Characterizations of the Existence and Removable Singularities of Divergence-measure Vector Fields

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ABSTRACT. We study the solvability and removable singularities of the equation $\operatorname{div} F = \mu$, with measure data μ, in the class of continuous or L^p vector fields F, where $1 \le p \le \infty$. In particular, we show that, for a signed measure μ, the equation $\operatorname{div} F = \mu$ has a solution $F \in L^\infty(\mathbb{R}^n)$ if and only if $|\mu(U)| \le C\mathcal{H}^{n-1}(\partial U)$ for any open set U with smooth boundary. For non-negative measures μ, we obtain explicit characterizations of the solvability of $\operatorname{div} F = \mu$ in terms of potential energies of μ for $p \ne \infty$, and in terms of densities of μ for continuous vector fields. These existence results allow us to characterize the removable singularities of the corresponding equation $\operatorname{div} F = \mu$ with signed measures μ.

1. INTRODUCTION

The main goal of this paper is to characterize the solvability and removable singularities of the equation

$$\operatorname{div} F = \mu \tag{1.1}$$

with measure data μ and continuous or L^p vector fields F. We deduce sharp necessary and sufficient conditions on the size of removable sets from explicit criteria for the solvability of equation (1.1) and fine properties of divergence-measure fields.

Divergence-measure fields arise naturally in some areas of partial differential equations such as the field of non-linear conservation laws. It is known that entropy solutions $\mathbf{u}(t,x)$ to the system

$$\mathbf{u}_t + \operatorname{div}_x \mathbf{f}(\mathbf{u}) = 0, \quad x \in \mathbb{R}^d,$$

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where $\mathbf{u} : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^m$ and $\mathbf{f} = (\mathbf{f}_1, \dots, \mathbf{f}_m)$ with $\mathbf{f}_i : \mathbb{R}^m \to \mathbb{R}^d$, $i = 1, \dots, m$, belong to some $L^p(\mathbb{R}_+^{d+1})$ space, $1 \le p \le \infty$, or they could even become measures. Moreover, $\mathbf{u}$ satisfies the entropy inequality

$$\eta(\mathbf{u})_t + \operatorname{div}_x \mathbf{q}(\mathbf{u}) \le 0$$

in the distributional sense, for any convex entropy-entropy flux pair $(\eta, \mathbf{q})$ (see [12]). The entropy inequality and the Riesz representation theorem imply that there exists a measure $\mu_{\eta,\mathbf{q}}$ in $\mathbb{R}_+^{d+1}$ such that

$$\operatorname{div}_{t,x}(\eta(\mathbf{u}), \mathbf{q}(\mathbf{u})) = \mu_{\eta,\mathbf{q}}$$

and therefore $F_{\mathbf{u}} := (\eta(\mathbf{u}), \mathbf{q}(\mathbf{u}))$ is a divergence-measure field.

Divergence-measure fields have been investigated by several authors and we refer the readers to the papers Chen-Frid [7, 8], Chen-Torres [9], Chen-Torres-Ziemer [10], Ambrosio-Crippa-Maniglia [3], Šilhavý [26], and the references therein. For a more detailed explanation on the connection and applications of divergence-measure fields to conservation laws we refer the readers to Chen-Torres-Ziemer [11] and the references therein.

We now discuss the solvability results in this paper for the L^p case, $1 \le p \le \infty$. For $p = \infty$, we show in Theorem 3.5 that the solvability of

$$\operatorname{div} F = \mu \quad \text{in } \mathbb{R}^n,\ \mu \text{ signed Radon measure,} \tag{1.2}$$

is strongly connected to the existence of normal traces, over boundaries of sets of finite perimeter, for divergence-measure fields. The trace theorem (see Theorem 2.6) obtained recently in Chen-Torres [9] and Chen-Torres-Ziemer [10] enables us to deal with signed measures in equation (1.2). In particular, we show that (1.2) has a global solution $F \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$ if and only if

$$|\mu(U)| \le C\mathcal{H}^{n-1}(\partial U), \tag{1.3}$$

for any bounded open (or closed) set U with smooth boundary. In fact, the new results obtained in Theorem 3.5 also characterize all signed measures $\mu \in BV(\mathbb{R}^n)^*$.

On the other hand, in Meyers-Ziemer ([22, Theorem 4.7]), the authors showed that property (1.3), with U replaced by balls, characterizes all non-negative measures μ in $BV(\mathbb{R}^n)^*$. In Theorem 3.3, we prove that this condition also characterizes the solvability of equation (1.2) for nonnegative measures. That is, we show that for nonnegative measures μ, equation (1.2) has a global solution $F \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$ if and only if

$$\mu(B_r) \le Cr^{n-1}$$

for any ball B_r.

It is still an open problem to us to characterize the solvability of (1.2) in the class of L^p vector fields F, $1 \le p < \infty$. We refer to [21] for a related and difficult problem involving signed measures (or even complex distributions) that has been solved by Maz'ya and Verbitsky. In this paper, for $p \ne \infty$, we study the solvability of equation (1.2) for non-negative measures μ. Using the Gauss-Green formula in Theorem 2.10 and the boundedness of Riesz transform, we show that the equation

$$\operatorname{div} F = \mu \quad \text{in } \mathbb{R}^n, \ \mu \text{ non-negative measure,} \tag{1.4}$$

has a global solution in $F \in L^p(\mathbb{R}^n, \mathbb{R}^n)$, $1 \le p \le n/(n-1)$, if and only if $\mu \equiv 0$. Moreover, for $n/(n-1) < p < \infty$, the equation (1.4) has a solution if and only if $I_1\mu \in L^p(\mathbb{R}^n)$, where $I_1\mu$ is the Riesz potential of order 1 of μ defined by

$$I_1\mu(x) = \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-1}} \, d\mu(y), \quad x \in \mathbb{R}^n.$$

The situation is more subtle if one looks for continuous vector fields F, i.e., if one considers the equation

$$\operatorname{div} F = \mu \quad \text{in } U, \ F \in C(U). \tag{1.5}$$

In the case $d\mu = f \, dx$ where $f \in L^n_{\mathrm{loc}}(U)$, Brezis and Bourgain [5] showed that the existence of a solution F to (1.5) follows from the closed-range theorem. In a recent paper, motivated by [5], De Pauw and Pfeffer [14] proved that equation (1.5) has a solution if and only if μ is a strong charge, i.e., given $\varepsilon > 0$ and a compact set $K \subset U$, there is $\theta > 0$ such that

$$\int_U \varphi \, d\mu \le \varepsilon \|\nabla\varphi\|_{L^1} + \theta\|\varphi\|_{L^1},$$

for any smooth function φ compactly supported on K. Equivalently, equation (1.5) has a solution if and only if for each compactly supported sequence $\{E_i\}$ of BV sets in U,

$$\frac{\int (\chi_{E_i})^* \, d\mu}{\|E_i\|} \to 0 \quad \text{whenever } |E_i| \to 0,$$

where $(\chi_{E_i})^*$ denotes the precise representation of χ_{E_i} and $\|E_i\|$, $|E_i|$ stand for the perimeter and the Lebesgue measure of E_i respectively (see [14]). In this paper, when μ is a non-negative measure, we show that one can replace the BV sets E_i with balls. That is, the equation (1.5), with non-negative measure μ, has a solution if and only if for each compact set $K \subset U$

$$\lim_{r \to 0} \frac{\mu(B_r(x))}{r^{n-1}} = 0, \quad x \in K \tag{1.6}$$

and the limit is uniform on K.

With these solvability results at hand, we show that singularities of the equation

$$\operatorname{div} F = \mu \quad \text{in } U,\ F \in L^p_{\mathrm{loc}}(U),\ \mu \text{ signed Radon measure}$$

can be removed if and only if they form a set of zero $(1, p')$-capacity, where $p' = p/(p-1)$ for $n/(n-1) < p < \infty$, and of zero Hausdorff measure $\mathcal{H}^{n-1}$ in the case $p = \infty$; see Theorem 5.1 below. In particular, if E is a compact set with $\mathcal{H}^{n-1}(E) > 0$, we find a bounded vector field F on U such that $\operatorname{div} F = 0$ in $U \setminus E$ but not in U. This existence result was also obtained in [15] by a different method. It was also shown recently in [23] that there exists a vector field $F \in L^\infty(U) \cap C^\infty(U \setminus E)$ such that $\operatorname{div} F = 0$ in $U \setminus E$ but not in U.

Finally, the new characterization with balls in (1.6) for the existence of continuous vector fields allows us to show that the Hausdorff dimension of removable sets of the equation

$$\operatorname{div} F = \mu \quad \text{in } U,\ F \in C(U).$$

cannot exceed $n - 1$; see Theorem 5.2.

2. PRELIMINARIES

In this section, we introduce our notation, as well as some definitions and earlier results needed for later development. If U is an open set, $\mathcal{M}(U)$ is the set of all locally finite signed Radon measures in U and $\mathcal{M}_+(U) \subset \mathcal{M}(U)$ consists only of non-negative measures. If $\mu \in \mathcal{M}(U)$, then $|\mu|$ denotes the total variation of μ. The open ball of radius r centered at $x \in \mathbb{R}^n$ will be denoted as $B_r(x)$. Given $1 \le p \le \infty$, we denote by $p' := p/(p-1)$ the conjugate of p. The capacity associated to the Sobolev space $W^{1,p'}(\mathbb{R}^n)$ is defined by

$$\operatorname{cap}_{1,p'}(K) = \inf\left\{ \int_{\mathbb{R}^n} |\nabla\varphi|^{p'}\,\mathrm{d}x \mid \varphi \in C_0^\infty(\mathbb{R}^n),\ \chi_K \le \varphi \le 1 \right\}, \tag{2.1}$$

for each compact set $K \subset \mathbb{R}^n$. In the case $1 \le p' < n$, i.e., $n/(n-1) < p \le \infty$ this capacity is known to be locally equivalent to the capacity

$$\operatorname{Cap}_{1,p'}(K) = \inf\left\{ \int_{\mathbb{R}^n} \varphi^{p'}\,\mathrm{d}x + \int_{\mathbb{R}^n} |\nabla\varphi|^{p'}\,\mathrm{d}x \mid \varphi \in C_0^\infty(\mathbb{R}^n),\ \chi_K \le \varphi \le 1 \right\}.$$

However, $\operatorname{cap}_{1,p'}(\cdot) = 0$ in the case $p' \ge n$, whereas $\operatorname{Cap}_{1,p'}$ is nondegenerate, i.e., $\operatorname{Cap}_{1,p'}(K) > 0$ for any nonempty compact set $K \subset \mathbb{R}^n$ when $p' > n$ (see [2, Proposition 2.6.1]).

Remark 2.1. For $p' = 1$ (i.e., $p = \infty$), $\operatorname{cap}_{1,p'}(E) = 0$ if and only if $\mathcal{H}^{n-1}(E) = 0$ (see [28, Theorem 3.5.5] and [2, Proposition 5.1.5]).

Definition 2.2. Let $1 < p < \infty$. We say that $\mu \in \mathcal{M}_+(\mathbb{R}^n)$ has finite $(1,p)$-energy if

$$\int_{\mathbb{R}^n} [I_1\mu(x)]^p \,\mathrm{d}x < \infty,$$

where I_1 is the Riesz potential of order 1 defined by

$$I_1\mu(x) = \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-1}} \,\mathrm{d}\mu(y).$$

Remark 2.3. Since for any $R > 0$,

$$I_1\mu(x) \geq \int_{B_R(O)} \frac{1}{|x-y|^{n-1}} \,\mathrm{d}\mu(y) \geq \frac{c\mu(B_R(O))}{(|x|+R)^{n-1}}, \tag{2.2}$$

where O is the origin of $\mathbb{R}^n$, we see in the case $1 < p \leq n/(n-1)$ that $\mu \equiv 0$ is the only measure in $\mathcal{M}_+(\mathbb{R}^n)$ that has finite $(1,p)$-energy.

We recall that the space $BV(\mathbb{R}^n)$ consists of all functions $u \in L^1(\mathbb{R}^n)$ such that the distributional gradient ∇u of u is a signed Radon measure with finite total variation in $\mathbb{R}^n$. In what follows, we consider the space $BV(\mathbb{R}^n)$ with the norm $\|u\|_{BV} = |\nabla u|(\mathbb{R}^n) = \int_{\mathbb{R}^n} |\nabla u|$. The space $BV_c^\infty(\mathbb{R}^n)$ consists of all bounded functions in $BV(\mathbb{R}^n)$ with compact support.

Definition 2.4. Let $E \subset \mathbb{R}^n$ be an $\mathcal{L}^n$-measurable subset. We say that E is a *set of finite perimeter* if $\chi_E \in BV(\mathbb{R}^n)$. Thus, $\nabla\chi_E$ is a measure with total variation $|\nabla\chi_E|$. The measure-theoretic interior of a set of finite perimeter E is denoted as E^1 and defined as

$$E^1 := \left\{ y \in \mathbb{R}^n \mid \lim_{r\to 0} \frac{|E \cap B_r(y)|}{|B_r(y)|} = 1 \right\}, \tag{2.3}$$

Definition 2.5. Let $E \subset \mathbb{R}^n$ be a set of finite perimeter. The *reduced boundary* of E, denoted as $\partial^* E$, is the set of all points $y \in \mathbb{R}^n$ such that

(i) $|\nabla\chi_E|(B_r(y)) > 0$ for all $r > 0$;
(ii) The limit $\nu_E(y) := \lim_{r\to 0} \nabla\chi_E(B_r(y))/(|\nabla\chi_E|(B_r(y)))$ exists and $|\nu_E(y)| = 1$.

The unit vector, $\nu_E(y)$, is called the measure-theoretic interior unit normal to E at y. We have $|\nabla\chi_E| = \mathcal{H}^{n-1} \llcorner \partial^* E$ (see [4, Theorem 3.59]).

The following Gauss-Green formula for bounded divergence-measure vector fields was proven in Chen-Torres [9] and Chen-Torres-Ziemer [10] (see also Šilhavý [26]).

Theorem 2.6. *Let $F \in L^\infty(U,\mathbb{R}^n)$ with* $\operatorname{div} F = \mu$ *for some signed Radon measure $\mu \in \mathcal{M}(U)$. Then, for every bounded set of finite perimeter $E \Subset U$, there exist functions $\mathcal{F}_i \cdot \nu \in L^\infty(\partial^* E)$ and $\mathcal{F}_e \cdot \nu \in L^\infty(\partial^* E)$ such that*

$$\int_{E^1} \operatorname{div} F = \int_{\partial^* E} (\mathcal{F}_i \cdot \nu)(y)\,\mathrm{d}\mathcal{H}^{n-1}(y)$$

and

$$\int_{E^1 \cup \partial^* E} \operatorname{div} F = \int_{\partial^* E} (\mathcal{F}_e \cdot \nu)(y)\,\mathrm{d}\mathcal{H}^{n-1}(y).$$

Moreover, $\|\mathcal{F}_i \cdot \nu\|_{L^\infty(\partial^* E)} \le \|F\|_\infty$ and $\|\mathcal{F}_e \cdot \nu\|_{L^\infty(\partial^* E)} \le \|F\|_\infty$.

A proof of the following result can be found in Chen-Frid [7, Proposition 3.1].

Theorem 2.7. *Let $F \in L^\infty(U,\mathbb{R}^n)$ with* $\operatorname{div} F = \mu$ *for some signed Radon measure $\mu \in \mathcal{M}(U)$. Then, $|\mu| \ll \mathcal{H}^{n-1}$ in U; that is, if $B \subset U$ is a Borel set satisfying $\mathcal{H}^{n-1}(B) = 0$, then $|\mu|(B) = 0$.*

The L^p analogue of the above theorem is stated as follows.

Theorem 2.8. *Let $F \in L^p(U,\mathbb{R}^n)$, $1 < p < \infty$, with* $\operatorname{div} F = \mu$ *for some signed Radon measure $\mu \in \mathcal{M}(U)$. Then, $|\mu| \ll \operatorname{Cap}_{1,p'}$ in U; that is, if $B \subset U$ is a Borel set satisfying $\operatorname{Cap}_{1,p'}(B) = 0$, then $|\mu|(B) = 0$.*

Proof. Let $K \subset U$ be a compact set such that $\operatorname{Cap}_{1,p'}(K) = 0$. It is enough to show that $\mu(K) = 0$. By Corollary 2.39 in [18], we see that $\operatorname{cap}_{1,p'}(K,O) = 0$ for any open set O containing K, where

$$\operatorname{cap}_{1,p'}(K,O) = \inf\left\{ \int_{\mathbb{R}^n} |\nabla\varphi|^{p'}\,\mathrm{d}x \mid \varphi \in C_0^\infty(O),\ \chi_K \le \varphi \le 1 \right\}.$$

Let $\{O_j\}$ be a sequence of open sets containing K such that $O_1 \supset O_2 \supset \cdots$ and $\bigcap_j O_j = K$. Since $\operatorname{cap}_{1,p'}(K,O_j) = 0$ for each j, we can find $\varphi_j \in C_0^\infty(O_j)$, $0 \le \varphi_j \le 1$, $\varphi_j \equiv 1$ in a neighborhood of K such that

$$\|\nabla\varphi_j\|_{L^{p'}} \to 0 \quad \text{as } j \to \infty. \tag{2.4}$$

Since $\operatorname{div} F = \mu$, we have $\mu(K) + \int_{U \setminus K} \varphi_j\,\mathrm{d}\mu = \int_U F \cdot \nabla\varphi_j \le \|F\|_{L^p(U)}\,\|\nabla\varphi_j\|_{L^{p'}}$. Thus

$$|\mu(K)| \le |\mu|(O_j \setminus K) + \|F\|_{L^p(U)}\,\|\nabla\varphi_j\|_{L^{p'}} \to 0$$

as $j \to \infty$ by (2.4). □

We refer to Theorem 3.96 in [4] for a proof of the following chain rule.

Theorem 2.9. *Let $u \in BV(U)$ and $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function satisfying $f(0) = 0$ if $|U| = \infty$. Then, $v = f \circ u$ belongs to $BV(U)$ and*

$$|\nabla v| \le M|\nabla u| \quad \text{where } M = \sup_z |\nabla f(z)|_\infty.$$

We also need the following Gauss-Green formula for L^1_{loc} vector fields (see [13, Theorem 5.4]):

Theorem 2.10. *Let $F \in L^1_{\mathrm{loc}}(U, \mathbb{R}^n)$ such that $\operatorname{div} F = \mu$, for some signed Radon measure $\mu \in \mathcal{M}(\mathbb{R}^n)$. Then for each $x \in \mathbb{R}^n$ and for* almost every $r > 0$,

$$\int_{B_r(x)} \operatorname{div} F = \int_{\partial B_r(x)} F(y) \cdot \nu(y) \, \mathrm{d}\mathcal{H}^{n-1}(y).$$

In this Gauss-Green formula, $F(y) \cdot \nu(y)$ denotes the standard inner product of F (which is defined on $\partial B_r(x)$ for almost every $r > 0$) and the outer unit normal $\nu(y)$.

The next result, originally due to Gustin [19], is known as the boxing inequality (see [29, Chapter 5, Lemma 5.9.3]).

Theorem 2.11. *Let $n > 1$ and $0 < \tau < \frac{1}{2}$. Suppose E is a set of finite perimeter such that* $\limsup_{r \to 0} |E \cap B_r(x)|/|B_r(x)| > \tau$ *whenever $x \in E$. Then there exists a constant $C = C(r, n)$ and a sequence of closed balls $\bar{B}_{r_i}(x_i)$ with $x_i \in E$ such that*

$$E \subset \bigcup_{i=1}^{\infty} \bar{B}_{r_i}(x_i)$$

and

$$\sum_{i=1}^{\infty} r_i^{n-1} \le C\mathcal{H}^{n-1}(\partial^* E).$$

Remark 2.12. If E is an open set, the proof of Theorem 2.11 actually shows that we can take $\tau = \frac{1}{2}$. Moreover, the covering $\{B_{r_i}\}$ can be chosen in such a way that

$$\frac{|B_{r_i/5} \cap E|}{|B_{r_i/5}|} = \frac{1}{2}.$$

This fact will be used in the proof of Theorem 4.4 below.

3. The L^p Case

In this section, we study the solvability of the equation $\operatorname{div} F = \mu$, where $\mu \in \mathcal{M}_+(\mathbb{R}^n)$ and $F \in L^p(\mathbb{R}^n, \mathbb{R}^n)$ with $1 \le p \le \infty$. Our first result shows that $n/(n-1)$ is a critical exponent of the problem since for $1 \le p \le n/(n-1)$ the equation admits no solution unless $\mu \equiv 0$.

Theorem 3.1. *Assume that* $1 \le p \le n/(n-1)$. *If* $F \in L^p(\mathbb{R}^n, \mathbb{R}^n)$ *satisfies* $\operatorname{div} F = \mu$, *for some* $\mu \in \mathcal{M}_+(\mathbb{R}^n)$, *then* $\mu \equiv 0$.

Proof. The case $p = 1$ will be treated slightly different from the case $1 < p \le n/(n-1)$. We first note that Fubini's theorem implies, for $1 < p < \infty$:

$$I_1\mu(x) = (n-1)\int_0^\infty \frac{\mu(B_r(x))}{r^n}\,\mathrm{d}r = (n-1)\lim_{\varepsilon\to 0^+}\int_\varepsilon^\infty \frac{\mu(B_r(x))}{r^n}\,\mathrm{d}r.$$

Thus by Theorem 2.10 and polar coordinates for L^1 functions we have

$$\begin{aligned} I_1\mu(x) &= (n-1)\lim_{\varepsilon\to 0^+}\int_\varepsilon^\infty \frac{1}{r^n}\int_{\partial B_r(x)} F\cdot\nu\,\mathrm{d}\mathcal{H}^{n-1}(y)\,\mathrm{d}r \\ &= (1-n)\lim_{\varepsilon\to 0^+}\int_\varepsilon^\infty \int_{\partial B_r(x)} F(y)\cdot\frac{(x-y)}{|x-y|^{n+1}}\,\mathrm{d}\mathcal{H}^{n-1}(y)\,\mathrm{d}r \\ &= (1-n)\lim_{\varepsilon\to 0^+}\int_{|x-y|>\varepsilon} F(y)\cdot\frac{(x-y)}{|x-y|^{n+1}}\,\mathrm{d}y. \end{aligned}$$

The last limit is known to exist for almost every $x \in \mathbb{R}^n$ and is equal to

$$c(n)\sum_{j=1}^n \mathcal{R}_j f_j(x),$$

where $F = (f_1, f_2, \dots, f_n)$ and $\mathcal{R}_j f_j$ denotes the j^{th}-Riesz transform of the function f_j (see [27, formula (5) on page 57]). Moreover, since

$$\|\mathcal{R}_j f\|_{L^p} \le C\|f\|_{L^p}$$

for $1 < p < \infty$, and

$$\|\mathcal{R}_j f\|_{L^{1,\infty}} \le C\|f\|_{L^1}$$

(see [27, Chapter II]), we conclude that

$$\|I_1\mu\|_{L^p} = c(n)\Big\|\sum_{j=1}^n \mathcal{R}_j f_j(x)\Big\|_{L^p} \le C\|F\|_{L^p} < +\infty \tag{3.1}$$

for $1 < p < \infty$, and

$$\|I_1\mu\|_{L^{1,\infty}} = c(n)\Big\|\sum_{j=1}^n \mathcal{R}_j f_j(x)\Big\|_{L^{1,\infty}} \le C\|F\|_{L^1} < +\infty. \tag{3.2}$$

Here $L^{1,\infty}$ is the weak L^1 space defined as

$$L^{1,\infty} = \{f \mid f \text{ measurable on } \mathbb{R}^n,\ \|f\|_{L^{1,\infty}} < \infty\},$$

where

$$\|f\|_{L^{1,\infty}} = \sup_{t>0} t|\{x \in \mathbb{R}^n : |f(x)| > t\}|.$$

Thus, in view of (2.2), (3.1) and (3.2), μ must be identically zero. $\square$

We next consider the case where $1 < p' < n$, i.e., $n/(n-1) < p < \infty$.

Theorem 3.2. *Assume that $n/(n-1) < p < \infty$. If $F \in L^p(\mathbb{R}^n, \mathbb{R}^n)$ satisfies* $\operatorname{div} F = \mu$, *for some $\mu \in \mathcal{M}_+(\mathbb{R}^n)$, then μ has finite $(1, p)$-energy. Conversely, if $\mu \in \mathcal{M}_+(\mathbb{R}^n)$ has finite $(1, p)$-energy, then there is a vector field $F \in L^p(\mathbb{R}^n, \mathbb{R}^n)$ such that* $\operatorname{div} F = \mu$.

Proof. The necessity part follows from (3.1) in the proof of the Theorem 3.1. To prove the sufficiency part, we consider the space $w^{1,p'}$ defined as the completion of $\mathcal{D}(\mathbb{R}^n)$ under the Dirichlet norm $\|\nabla u\|_{L^{p'}}$, where $p' = p/(p-1)$. In the case $1 < p' < n$, one has $w^{1,p'} = h^{1,p'}$, where $h^{1,p'}$ is the completion of $\mathcal{D}(\mathbb{R}^n)$ under the norm $\|\Delta^{1/2} u\|_{L^{p'}}$ (see [20, Section 7.1.2, Theorem 2]). Moreover, by Sobolev's imbedding theorem, in this case $w^{1,p'}$ can also be realized as

$$w^{1,p'} = \{u \in L^{np'/(n-p')}(\mathbb{R}^n) \mid \nabla u \in L^{p'}(\mathbb{R}^n)\}.$$

Let $X = w^{1,p'}$ and $Y = L^{p'}(\mathbb{R}^n, \mathbb{R}^n)$. We define an operator $A : X \to Y$ by

$$A(u) = -\nabla u, \quad u \in X.$$

Obviously, $\|Au\|_Y = \|u\|_X$ and in particular A is a bounded and injective linear operator. Thus its adjoint A^* is surjective, where

$$A^* : Y^* \to X^*,$$

with $Y^* = L^p(\mathbb{R}^n, \mathbb{R}^n)$ and $X^* = (w^{1,p'})^*$. We now note that for any $u \in \mathcal{D}(\mathbb{R}^n)$,

$$\langle A^* F, u \rangle_{X^*,X} = \langle F, Au \rangle_{Y^*,Y} = -\int_{\mathbb{R}^n} F \cdot \nabla u,$$

which implies that

$$A^* F = \operatorname{div} F \quad \text{in } \mathcal{D}'(\mathbb{R}^n).$$

In our situation, since $\mu \in \mathcal{M}_+(\mathbb{R}^n)$ has finite $(1, p)$-energy, using the pointwise estimate

$$u(x) \le C I_1(|\nabla u|)(x), \quad u \in \mathcal{D}(\mathbb{R}^n),$$

(see [27, formula (18) on page 125]), we deduce that

$$\begin{aligned}\left|\int_{\mathbb{R}^n} u\,\mathrm{d}\mu\right| &\le C\int_{\mathbb{R}^n} I_1(|\nabla u|)\,\mathrm{d}\mu\\ &= C\int_{\mathbb{R}^n} |\nabla u| I_1(\mu)\,\mathrm{d}x\\ &\le C\Big(\int_{\mathbb{R}^n} |\nabla u|^{p'}\,\mathrm{d}x\Big)^{1/p'}\end{aligned}$$

by Hölder's inequality. Thus, by density $\mu \in X^*$ and this yields the desired result since A^* is surjective. □

The analogue of Theorem 3.2 for the case $F \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$ is the following theorem.

Theorem 3.3. *Let $\mu \in \mathcal{M}_+(\mathbb{R}^n)$ with the property*

$$\mu(B_r(x)) \le Mr^{n-1}, \quad \text{for all } r > 0,\ x \in \mathbb{R}^n, \tag{3.3}$$

for some constant M independent of x and r. Then there exists $F \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$ satisfying $\operatorname{div} F = \mu$. *Conversely, if $F \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$ is such that* $\operatorname{div} F = \mu$ *for some $\mu \in M_+(\mathbb{R}^n)$, then μ has the property* (3.3).

Proof. For the necessity part, if $F \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$ satisfies $\operatorname{div} F = \mu$ then, for any ball $B_r(x)$, we have by Gauss-Green formula (Theorem 2.6),

$$\mu(B_r(x)) = \int_{B_r(x)} \operatorname{div} F = \int_{\partial B_r(x)} (\mathcal{F}_i \cdot \nu)(y)\,\mathrm{d}\mathcal{H}^{n-1}.$$

Now since $\|\mathcal{F}_i \cdot \nu\|_{L^\infty(\partial B_r(x))} \le \|F\|_{L^\infty}$ we obtain

$$\mu(B_r(x)) \le C(n)\|F\|_{L^\infty} r^{n-1},$$

as desired.

To prove the sufficiency part, as before we consider the space $w^{1,1}$ defined as the completion of $\mathcal{D}(\mathbb{R}^n)$ under the norm $\|\nabla u\|_{L^1}$. Let $X = w^{1,1}$, $Y = L^1(\mathbb{R}^n, \mathbb{R}^n)$ and let $A : w^{1,1} \to L^1(\mathbb{R}^n, \mathbb{R}^n)$ be an operator defined by

$$A(u) = -\nabla u.$$

Since $\|Au\|_Y = \|u\|_X$, we see that A is bounded and injective. Thus its adjoint A^* is surjective, where

$$A^* : L^\infty(\mathbb{R}^n, \mathbb{R}^n) \to (w^{1,1})^*$$

is given by

$$A^* F = \operatorname{div} F \quad \text{in } \mathcal{D}'(\mathbb{R}^n).$$

Therefore, to conclude the proof of the theorem it is enough to show that if μ satisfies property (3.3), then $\mu \in (w^{1,1})^*$, i.e.,

$$\left| \int_{\mathbb{R}^n} u \, d\mu \right| \leq C \int_{\mathbb{R}^n} |\nabla u| \, dx,$$

for all $u \in \mathcal{D}(\mathbb{R}^n)$. This fact can be proved using the boxing inequality (Theorem 2.11) and the coarea formula as it can be found in [22, Theorem 4.7]. Thus the proof of the theorem is completed. $\square$

Lemma 3.4. *$BV_c^\infty(\mathbb{R}^n)$ is dense in $BV(\mathbb{R}^n)$.*

Proof. We will prove that $BV_c^\infty(\mathbb{R}^n)$ is dense in $BV(\mathbb{R}^n)$ by showing that BV_c is dense in BV and BV_c^∞ is dense in BV_c. We let $u \in BV(\mathbb{R}^n)$ and $\varphi_k \in C_0^\infty(\mathbb{R}^n)$ a sequence of smooth functions satisfying:

$$0 \leq \varphi_k \leq 1, \ \varphi_k \equiv 1 \text{ on } B_k(0), \ \varphi_k \equiv 0 \text{ on } B_{2k}(0), \text{ and } \quad |\nabla \varphi_k| \leq \frac{C}{k}. \tag{3.4}$$

The product rule for BV functions gives that $\varphi_k u \in BV(\mathbb{R}^n)$ and $\nabla(\varphi_k u) = \varphi_k \nabla u + u \nabla \varphi$ (as measures). Thus

$$\begin{aligned} \int_{\mathbb{R}^n} |\nabla(\varphi_k u - u)| &= \int_{\mathbb{R}^n} |\varphi_k \nabla u - \nabla u + u \nabla \varphi_k| \\ &\leq \int_{\mathbb{R}^n} |\varphi_k - 1| \, |\nabla u| + \int_{\mathbb{R}^n} |u| \, |\nabla \varphi_k|. \end{aligned} \tag{3.5}$$

We let $k \to \infty$ in (3.5) and use (3.4) and the dominated convergence theorem to conclude

$$\lim_{k \to \infty} \int_{\mathbb{R}^n} |\nabla(\varphi_k u - u)| = 0,$$

which shows that $BV_c(\mathbb{R}^n)$ is dense in $BV(\mathbb{R}^n)$. For $u \in BV_c^+$ we define

$$u_k := u \wedge k, \quad k = 1, 2, \ldots .$$

We will show that $u_k \to u$ in $BV(\mathbb{R}^n)$. The coarea formula yields

$$\begin{aligned} \int_{\mathbb{R}^n} |\nabla(u - u_k)| \, dx &= \int_0^\infty \mathcal{H}^{n-1}(\partial^* \{u - u_k > t\}) \, dt \\ &= \int_0^\infty \mathcal{H}^{n-1}(\partial^* \{u - k > t\}) \, dt \\ &= \int_0^\infty \mathcal{H}^{n-1}(\partial^* \{u > k + t\}) \, dt \\ &= \int_k^\infty \mathcal{H}^{n-1}(\partial^* \{u > s\}) \, ds. \end{aligned}$$

Since $\int_0^{\infty} \mathcal{H}^{n-1}(\partial^*\{u > s\})\,\mathrm{d}s < \infty$, the Lebesgue dominated convergence theorem implies that

$$\int_{\mathbb{R}^n} |\nabla(u - u_k)| \to 0 \quad \text{as } k \to \infty. \tag{3.6}$$

If $u \in BV_c$, we write $u = u^+ - u^-$. We define $f_k = u^+ \wedge k$ and $g_k = u^- \wedge k$. Thus $f_k - g_k \in BV_c$ and

$$\begin{aligned}\int_{\mathbb{R}^n} |\nabla(u - (f_k - g_k))| &= \int_{\mathbb{R}^n} |\nabla u^+ - \nabla u^- - \nabla f_k + \nabla g_k| \\ &\le \int_{\mathbb{R}^n} |\nabla(u^+ - f_k)| + \int_{\mathbb{R}^n} |\nabla(u^- - g_k)| \\ &\to 0 \quad \text{as } k \to \infty,\end{aligned}$$

due to (3.6). That completes the proof of the lemma. □

We next proceed to prove an analogue of Theorem 3.3 for signed measures.

Theorem 3.5. *Let $\mu \in \mathcal{M}(\mathbb{R}^n)$. The following are equivalent:*

(i) *There exists a vector field $F \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$ such that* $\operatorname{div} F = \mu$.

(ii) *For any bounded set of finite perimeter E, there exist functions $\mathcal{F}_i \cdot \nu$ and $\mathcal{F}_e \cdot \nu$ in $L^\infty(\partial^* E)$ such that*

$$\mu(E^1) = \int_{\partial^* E} (\mathcal{F}_i \cdot \nu)(y)\,\mathrm{d}\mathcal{H}^{n-1}(y)$$

and

$$\mu(E^1 \cup \partial^* E) = \int_{\partial^* E} (\mathcal{F}_e \cdot \nu)(y)\,\mathrm{d}\mathcal{H}^{n-1}(y).$$

Moreover, $\|\mathcal{F}_i \cdot \nu\|_{L^\infty(\partial^* E)} \le \|F\|_\infty$ and $\|\mathcal{F}_e \cdot \nu\|_{L^\infty(\partial^* E)} \le \|F\|_\infty$.

(iii) *There is a constant C such that*

$$\max\{|\mu(E^1)|, |\mu(\partial^* E)|\} \le C\mathcal{H}^{n-1}(\partial^* E)$$

for any bounded set of finite perimeter E.

(iv) *There is a constant C such that*

$$|\mu(U)| \le C\mathcal{H}^{n-1}(\partial U)$$

for any smooth bounded open (or closed) set U with $\mathcal{H}^{n-1}(\partial U) < +\infty$.

(v) *$\mathcal{H}^{n-1}(A) = 0$ implies $\mu(A) = 0$ for all Borel sets A, and there is a constant C such that, for all $u \in BV_c^\infty(\mathbb{R}^n)$,*

$$|\langle \mu, u \rangle| := \left| \int_{\mathbb{R}^n} u^*\,\mathrm{d}\mu \right| \le C \int_{\mathbb{R}^n} |\nabla u|\,\mathrm{d}x,$$

where u^* is the representative in the class of u that is defined $\mathcal{H}^{n-1}$-almost everywhere.

(vi) $\mu \in BV(\mathbb{R}^n)^*$. *The action of μ on any $u \in BV(\mathbb{R}^n)$ is defined (uniquely) as*

$$\langle \mu, u \rangle := \lim_{k\to\infty} \langle \mu, u_k \rangle,$$

where $u_k \in BV_c^\infty(\mathbb{R}^n)$ converges to u in $BV(\mathbb{R}^n)$. Moreover, if μ is a non-negative measure then, for all $u \in BV(\mathbb{R}^n)$,

$$\langle \mu, u \rangle = \int_{\mathbb{R}^n} u^* \, \mathrm{d}\mu.$$

(vii) *There is a constant C such that*

$$\left| \int_{\mathbb{R}^n} u \, \mathrm{d}\mu \right| \le C \int_{\mathbb{R}^n} |\nabla u| \, \mathrm{d}x$$

for all $u \in C_0^\infty(\mathbb{R}^n)$.

Proof. Suppose (i) holds. If $F \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$ satisfies $\operatorname{div} F = \mu$ then, for any bounded set of finite perimeter E, the Gauss-Green formula (Theorem 2.6) yields,

$$\mu(E^1 \cup \partial^* E) = \int_{E^1 \cup \partial^* E} \operatorname{div} F = \int_{\partial^* E} (\mathcal{F}_e \cdot \nu)(y) \, \mathrm{d}\mathcal{H}^{n-1}(y)$$

and

$$\mu(E^1) = \int_{E^1} \operatorname{div} F = \int_{\partial^* E} (\mathcal{F}_i \cdot \nu)(y) \, \mathrm{d}\mathcal{H}^{n-1}(y),$$

which are the equalities in (ii). The estimates

$$\|\mathcal{F}_e \cdot \nu\|_{L^\infty(\partial^* E)} \le \|F\|_{L^\infty} \quad \text{and} \quad \|\mathcal{F}_i \cdot \nu\|_{L^\infty(\partial^* E)} \le \|F\|_{L^\infty}$$

give

$$|\mu(E^1 \cup \partial^* E)| = |\mu(E^1) + \mu(\partial^* E)| \le \|F\|_{L^\infty} \mathcal{H}^{N-1}(\partial^* E)$$

and

$$|\mu(E^1)| \le \|F\|_{L^\infty} \mathcal{H}^{N-1}(\partial^* E).$$

Therefore,

$$|\mu(\partial^* E)| \le \|F\|_{L^\infty} \mathcal{H}^{N-1}(\partial^* E) + |\mu(E^1)| \le 2\|F\|_{L^\infty} \mathcal{H}^{N-1}(\partial^* E),$$

which gives (iii). We note that (iii)⇒(iv) since for any bounded open (resp. closed) set U with smooth boundary we have $U = U^1$ (resp. $U = U^1 \cup \partial^* U$).

We proceed now to show that (iv)⇒(v). Since (iv) gives $|\mu(B_r(x))| \le Cr^{n-1}$ for every ball $B_r(x)$, a standard convering argument (see e.g., Theorem 2.56 in

[4]) shows that $\mu \ll \mathcal{H}^{n-1}$ (see also Theorem 2.7). We let $u \in BV_c^\infty(\mathbb{R}^n)$ and we consider its positive and negative parts u^+ and u^-. We note that

$$u^+ = f_{\max}(u), \quad \text{where } f_{\max}(x) = \max(0, x)$$

and

$$u^- = f_{\min}(u), \quad \text{where } f_{\min}(x) = \min(0, x).$$

Since $f_{\max}$ and $f_{\min}$ are Lipschitz functions, Theorem 2.9 implies that both u^+ and u^- belong to $BV_c^\infty(\mathbb{R}^n)$. Let ρ_ε be a stardard sequence of mollifiers. We consider the convolutions $\rho_\varepsilon * u^+$ and define $A_t^\varepsilon := \{\rho_\varepsilon * u^+ > t\}$. Since $\rho_\varepsilon * u^+ \in C_0^\infty(\mathbb{R}^n)$, it follows that ∂A_t^ε is smooth for a.e. t and thus, for each ε and almost every t:

$$A_t^\varepsilon = (A_t^\varepsilon)^1. \tag{3.7}$$

Since $\rho_\varepsilon * u^+ \geq 0$ we can now compute:

$$\begin{aligned} \left| \int_{\mathbb{R}^n} \rho_\varepsilon * u^+ \,\mathrm{d}\mu \right| &= \left| \int_0^\infty \mu(A_t^\varepsilon)\,\mathrm{d}t \right| = \left| \int_0^\infty \mu((A_t^\varepsilon)^1)\,\mathrm{d}t \right| \\ &\leq \left| \int_0^\infty C\mathcal{H}^{n-1}(\partial A_t^\varepsilon)\,\mathrm{d}t \right| \\ &= C \int_{\mathbb{R}^n} |\nabla(\rho_\varepsilon * u^+)|\,\mathrm{d}x \\ &\leq C \int_{\mathbb{R}^n} |\nabla u^+|\,\mathrm{d}x \leq C \int_{\mathbb{R}^n} |\nabla u|\,\mathrm{d}x. \end{aligned} \tag{3.8}$$

In the same way we obtain

$$\left| \int_{\mathbb{R}^n} \rho_\varepsilon * u^- \,\mathrm{d}\mu \right| \leq C \int_{\mathbb{R}^n} |\nabla u|\,\mathrm{d}x. \tag{3.9}$$

We let u^* denote the precise representative of u. We have that (see [4, Chapter 3, Corollary 3.80]):

$$\rho_\varepsilon * u \to u^* \quad \mathcal{H}^{n-1}\text{-almost everywhere.} \tag{3.10}$$

From (3.8) and (3.9) we obtain

$$\left| \int_{\mathbb{R}^n} \rho_\varepsilon * u \,\mathrm{d}\mu \right| = \left| \int_{\mathbb{R}^n} (\rho_\varepsilon * u^+ - \rho_\varepsilon * u^-)\,\mathrm{d}\mu \right| \leq 2C \int_{\mathbb{R}^n} |\nabla u|. \tag{3.11}$$

We now let $\varepsilon \to 0$ in (3.11). Since u is bounded, (3.10), Theorem 2.7 and the dominated convergence theorem yield

$$\left| \int_{\mathbb{R}^n} u^* \,\mathrm{d}\mu \right| \leq C \int_{\mathbb{R}^n} |\nabla u|\,\mathrm{d}x, \tag{3.12}$$

which proves (v). From (v) we obtain that the linear operator

$$T(u) := \langle \mu, u \rangle = \int_{\mathbb{R}^n} u^* \, d\mu, \quad u \in BV_c^\infty(\mathbb{R}^n)$$

is continuous and hence it can be uniquely extended, since $BV_c^\infty(\mathbb{R}^n)$ is dense in $BV(\mathbb{R}^n)$ (Lemma 3.4), to the space $BV(\mathbb{R}^n)$. Assume now that μ is non-negative. We take $u \in BV(\mathbb{R}^n)$ and consider the positive and negative parts $(u^*)^+$ and $(u^*)^-$ of the representative u^*. With the notation of Lemma 3.4 we find that the sequence

$$u_k := \varphi_k((u^*)^+ \wedge k)$$

belongs to $BV_c^\infty(\mathbb{R}^n)$ and converges to u in the BV norm. Therefore, the monotone convergence theorem yields

$$T((u^*)^+) = \lim_{k\to\infty} \int_{\mathbb{R}^n} u_k \, d\mu = \int_{\mathbb{R}^n} (u^*)^+ \, d\mu.$$

We proceed in the same way for $(u^*)^-$ and thus we conclude

$$T(u) = T((u^*)^+) - T((u^*)^-) = \int_{\mathbb{R}^n} (u^*)^+ - (u^*)^- \, d\mu = \int_{\mathbb{R}^n} u^* \, d\mu.$$

It is clear that (vi)⇒(vii). We note now that if $\mu \in \mathcal{M}(\mathbb{R}^n)$ satisfies (vii), then μ belongs to the space $(w^{1,1})^*$ defined in Theorem 3.3, and since $A^* : L^\infty(\mathbb{R}^n, \mathbb{R}^n) \to (w^{1,1})^*$ is surjective, we obtain (i); that is, there is a vector field $F \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$ satisfying $\operatorname{div} F = \mu$. □

Remark 3.6. In Meyers-Ziemer ([22, Theorem 4.7]), the authors showed that property (3.3) characterizes all positive measures in $BV(\mathbb{R}^n)^*$. In Theorem 3.3, we have shown that property (3.3) also characterizes the solvability of the equation $\operatorname{div} F = \mu$. Moreover, the results obtained in Theorem 3.5 allow us to characterize even signed measures $\mu \in BV(\mathbb{R}^n)^*$ and in particular include the Meyers-Ziemer result for non-negative measures.

4. The Continuous Case

We begin this section by quoting the following corollary of a result due to De Pauw and Pfeffer [14] (see also [25]).

Theorem 4.1. *Let μ be a signed Radon measure in a nonempty open set $U \subset \mathbb{R}^n$. Then the following properties are equivalent.*

(i) *The equation* $\operatorname{div} F = \mu$ *has a* continuous *solution* $F : U \to \mathbb{R}^n$.
(ii) *Given $\varepsilon > 0$ and a compact set $K \Subset U$, there is $\theta > 0$ such that*

$$\left| \int_U \varphi \, d\mu \right| \le \varepsilon \int_{\mathbb{R}^n} |\nabla \varphi| \, dx + \theta \int_{\mathbb{R}^n} |\varphi| \, dx$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $\operatorname{supp} \varphi \subset K$.

(iii) *For each compactly supported sequence* $\{E_i\}$ *of BV sets in* U,

$$\frac{\int (\chi_{E_i})^* \,\mathrm{d}\mu}{\|E_i\|} \to 0 \quad \text{whenever } |E_i| \to 0.$$

Here $(\chi_{E_i})^*$ *denotes the precise representation of* χ_{E_i}, *whereas* $\|E_i\|$ *and* $|E_i|$ *stand for the perimeter and the Lebesgue measure of* E_i *respectively.*

Our goal in this section is to consider the case where μ is a non-negative measure in the above theorem and simplify property (iii) there by replacing *BV* sets E_i with balls. Thereby, we obtain new criteria for the solvability of $\operatorname{div} F = \mu$ in the class of continuous vector fields F when μ is a non-negative measure. We remark that De Pauw-Pfeffer's result in [14] deals with distributions called *strong charges* and Radon measures are not necessarily strong charges. In order to obtain our main result, Theorem 4.5 below, we first show the following result.

Theorem 4.2. *Let* $U \subset \mathbb{R}^n$ *be an open set and let* μ *be a signed Radon measure in* U. *Suppose that for any compact set* $K \Subset U$,

$$\lim_{\delta\to 0^+} \sup_{x_0\in K} \sup \left\{ \left| \int_{\mathbb{R}^n} u \,\mathrm{d}\mu \right| : u \in C_0^\infty(B_\delta(x_0)),\ \|\nabla u\|_{L^1} \le 1 \right\} = 0.$$

Then, given $\varepsilon > 0$ *and a compact set* $K \Subset U$, *there is* $\theta > 0$ *such that*

$$\left| \int_{\mathbb{R}^n} \varphi \,\mathrm{d}\mu \right| \le \varepsilon \int_{\mathbb{R}^n} |\nabla\varphi| \,\mathrm{d}x + \theta \int_{\mathbb{R}^n} |\varphi| \,\mathrm{d}x \tag{4.1}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n)$ *with* $\operatorname{supp}\varphi \subset K$.

Proof. We let $\varepsilon > 0$ and $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $\operatorname{supp}\varphi \subset K \Subset U$. Let $d(K) = \operatorname{dist}(K, \partial U)$, and set

$$K_{d(K)/2} = \left\{ x_0 \in U \mid \operatorname{dist}(x_0, K) \le \frac{d(K)}{2} \right\}.$$

By hypothesis, there exists $0 < \delta = \delta(\varepsilon) < d(K)/2$ such that for any $x_0 \in K_{d(K)/2}$

$$\left| \int_{\mathbb{R}^n} u \,\mathrm{d}\mu \right| \le \varepsilon \int_{\mathbb{R}^n} |\nabla u| \,\mathrm{d}x, \tag{4.2}$$

for all $u \in C_0^\infty(B_\delta(x_0))$.

Let $\eta \in C_0^\infty(B_1(0))$ be a cut-off function such that $0 \le \eta(x) \le 1$, $\eta(x) = 1$ for $|x| \le \frac{1}{2}$, and $|\nabla\eta(x)| \le C(n)$. We next choose $x_i \in \mathbb{R}^n$, $i = 1, 2, \ldots$, so that $\{x_i\}$ form a cubic lattice with grid distance $\delta/(3\sqrt{n})$ and let $\eta_i(x) = \eta(x - x_i)/\delta$. Note that $\eta_i \in C_0^\infty(B_\delta(x_i))$, $\eta_i(x) = 1$ for $|x - x_i| < \delta/2$, and

$$|\nabla \eta_i| \le \frac{C}{\delta}. \tag{4.3}$$

Let

$$\varphi(x) = \sum_i \eta_i(x), \tag{4.4}$$

where the sum is taken over a finite number of indices i such that

$$K_{d(K)/2} \subset \bigcup_i B_{\delta/2}(x_i).$$

Since each point $x \in K_{d(K)/2}$ is contained in at most $N = N(n)$ balls $B_{\delta/2}(x_i)$, we have that $1 \le \varphi(x) \le N(n)$ on $K_{d(K)/2}$, and by (4.3) and (4.4),

$$|\nabla \varphi(x)| \le \sum_i |\nabla \eta_i(x)| \le \frac{C(n)}{\delta} \quad \text{on } K_{d(K)/2}. \tag{4.5}$$

We now define

$$\xi_i(x) = \frac{\eta_i(x)}{\varphi(x)} \tag{4.6}$$

and note that $\sum_i \xi_i(x) = 1$ on K. Since each $\xi_i\varphi \in C_0^\infty(B_\delta(x_i))$, from (4.2) we obtain

$$\begin{aligned}
\left| \int_{\mathbb{R}^n} \varphi \,\mathrm{d}\mu \right| &= \left| \int_{\mathbb{R}^n} \sum_i \xi_i \varphi \,\mathrm{d}\mu \right| \le \varepsilon \sum_i \int_{\mathbb{R}^n} |\nabla(\xi_i\varphi)| \,\mathrm{d}x \\
&\le \varepsilon \sum_i \int_{\mathbb{R}^n} |\nabla \xi_i|\,|\varphi| \,\mathrm{d}x + \varepsilon \sum_i \int_{\mathbb{R}^n} |\nabla \varphi| \xi_i \,\mathrm{d}x \\
&= \varepsilon \sum_i \int_{\mathbb{R}^n} |\nabla \varphi| \,\mathrm{d}x + \varepsilon \int_{\mathbb{R}^n} |\varphi| \sum_i |\nabla \xi_i| \,\mathrm{d}x.
\end{aligned}$$

On the other hand, using (4.5) and (4.6) we can estimate the last sum by

$$\begin{aligned}
\sum_i |\nabla \xi_i(x)| &\le \sum_i \frac{|\nabla \eta_i(x)|}{\varphi(x)} + \sum_i \frac{|\nabla \varphi(x)| \eta_i(x)}{(\varphi(x))^2} \\
&\le \sum_i \frac{|\nabla \eta_i(x)|}{\varphi(x)} + \frac{|\nabla \varphi(x)|}{\varphi(x)} \\
&\le \frac{C_1(n)}{\delta} + \frac{C_2(n)}{\delta} = \frac{C(n)}{\delta}.
\end{aligned}$$

Hence

$$\left| \int_{\mathbb{R}^n} \varphi \, \mathrm{d}\mu \right| \le \varepsilon \int_{\mathbb{R}^n} |\nabla \varphi| \, \mathrm{d}x + \frac{\varepsilon C(n)}{\delta} \int_{\mathbb{R}^n} |\varphi| \, \mathrm{d}x,$$

which gives inequality (4.1) with $\theta = \varepsilon C(n)/\delta$. This completes the proof of the theorem. ◻

Remark 4.3. Theorem 4.2 holds also for general distributions μ with the same proof.

We next prove the following result.

Theorem 4.4. *Let $U \subset \mathbb{R}^n$ be an open set and let μ be a non-negative measure in U. Suppose that for any compact set $K \Subset U$,*

$$\lim_{\delta \to 0^+} \sup_{x_0 \in K} \sup \left\{ \frac{\mu(B_r(x_0))}{r^{n-1}}, \; 0 < r < \delta \right\} = 0.$$

Then, for any compact set $K \Subset U$,

$$\lim_{\delta \to 0^+} \sup_{x_0 \in K} \sup \left\{ \left| \int u \, \mathrm{d}\mu \right| : u \in C_0^\infty(B_\delta(x_0)), \; \|\nabla u\|_{L^1} \le 1 \right\} = 0.$$

Proof. Let $K \Subset U$ be a compact set. Let $\varepsilon > 0$ and $d(K) = \operatorname{dist}(K, \partial U)$. We define $K_{d(K)/2} = \{x_0 \in U \mid \operatorname{dist}(x_0, K) \le d(K)/2\}$. By hypothesis, there exists $0 < \delta_1 < d(K)/2$ such that

$$\mu(B_{2r}(x_0)) \le \varepsilon r^{n-1}, \tag{4.7}$$

for all $x_0 \in K_{d(K)/2}$ and $0 < r < \delta_1$. Now let $u \in C_0^\infty(B_\delta(x_0))$ with $\|\nabla u\|_{L^1} \le 1$, $\delta < \delta_1/10$ and $x_0 \in K$. We consider u^+ and u^-, the positive and negative parts of u, which are continuous functions. By applying boxing inequality (Theorem 2.11) to the open set $\{u^+ > t\}$ we can find a covering $\{\bar{B}_{r_{i,t}}(x_{i,t})\}$ of $\{u^+ > t\}$ such that

$$\sum_i r_{i,t}^{n-1} \le C(n) \mathcal{H}^{n-1}(\partial^* \{u^+ > t\}), \tag{4.8}$$

where $C(n)$ is independent of t.

Note that since u^+ is compactly supported in $B_\delta(x_0)$, the set $\{u^+ > t\} \subset B_\delta(x_0)$. Also, from the proof of the boxing inequality (see Remark 2.12) we have that the covering $\{\bar{B}_{r_{i,t}}(x_{i,t})\}$ is chosen in such a way that

$$2|B_{r_{i,t}/5}(x_{i,t}) \cap \{u^+ > t\}| = |B_{r_{i,t}/5}(x_{i,t})|.$$

But since $\{u^+ > t\} \subset B_\delta(x_0)$, we conclude that each radius $r_{i,t}$ is less than 10δ. That is, we have $x_{i,t} \in K_{d(K)/2}$ and $r_{i,t} < 10\delta < \delta_1$, which implies, in view of (4.7), that

$$\mu(B_{2r_{i,t}}(x_{i,t})) \le \varepsilon r_{i,t}^{n-1}, \quad \text{for all } i. \tag{4.9}$$

Therefore, from (4.8), (4.9) and the coarea formula we obtain

$$\begin{aligned}
\left| \int_{\mathbb{R}^n} u^+ \, \mathrm{d}\mu \right| &= \left| \int_0^\infty \mu(\{u^+ > t\}) \, \mathrm{d}t \right| \\
&\le \int_0^\infty \sum_i \mu(B_{2r_{i,t}}(x_{i,t})) \, \mathrm{d}t \\
&\le \varepsilon \int_0^\infty \sum_i r_{i,t}^{n-1} \, \mathrm{d}t \\
&\le C(n)\varepsilon \int_0^\infty \mathcal{H}^{n-1}(\partial^* \{u^+ > t\}) \, \mathrm{d}t \\
&= C(n)\varepsilon \int_{\mathbb{R}^n} |\nabla u^+| \, \mathrm{d}x \le C(n)\varepsilon \int_{\mathbb{R}^n} |\nabla u| \, \mathrm{d}x \le C(n)\varepsilon.
\end{aligned} \tag{4.10}$$

In the same way we obtain

$$\left| \int_{\mathbb{R}^n} u^- \, \mathrm{d}\mu \right| \le C(n)\varepsilon \int_{\mathbb{R}^n} |\nabla u^-| \, \mathrm{d}x \le C(n)\varepsilon \int_{\mathbb{R}^n} |\nabla u| \, \mathrm{d}x \le C(n)\varepsilon. \tag{4.11}$$

From (4.10) and (4.11) we conclude

$$\left| \int_{\mathbb{R}^n} u \, \mathrm{d}\mu \right| = \left| \int_{\mathbb{R}^n} (u^+ - u^-) \, \mathrm{d}\mu \right| \le 2C\varepsilon. \tag{4.12}$$

which yields the theorem. □

We can now put together the previous results to obtain the following equivalences.

Theorem 4.5. *Let μ be a non-negative measure on a nonempty open set $U \subset \mathbb{R}^n$. Then the following properties are equivalent.*

(i) *The equation $\operatorname{div} F = \mu$ has a continuous solution $F : U \to \mathbb{R}^n$.*

(ii) *Given $\varepsilon > 0$ and a compact set $K \Subset U$, there is $\theta > 0$ such that*

$$\left| \int_{\mathbb{R}^n} \varphi \, \mathrm{d}\mu \right| \le \varepsilon \int_{\mathbb{R}^n} |\nabla \varphi| \, \mathrm{d}x + \theta \int_{\mathbb{R}^n} |\varphi| \, \mathrm{d}x,$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $\operatorname{supp} \varphi \subset K$.

(iii) *For any compact set $K \Subset U$,*

$$\lim_{\delta\to 0^+} \sup_{x_0\in K} \sup \left\{ \frac{\mu(B_r(x_0))}{r^{n-1}},\ 0<r<\delta \right\} = 0.$$

(iv) *For any compact set $K \Subset U$,*

$$\lim_{\delta\to 0^+} \sup_{x_0\in K} \sup \left\{ \left| \int_{\mathbb{R}^n} u \, \mathrm{d}\mu \right| : u \in C_0^\infty(B_\delta(x_0)),\ \|\nabla u\|_{L^1} \le 1 \right\} = 0.$$

Proof. By Theorem 4.1 we have (i) $\iff$ (ii). Thus it is enough to show that (ii)$\Rightarrow$(iii) since by Theorem 4.4 and Theorem 4.2 we have (iii)$\Rightarrow$(vi) and (vi)$\Rightarrow$(ii). To this end, let $\varepsilon > 0$ and $K \Subset U$. As before, we define $d(K) = \operatorname{dist}(K, \partial U)$ and set

$$K_{d(K)/2} = \left\{ x_0 \in U \mid \operatorname{dist}(x_0, K) \le \frac{d(K)}{2} \right\}.$$

By property (ii), there exists $\theta(\varepsilon) > 0$ such that

$$\left| \int_{\mathbb{R}^n} \varphi \, \mathrm{d}\mu \right| \le \varepsilon \int_{\mathbb{R}^n} |\nabla\varphi| \, \mathrm{d}x + \theta(\varepsilon) \int_{\mathbb{R}^n} |\varphi| \, \mathrm{d}x \tag{4.13}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $\operatorname{supp}\varphi \subset K_{d(K)/2}$. Let $x_0 \in K$ and let $0 < r < \delta$ where $\delta = \min\{d(K)/4, \varepsilon/(2\theta(\varepsilon))\}$. We next choose a cut-off function $\varphi \in C_0^\infty(B_{2r}(x_0))$ with $0 \le \varphi \le 1$, $\varphi = 1$ on $B_r(x_0)$, and $|\nabla\varphi(x)| \le C(n)/r$. Since $\operatorname{supp}\varphi \subset K_{d(K)/2}$, we can use it to "test" (4.13) to obtain

$$\begin{aligned}
\mu(B_r(x_0)) &\le \int_{B_{2r}(x_0)} \varphi \, \mathrm{d}\mu \\
&\le \varepsilon \int_{B_{2r}(x_0)} |\nabla\varphi| \, \mathrm{d}x + \theta(\varepsilon) \int_{B_{2r}(x_0)} \varphi \, \mathrm{d}x \\
&\le \varepsilon C(n) r^{n-1} + \theta(\varepsilon) C(n) r^n \\
&\le \varepsilon C(n) r^{n-1},
\end{aligned}$$

by our choice of δ. Thus, we get (ii) and the theorem is completely proved. $\square$

5. REMOVABLE SINGULARITIES

We give in this section an application of the previous results to the removability of singularities for the equation $\operatorname{div} F = \mu$, for both $F \in L^p_{\mathrm{loc}}$, $n/(n-1) < p \le \infty$, and F continuous. As it turns out, removable sets of such equations can be characterized by the capacity associated to the Sobolev space $W^{1,p'}(\mathbb{R}^n)$ defined in (2.1). To emphasize our next result for $p = \infty$, we recall from Remark 2.1 that $\operatorname{cap}_{1,1}(E) = 0$ if and only if $\mathcal{H}^{n-1}(E) = 0$.

Theorem 5.1. *Let E be a compact set contained in an open set $U \subset \mathbb{R}^n$. Let $\mu \in \mathcal{M}(U)$ such that $\mu(E) = 0$, and let $n/(n-1) < p \leq \infty$. If $\operatorname{cap}_{1,p'}(E) = 0$; then every solution F to*

$$\operatorname{div} F = \mu \quad \text{in } U \setminus E,\ F \in L^p_{\mathrm{loc}}(U) \tag{5.1}$$

is a solution to

$$\operatorname{div} F = \mu \quad \text{in } U,\ F \in L^p_{\mathrm{loc}}(U). \tag{5.2}$$

Conversely, assume there is at least one vector field $\tilde{F}$ that solves (5.2) and suppose that every solution to (5.1) is also a solution to (5.2), then necessarily $\operatorname{cap}_{1.p'}(E) = 0$.

Proof. We prove first the sufficiency part and assume now that $\operatorname{cap}_{1,p'}(E) = 0$. Let F be a solution to (5.1). Thus,

$$\int_U F \cdot \nabla\varphi \, \mathrm{d}x = -\int_U \varphi \, \mathrm{d}\mu, \quad \varphi \in C_0^\infty(U \setminus E). \tag{5.3}$$

Since $\operatorname{cap}_{1,p'}(E) = 0$, we can find a sequence $u_k \in C_0^\infty(U)$ such that $u_k \equiv 1$ on E and $\|\nabla u_k\|_{L^{p'}} \to 0$. Moreover, u_k can be chosen so that $0 \leq u_k \leq 1$, and $u_k \to 0$ pointwise on U, except possibly on a set $\mathcal{N} \subset U$ with $\operatorname{cap}_{1,p'}(\mathcal{N}) = 0$ (see [24]). We need to show that

$$\int_U F \cdot \nabla\psi \, \mathrm{d}x = \int_U \psi \, \mathrm{d}\mu, \quad \text{for all } \psi \in C_0^\infty(U). \tag{5.4}$$

To this end, we approximate ψ by the sequence of functions

$$\psi_k := \psi(1 - u_k) \in C_0^\infty(U \setminus E). \tag{5.5}$$

We have

$$\nabla\psi_k = \nabla\psi(1 - u_k) - \psi\nabla u_k$$

and hence

$$\begin{aligned} \|\nabla\psi_k - \nabla\psi\|_{L^{p'}} &= \| - u_k\nabla\psi - \psi\nabla u_k\|_{L^{p'}} \\ &\leq \|u_k\nabla\psi\|_{L^{p'}} + \|\psi\nabla u_k\|_{L^{p'}} \\ &\to 0. \end{aligned} \tag{5.6}$$

From (5.3) and (5.5) we get, for all k,

$$\int_U F \cdot \nabla\psi_k = \int_U \psi_k \, \mathrm{d}\mu. \tag{5.7}$$

As $k \to \infty$, Hölder's inequality and (5.6) yield

$$\int_U F \cdot \nabla \psi_k \, dx \to \int_U F \cdot \nabla \psi \, dx. \tag{5.8}$$

Moreover, since $\mu = \operatorname{div} F$ on the open set $U \setminus E$, Theorems 2.7 and 2.8 imply that $\mu \ll \mathcal{H}^{n-1}$ on $U \setminus E$ (for $p = \infty$) and $\mu \ll \operatorname{cap}_{1,p'}$ on $U \setminus E$ (for $p < \infty$). In any case, we have $\mu(\mathcal{N}) = 0$ since $\mu(E) = 0$. Thus, $\psi_k \to \psi$ μ-e.a. and the dominated convergence theorem then gives

$$\int_U \psi_k \, d\mu \to \int_U \psi \, d\mu \tag{5.9}$$

as $k \to \infty$. Combining (5.7), (5.8) and (5.9) we obtain (5.4).

We now proceed to prove the necessity part and we consider first the case $p = \infty$. If $\mathcal{H}^{n-1}(E) > 0$, then Frostman's lemma (see [6, Theorem 1 on page 7]) gives the existence of a non-trivial positive measure σ supported on E such that for any ball B_r,

$$\sigma(B_r) \le C r^{n-1}.$$

Thus by Theorem 3.3, there is $F_\sigma \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$ such that $\operatorname{div} F_\sigma = \sigma$. We now let

$$F = \bar{F} + F_\sigma.$$

Then $\operatorname{div} F = \mu$ in $\mathcal{D}'(U \setminus E)$ but $\operatorname{div} F = \mu + \sigma \neq \mu$ in $\mathcal{D}'(U)$, which gives a contradiction. For the case $p \neq \infty$, we assume now that every solution to (5.1) is also a solution to (5.2) but $\operatorname{cap}_{1,p'}(E) > 0$. Since $\operatorname{cap}_{1,p'}(E) > 0$, there is a non-trivial non-negative measure σ supported on E such that σ has finite $(1,p)$-energy (see [2, Theorem 2.5.3]). By Theorem 3.2, there is $F \in L^p(\mathbb{R}^n, \mathbb{R}^n)$ such that $\operatorname{div} F = \sigma$. If we let $F = \bar{F} + F_\sigma$, then as before we obtain a contradiction since $\operatorname{div} F = \mu$ in $\mathcal{D}'(U \setminus E)$ but not in $\mathcal{D}'(U)$. This completes the proof of the theorem. ∎

Theorem 5.2. *Let E be a compact set contained in an open set $U \subset \mathbb{R}^n$. Let $\mu \in \mathcal{M}(U)$ such that $\mu(E) = 0$. If $\mathcal{H}^{n-1}(E) = 0$, then every solution F to*

$$\operatorname{div} F = \mu \quad \text{in } U \setminus E,\ F \in C(U) \tag{5.10}$$

is a solution to

$$\operatorname{div} F = \mu \quad \text{in } U,\ F \in C(U). \tag{5.11}$$

Conversely, assume there is at least one vector field $\bar{F}$ that solves (5.11) and suppose that every solution to (5.10) is also a solution to (5.11); then

$$\mathcal{H}^{n-1+\varepsilon}(E) = 0$$

for any $\varepsilon > 0$. That is, the Hausdorff dimension of E cannot exceed $n-1$.

Proof. The proof of the sufficiency part is the same as that of Theorem 5.1 since $F \in L^\infty_{\text{loc}}(U)$. To prove the necessity part, we let $\varepsilon > 0$ and assume that $\mathcal{H}^{n-1+\varepsilon}(E) > 0$. Then by Frostman's lemma there exists a non-trivial positive measure σ supported on E such that for any ball B_r,

$$\sigma(B_r) \leq Cr^{n-1+\varepsilon}.$$

Thus,

$$\lim_{r\to 0} \frac{\sigma(B_r)}{r^{n-1}} = 0$$

which, in view of Theorem 4.5, implies that there is $F_\sigma \in C(\mathbb{R}^n, \mathbb{R}^n)$ such that $\operatorname{div} F_\sigma = \sigma$. We now let

$$F = \tilde{F} + F_\sigma.$$

Then $\operatorname{div} F = \mu$ in $\mathcal{D}'(U \setminus E)$ but $\operatorname{div} F = \mu + \sigma \neq \mu$ in $\mathcal{D}'(U)$, which gives a contradiction. ◻

Remark 5.3. A similar result on removable singularities was obtained in [23] by a different method, where it is shown that if $\mathcal{H}^{n-1}(E) = 0$ then every solution F to

$$\operatorname{div} F = 0 \text{ in } U \setminus E, \quad F \in L^\infty_{\text{loc}}(U) \cap C^\infty(U \setminus E) \tag{5.12}$$

is a solution to

$$\operatorname{div} F = 0 \text{ in } U, \quad F \in L^\infty_{\text{loc}}(U) \cap C^\infty(U \setminus E). \tag{5.13}$$

Conversely, if $\mathcal{H}^{n-1}(E) > 0$, then there exists a vector field $F \in L^\infty_{\text{loc}}(U) \cap C^\infty(U \setminus E)$ that solves (5.12) but not (5.13). This strengthens our result in Theorem 5.1 for $p = \infty$ in the necessity direction.

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References

[1] D. R. Adams, *A note on Choquet integrals with respect to Hausdorff capacity*, Proc. Function Spaces and Applications, Lund, 1986, Lecture Notes in Math., vol. 1302, Springer, Berlin, 1988, pp. 115–124. MR 942261 (90c:26028)

[2] D. R. Adams and L. I. Hedberg, *Function Spaces and Potential Theory*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 314, Springer-Verlag, Berlin, 1996, ISBN 3-540-57060-8. MR 1411441 (97j:46024)

[3] L. Ambrosio, G. Crippa, and S. Maniglia, *Traces and fine properties of a BD class of vector fields and applications*, Ann. Fac. Sci. Toulouse Math. (6) **14** (2005), 527–561. MR 2188582 (2007b:35040) (English, with English and French summaries)

[4] L. AMBROSIO, N. FUSCO, and D. PALLARA, *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 2000, ISBN 0-19-850245-1. MR 1857292 (2003a:49002)

[5] J. BOURGAIN and H. BREZIS, *On the equation* $\operatorname{div} Y = f$ *and application to control of phases*, J. Amer. Math. Soc. **16** (2003), 393–426 (electronic). MR 1949165 (2004d:35032)

[6] L. CARLESON, *Selected Problems on Exceptional Sets*, Van Nostrand Mathematical Studies, No. 13, D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto, Ont.-London, 1967. MR 0225986 (37 #1576)

[7] G.-Q. CHEN and H. FRID, *Divergence-measure fields and hyperbolic conservation laws*, Arch. Ration. Mech. Anal. **147** (1999), 89–118, http://dx.doi.org/10.1007/s002050050146. MR 1702637 (2000d:35136)

[8] ______, *Extended divergence-measure fields and the Euler equations for gas dynamics*, Comm. Math. Phys. **236** (2003), 251–280, http://dx.doi.org/10.1007/s00220-003-0823-7. MR 1981992 (2004f:35113)

[9] G.-Q. CHEN and M. TORRES, *Divergence-measure fields, sets of finite perimeter, and conservation laws*, Arch. Ration. Mech. Anal. **175** (2005), 245–267, http://dx.doi.org/10.1007/s00205-004-0346-1. MR 2118477 (2005j:28011)

[10] G.-Q. CHEN, M. TORRES, and W. P. ZIEMER, *Gauss-Green theorem for weakly differentiable vector fields, sets of finite perimeter, and balance laws*, Comm. Pure and Applied Math. (2008) (to appear).

[11] ______, *Measure-theoretic analysis and nonlinear conservation laws*, Pure Appl. Math. Q. **3** (2007), 841–879. MR 2351648

[12] C. M. DAFERMOS, *Hyperbolic Conservation Laws in Continuum Physics*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 325, Springer-Verlag, Berlin, 2000, ISBN 3-540-64914-X. MR 1763936 (2001m:35212)

[13] M. DEGIOVANNI, A. MARZOCCHI, and A. MUSESTI, *Cauchy fluxes associated with tensor fields having divergence measure*, Arch. Ration. Mech. Anal. **147** (1999), 197–223, http://dx.doi.org/10.1007/s002050050149. MR 1709215 (2000g:74005)

[14] T. DE PAUW and W. F. PFEFFER, *Distributions for which* $\operatorname{div} v = F$ *has a continuous solution*, Comm. Pure Appl. Math. **61** (2008), 230–260, http://dx.doi.org/10.1002/cpa.20204. MR 2368375

[15] T. DE PAUW, *On the exceptional sets of the flux of a bounded vectorfield*, J. Math. Pures Appl. (9) **82** (2003), 1191–1217. MR 2012808 (2004m:42027)

[16] H. FEDERER and W. P. ZIEMER, *The Lebesgue set of a function whose distribution derivatives are* p*-th power summable*, Indiana Univ. Math. J. **22** (1972/73), 139–158, http://dx.doi.org/10.1512/iumj.1972.22.22013. MR 0435361 (55 #8321)

[17] B. FUGLEDE, *Extremal length and functional completion*, Acta Math. **98** (1957), 171–219, http://dx.doi.org/10.1007/BF02404474. MR 0097720 (20 #4187)

[18] J. HEINONEN, T. KILPELÄINEN, and O. MARTIO, *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1993, ISBN 0-19-853669-0, Oxford Science Publications. MR 1207810 (94e:31003)

[19] W. GUSTIN, *Boxing inequalities*, J. Math. Mech. **9** (1960), 229–239. MR 0123680 (23 #A1004)

[20] V. G. MAZ'JA, *Sobolev Spaces* (1985), xix+486, Translated from the Russian by T. O. Shaposhnikova. MR 817985 (87g:46056)

[21] V. G. MAZ'YA and I. E. VERBITSKY, *The Schrödinger operator on the energy space: boundedness and compactness criteria*, Acta Math. **188** (2002), 263–302, http://dx.doi.org/10.1007/BF02392684. MR 1947894 (2004b:35050)

[22] N. G. MEYERS and W. P. ZIEMER, *Integral inequalities of Poincaré and Wirtinger type for BV functions*, Amer. J. Math. **99** (1977), 1345–1360, http://dx.doi.org/10.2307/2374028. MR 0507433 (58 #22443)

[23] L. MOONENS, *Removable singularities for the equation* div $v = 0$, Real Anal. Exchange (2006), 125–132, 30th Summer Symposium Conference. MR 2323837

[24] P. BARAS and M. PIERRE, *Singularités éliminables pour des équations semi-linéaires*, Ann. Inst. Fourier (Grenoble) **34** (1984), 185–206. MR 743627 (86j:35063) (French, with English summary)

[25] W. F. PFEFFER, *Derivation and Integration*, Cambridge Tracts in Mathematics, vol. 140, Cambridge University Press, Cambridge, 2001, ISBN 0-521-79268-1. MR 1816996 (2001m:26018)

[26] M. ŠILHAVÝ, *Divergence-measure fields and Cauchy's stress theorem*, Rend. Sem. Mat. Univ. Padova **113** (2005), 15–45.

[27] E. M. STEIN, *Singular Integrals and Differentiability Properties of Functions*, Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, N.J., 1970. MR 0290095 (44 #7280)

[28] B. O. TURESSON, *Nonlinear Potential Theory and Wighted Sobolev Spaces*, Lecture Notes in Mathematics, vol. 1736, Springer-Verlag, Berlin, 2000, ISBN 3-540-67588-4. MR 1774162 (2002f:31027)

[29] W. P. ZIEMER, *Weakly Differentiable Functions*, Graduate Texts in Mathematics, vol. 120, Springer-Verlag, New York, 1989, ISBN 0-387-97017-7, Sobolev spaces and functions of bounded variation. MR 1014685 (91e:46046)

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