrm{CH}_0(\mathbb{R}^m)$, and then we will establish that $\mathcal{R}(\Phi)$ is closed in $\mathrm{CH}_0(\mathbb{R}^m)$ as an application of the closed range theorem.

In order to show that $\mathcal{R}(\Phi)$ is dense in $\mathrm{CH}_0(\mathbb{R}^m)$, it suffices to prove the following, according to the Hahn–Banach theorem. Every $\alpha \in \mathrm{CH}_0(\mathbb{R}^m)^*$ whose restriction to $\mathcal{R}(\Phi)$ is zero vanishes identically. Assume $\alpha \in \mathrm{CH}_0(\mathbb{R}^m)$ and $\langle \Phi(v), \alpha \rangle = 0$ for every $v \in C_0(\mathbb{R}^m;\mathbb{R}^m)$. It follows from proposition 5.1 that $\alpha = \mathrm{ev}(u)$ for some $u \in \mathrm{BV}_{1^*}(\mathbb{R}^m)$. Since

$$0 = \langle \Phi(v), \mathrm{ev}(u) \rangle = \int_{\mathbb{R}^m} \langle v, \mathrm{d}(\nabla u) \rangle$$

for every v, we infer that $\nabla u = 0$, and in turn $u = 0$. Thus, $\alpha = \mathrm{ev}(u) = 0$ and the proof that $\mathcal{R}(\Phi)$ is dense in $\mathrm{CH}_0(\mathbb{R}^m)$ is complete.

In order to show that $\mathcal{R}(\Phi)$ is closed in $\mathrm{CH}_0(\mathbb{R}^m)$, it suffices to show that $\mathcal{R}(\Phi^*)$ is closed in $C_0(\mathbb{R}^m;\mathbb{R}^m)^*$, according to the closed range theorem. We first need to identify the adjoint map Φ^* of Φ. Recall that $\mathrm{CH}_0(\mathbb{R}^m)^*$ is identified with $\mathrm{BV}_{1^*}(\mathbb{R}^m)$ through the evaluation map (see proposition 5.1), and that $C_0(\mathbb{R}^m;\mathbb{R}^m)^*$ is identified with $\mathcal{M}(\mathbb{R}^m;\mathbb{R}^m)$. Given $\alpha \in \mathrm{CH}_0(\mathbb{R}^m)^*$, we find $u \in \mathrm{BV}_{1^*}(\mathbb{R}^m)$ such that $\alpha = \mathrm{ev}(u)$. For each $v \in C_0(\mathbb{R}^m;\mathbb{R}^m)$, we have

$$\langle v, \Phi^*(\mathrm{ev}(u)) \rangle = \langle \Phi(v), \mathrm{ev}(u) \rangle = \langle u, \Phi(v) \rangle = -\int_{\mathbb{R}^m} \langle v, \mathrm{d}(\nabla u) \rangle.$$

Thus, $\Phi^* \circ \mathrm{ev} = -\nabla$. Now let $\{\alpha_j\}$ be a sequence in $\mathrm{CH}_0(\mathbb{R}^m)^*$ such that $\{\Phi^*(\alpha_j)\}$ converges to some $\mu \in \mathcal{M}(\mathbb{R}^m;\mathbb{R}^m)$. We ought to prove the existence of $u \in \mathrm{BV}_{1^*}(\mathbb{R}^m)$ such that $\mu = \nabla u$. Find a sequence $\{u_j\}$ in $\mathrm{BV}_{1^*}(\mathbb{R}^m)$ such that $\alpha_j = \mathrm{ev}(u_j)$. Observe that

$$\|\Phi^*(\alpha_j)\|_{\mathcal{M}} = \|(\Phi^* \circ \mathrm{ev})(u_j)\|_{\mathcal{M}} = \|\nabla u_j\|_{\mathcal{M}}.$$

Since $\{\Phi^*(\alpha_j)\}$ is bounded, we infer that $\sup_j \|\nabla u_j\|_{\mathcal{M}} < \infty$. Then there exists a subsequence $\{u_{j_k}\}$ and $u \in \mathrm{BV}_{1^*}(\mathbb{R}^m)$ such that $u - u_{j_k} \twoheadrightarrow 0$ according to proposition 2.6. In particular, for each $v \in \mathcal{D}(\mathbb{R}^m;\mathbb{R}^m)$, we have

$$\begin{aligned}\int_{\mathbb{R}^m} \langle v, \mathrm{d}(\nabla u) \rangle &= -\int_{\mathbb{R}^m} u \operatorname{div} v \\ &= -\lim_k \int_{\mathbb{R}^m} u_{j_k} \operatorname{div} v \\ &= \lim_k \int_{\mathbb{R}^m} \langle v, \mathrm{d}(\nabla u_{j_k}) \rangle.\end{aligned}$$

From this we infer that

$$\int_{\mathbb{R}^m} \langle v, \mathrm{d}(\nabla u) \rangle = \int_{\mathbb{R}^m} \langle v, \mathrm{d}\mu \rangle$$

because μ is the limit of $\{\nabla u_j\}$. Since $\mathcal{D}(\mathbb{R}^m;\mathbb{R}^m)$ is dense in $C_0(\mathbb{R}^m;\mathbb{R}^m)$ we conclude that $\nabla u = \mu$. $\square$

Corollary 6.2. *For every $f \in L^m(\mathbb{R}^m)$, there exists $v \in C_0(\mathbb{R}^m;\mathbb{R}^m)$ such that $\Lambda(f) = \Phi(v)$.*

7. Another proof

Here we provide an alternative approach based on the general theory developed in [3, § 3]. Our space $X = \mathrm{BV}_{1^*}(\mathbb{R}^m)$ is initially equipped with the locally convex linear topology $\mathfrak{T}$, which is the trace on $\mathrm{BV}_{1^*}(\mathbb{R}^m)$ of the weak topology of $L^{1^*}(\mathbb{R}^m)$. We further consider the linearly stable family $\mathcal{C}$ [3, definition 3.1] consisting of those convex sets

$$C_j := \mathrm{BV}_{1^*}(\mathbb{R}^m) \cap \{u\colon \|\nabla u\|_{\mathcal{M}} \leqslant j\}, \quad j = 1, 2, \dots .$$

The corresponding locally convex topology $\mathfrak{T}_{\mathcal{C}}$ on $\mathrm{BV}_{1^*}(\mathbb{R}^m)$ is described in [3, theorem 3.3].

We note that the bounded subsets of $L^{1^*}(\mathbb{R}^m)$ are weakly relatively compact (according to the Banach–Alaoglu theorem [5, theorem 3.15], because $L^{1^*}(\mathbb{R}^m)$ is reflexive) and that the restriction of the weak topology to such subsets is metrizable (because the dual of $L^{1^*}(\mathbb{R}^m)$ is separable [5, theorem 3.8(c)]). We infer from proposition 2.5 that the sets C_j defined above are weakly bounded. Thus, the C_j are $\mathfrak{T}$ relatively compact, the restriction of $\mathfrak{T}$ to C_j is sequential (in fact, metrizable) and, in turn, the C_j are $\mathfrak{T}$ compact according to proposition 2.3.

Next we infer from [3, proposition 3.8(1)] that a sequence $\{u_k\}$ in $\mathrm{BV}_{1^*}(\mathbb{R}^m)$ converges to zero in the sense of $\mathfrak{T}_{\mathcal{C}}$ if and only if it converges to zero in the sense of definition 2.2. Since the restriction of $\mathfrak{T}$ to each C_j is sequential, as noted above, the proof of [3, proposition 3.8(3)] shows that $\mathrm{CH}_0(\mathbb{R}^m) = \mathrm{BV}_{1^*}(\mathbb{R}^m)[\mathfrak{T}_{\mathcal{C}}]^*$.

The strong topology on $\mathrm{CH}_0(\mathbb{R}^m)$, i.e. the topology of uniform convergence on bounded subsets of $\mathrm{BV}_{1^*}(\mathbb{R}^m)[\mathfrak{T}_{\mathcal{C}}]$, is exactly the normed topology considered in proposition 3.2 according to [3, proposition 3.8(2)]. The $\mathfrak{T}$-compactness of the C_j then implies that $\mathrm{CH}_0(\mathbb{R}^m)^* \cong \mathrm{BV}_{1^*}(\mathbb{R}^m)$, via the evaluation map, according to [3, theorem 3.16]. In other words, proposition 5.1 is established in this abstract fashion. The proof of theorem 6.1 remains unchanged.

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