

Gauss-Green Theorem for Weakly Differentiable Vector Fields, Sets of Finite Perimeter, and Balance Laws

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Abstract

We analyze a class of weakly differentiable vector fields $\mathbf{F} : \mathbb{R}^N \rightarrow \mathbb{R}^N$ with the property that $\mathbf{F} \in L^\infty$ and $\operatorname{div} \mathbf{F}$ is a (signed) Radon measure. These fields are called *bounded divergence-measure fields*. The primary focus of our investigation is to introduce a suitable notion of the normal trace of any divergence-measure field \mathbf{F} over the boundary of an arbitrary set of finite perimeter that ensures the validity of the Gauss-Green theorem. To achieve this, we first establish a fundamental approximation theorem which states that, given a (signed) Radon measure μ that is absolutely continuous with respect to \mathcal{H}^{N-1} on \mathbb{R}^N , any set of finite perimeter can be approximated by a family of sets with smooth boundary essentially from the measure-theoretic interior of the set with respect to the measure $\|\mu\|$, the total variation measure. We employ this approximation theorem to derive the normal trace of \mathbf{F} on the boundary of any set of finite perimeter E as the limit of the normal traces of \mathbf{F} on the boundaries of the approximate sets with smooth boundary so that the Gauss-Green theorem for \mathbf{F} holds on E . With these results, we analyze the Cauchy flux that is bounded by a nonnegative Radon measure over any oriented surface (i.e., an $(N - 1)$ -dimensional surface that is a part of the boundary of a set of finite perimeter) and thereby develop a general mathematical formulation of the physical principle of the balance law through the Cauchy flux. Finally, we apply this framework to the derivation of systems of balance laws with measure-valued source terms from the formulation of the balance law. This framework also allows the recovery of Cauchy entropy flux through the Lax entropy inequality for entropy solutions of hyperbolic conservation laws. © 2008 Wiley Periodicals, Inc.

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1 Introduction

In this paper we analyze a class of weakly differentiable vector fields $\mathbf{F}: \mathbb{R}^N \rightarrow \mathbb{R}^N$ with the property that $\mathbf{F} \in L^\infty$ and $\operatorname{div} \mathbf{F}$ is a Radon measure μ (that is, a signed Borel measure with finite total variation on compact sets). These fields are called *bounded divergence-measure fields*, and the class is denoted by \mathcal{DM}^∞ . The primary focus of our investigation is to introduce a suitable notion of the normal trace of any divergence-measure field over the boundary of an arbitrary set of finite perimeter to obtain a general version of the Gauss-Green theorem.

Clearly, this investigation is closely related to the theory of BV functions in \mathbb{R}^N ; in fact, it would be completely subsumed by the BV theory if the fields were of the form $\mathbf{F} = (F_1, \dots, F_N)$ with each $F_k \in \operatorname{BV}(\mathbb{R}^N)$, since $\operatorname{div} \mathbf{F} = \sum_{k=1}^N \frac{\partial F_k}{\partial x_k}$ would then be a Radon measure (cf. [72, exercise 5.6]). However, in general, the condition $\operatorname{div} \mathbf{F} = \mu$ allows for cancellation, which thus makes the problem more difficult and accordingly more important for applications (see Sections 9–11). For the Gauss-Green theorem in the BV setting, we refer to Burago and Maz'ja [10], Volpert [66], and the references therein. The Gauss-Green theorem for Lipschitz vector fields over sets of finite perimeter was first obtained by De Giorgi [24, 25] and Federer [30, 31]. Also see Evans and Gariepy [29], Lin and Wang [51], and Simon [64].

Some earlier efforts were made on generalizing the Gauss-Green theorem for some special situations of divergence-measure fields, and relevant results can be found in Anzellotti [3] for an abstract formulation when $\mathbf{F} \in L^\infty$ over a set with C^1 boundary and Ziemer [71] for a related problem for $\operatorname{div} \mathbf{F} \in L^\infty$; also see [1, 2, 5, 9, 27, 46, 47, 48, 50, 56, 57, 58]. In Chen and Frid [16, 19], an explicit way to formulate the suitable normal trace over a Lipschitz deformable surface was first observed for $\mathbf{F} \in \mathcal{DM}^\infty$. In particular, it has been proved in [16, 19] that

the normal trace over a Lipschitz deformable surface, oriented by the unit normal vector \mathbf{v} , is determined completely by the neighborhood information from the positive side of the surface oriented by \mathbf{v} and is independent of the information from the other side. This is the primary motivation for our further investigation into divergence-measure fields. Chen and Torres [21] have established the normal trace for any bounded divergence-measure field over a set of finite perimeter E and the corresponding Gauss-Green theorem. One of the main results in this paper is to obtain this normal trace as the limit of the normal traces over the smooth boundaries that approximate the *reduced boundary* $\partial^* E$ of E (see Definition 2.6). In particular, the normal trace is determined completely by the neighborhood information essentially from the measure-theoretic interior of the set (see Theorem 5.2), so that the Gauss-Green theorem holds for any set of finite perimeter.

We recall a very general approach, initiated by Fuglede [39], in which the following result was established: If $\mathbf{F} \in L^p(\mathbb{R}^N; \mathbb{R}^N)$, $1 \leq p < \infty$, is a vector field with $\operatorname{div} \mathbf{F} \in L^p(\mathbb{R}^N)$, then

$$(1.1) \quad \int_E \operatorname{div} \mathbf{F} \, dy = - \int_{\partial^* E} \mathbf{F}(y) \cdot \mathbf{v}(y) \, d\mathcal{H}^{N-1}(y)$$

for “almost all” sets of finite perimeter E where \mathcal{H}^{N-1} is the $(N-1)$ -dimensional Hausdorff measure. The term “almost all” is expressed in terms of a condition that resembles “extremal length,” a concept used in complex analysis and potential theory (cf. [45, 68, 69, 70]). One way of summarizing our work in this paper is to say that we wish to extend Fuglede’s result so that (1.1) holds for *every* set E of finite perimeter. Of course, this requires a suitable notion of the normal trace of \mathbf{F} on $\partial^* E$. This is really the crux of the problem as \mathbf{F} , being only measurable, cannot be redefined on an arbitrary set of dimension $N-1$. The following example illustrates the subtlety of the problem.

Example. Let $N = 2$ and

$$\mathbf{F}(x_1, x_2) = \begin{cases} (1, 0) & \text{if } x_1 > 0, \\ (-1, 0) & \text{if } x_1 < 0. \end{cases}$$

Note that $\operatorname{div} \mathbf{F} = 2\mathcal{H}^1 \llcorner S$ with $S = \{0\} \times \mathbb{R}$, since, for any $\varphi \in C_0^1(\mathbb{R}^N)$,

$$\begin{aligned} \langle \operatorname{div} \mathbf{F}, \varphi \rangle &= - \int_{\mathbb{R}^N} \mathbf{F} \cdot \nabla \varphi = \int_{\{x_1 < 0\}} \varphi_{x_1} - \int_{\{x_1 > 0\}} \varphi_{x_1} \\ &= \int_S \varphi \, d\mathcal{H}^1 + \int_S \varphi \, d\mathcal{H}^1 = 2 \int_S \varphi \, d\mathcal{H}^1. \end{aligned}$$

By letting $E_+ = (0, 1) \times (0, 1)$ and $E_- = (-1, 0) \times (0, 1)$, there exist scalar functions f^+ and f^- , respectively, on $\partial^* E_+$ and $\partial^* E_-$ such that

$$\int_{E_{\pm}} \varphi \operatorname{div} \mathbf{F} + \int_{E_{\pm}} \mathbf{F} \cdot \nabla \varphi = \int_{\partial^* E_{\pm}} \varphi f^{\pm} d\mathcal{H}^1,$$

where

$$f^+ = \begin{cases} 1 & \text{on } \{0\} \times (0, 1), \\ 0 & \text{on } (0, 1) \times (0, 1), \\ -1 & \text{on } \{1\} \times (0, 1), \end{cases} \quad f^- = \begin{cases} 1 & \text{on } \{0\} \times (0, 1), \\ 0 & \text{on } (-1, 0) \times (0, 1), \\ -1 & \text{on } \{-1\} \times (0, 1). \end{cases}$$

However, note that the normal traces f^{\pm} are not opposite on $\{0\} \times (0, 1)$, as one might at first expect.

To achieve the goal of this paper, we first establish a fundamental approximation theorem which states that, given a Radon measure μ on \mathbb{R}^N such that $\mu \ll \mathcal{H}^{N-1}$, any set of finite perimeter can be approximated by a family of sets with smooth boundary essentially from the measure-theoretic interior of the set with respect to the measure $\|\mu\|$ (for example, we may take $\mu = \operatorname{div} \mathbf{F}$). Then we employ this approximation theorem to derive the normal trace of \mathbf{F} on the boundary of any set of finite perimeter as the limit of the normal trace of \mathbf{F} on the smooth boundaries of the approximating sets and then establish the Gauss-Green theorem for \mathbf{F} , which holds for an arbitrary set of finite perimeter.

With these results on divergence-measure fields and sets of finite perimeter, we analyze the Cauchy flux that is bounded by a nonnegative Radon measure σ over an oriented surface (i.e., an $(N - 1)$ -dimensional surface that is a part of the boundary of a set of finite perimeter) and develop a general mathematical formulation of the physical principle of balance law through the Cauchy flux. In the classical setting of the physical principle of balance law, Cauchy [12, 13] first discovered that the flux density is necessarily a linear function of the interior normal (equivalently, the exterior normal) under the assumption that the flux density through a surface depends on the surface solely through the normal at that point. It was shown in Noll [54] that Cauchy’s assumption follows from the balance law. Ziemer [71] provided a first formulation of the balance law for the flux function $\mathbf{F} \in L^\infty$ with $\operatorname{div} \mathbf{F} \in L^\infty$ at the level of generality with sets of finite perimeter. Also see Dafermos [22], Gurtin and Martins [42], and Gurtin and Williams [43]. One of the new features in our formulation is to allow the presence of exceptional surfaces, “shock waves,” across which the Cauchy flux has a jump. When the Radon measure μ reduces to the N -dimensional Lebesgue measure \mathcal{L}^N , the formulation reduces to Ziemer’s formulation in [71], which shows its consistency with the classical setting. We first show that, for a Cauchy flux \mathcal{F} bounded by a measure σ , there

exists a bounded divergence-measure field $\mathbf{F} : \mathbb{R}^N \rightarrow \mathbb{R}^N$, defined \mathcal{L}^N -a.e., such that

$$\mathcal{F}(S) = - \int_S \mathbf{F}(y) \cdot \mathbf{v}(y) d\mathcal{H}^{N-1}(y)$$

for almost any oriented surface S oriented by the interior unit normal \mathbf{v} . Then we employ our results on divergence-measure fields to recover the values of the Cauchy flux on the exceptional surfaces directly via the vector field \mathbf{F} . The value as the normal trace of \mathbf{F} on the exceptional surface is the unique limit of the normal trace of \mathbf{F} on the nonexceptional surfaces, which is defined essentially from the positive side of the exceptional surface oriented by \mathbf{v} . Finally, we apply this general framework to the derivation of systems of balance laws with measure-valued source terms from the mathematical formulation of the physical principle of balance law. We also apply the framework to the recovery of Cauchy entropy flux through the Lax entropy inequality for entropy solutions of hyperbolic conservation laws by capturing entropy dissipation.

We observe the recent important work by Bourgain and Brezis [8] and De Pauw and Pfeffer [28] (see also Phuc and Torres [59]) for the following problem with a different point of view: Find a continuous vector field that solves the equation

$$(1.2) \quad \operatorname{div} \mathbf{F} = \mu \quad \text{in } \Omega$$

for a given Radon measure μ . In the case $d\mu = f dx$ with $f \in L^N_{\text{loc}}(\Omega)$, the existence of a solution \mathbf{F} to (1.2) follows from the closed-range theorem as shown in [8]. It is proved in [28] that equation (1.2) has a continuous weak solution if and only if μ is a strong charge; i.e., given $\varepsilon > 0$ and a compact set $K \subset \Omega$, there is $\theta > 0$ such that

$$\int_{\Omega} \phi d\mu \leq \varepsilon \|\nabla \phi\|_{L^1} + \theta \|\phi\|_{L^1}$$

for any smooth function ϕ compactly supported on K .

The organization of this paper is as follows. In Section 2, we first recall some properties of Radon measures, sets of finite perimeter, and related BV functions, and then we introduce the notion of an oriented surface and develop some basic properties of divergence-measure fields. In Section 3, we develop an alternative way to obtain the Gauss-Green formula for a bounded divergence-measure field over any smooth boundary by a technique that motivates our further development for the general case. In Section 4, we establish a fundamental approximation theorem which states that, given a Radon measure μ on \mathbb{R}^N such that $\mu \ll \mathcal{H}^{N-1}$, any set of finite perimeter can be approximated by a sequence of sets with smooth boundary essentially from the interior of the set with respect to the measure $\|\mu\|$. In Section 5, we introduce the normal trace of a divergence-measure field \mathbf{F} on the boundary $\partial^* E$ of any set of finite perimeter as the limit of the normal traces of \mathbf{F} on the smooth surfaces that approximate $\partial^* E$ essentially from the measure-theoretic

interior of E with respect to the measure $\|\operatorname{div} \mathbf{F}\|$, constructed in Section 4, and then we establish the corresponding Gauss-Green theorem.

In Sections 6 and 7, we further analyze properties of divergence-measure fields, especially showing the representation of the divergence measures of jump sets via the normal traces and the consistency of our normal traces with the classical traces (i.e., values) when the vector field is continuous. In Section 8, we first show that, if the set of finite perimeter E satisfies (8.1) (which is similar to Lewis's *uniformly fat* condition in potential theory [49]), there exists a one-sided approximation to E , and we then show that an open set of finite perimeter is an extension domain for any bounded divergence-measure field. In Section 9, we first introduce a class of Cauchy fluxes that allow the presence of these exceptional surfaces or "shock waves," and we then prove that such a Cauchy flux induces a bounded divergence-measure (vector) field \mathbf{F} so that the Cauchy flux over *every* oriented surface with finite perimeter can be recovered through \mathbf{F} via the normal trace over the oriented surface.

In Section 10, we apply the results established in Sections 3 through 9 to the mathematical formulation of the physical principle of balance law and the rigorous derivation of systems of balance laws with measure-valued source terms from that formulation. Finally, in Section 11, we apply our results to the recovery of Cauchy entropy flux through the Lax entropy inequality for entropy solutions of hyperbolic conservation laws by capturing entropy dissipation.

Added in proof. We recently learned of the paper by Šilhavý [63], which has some overlap with our work. However, the techniques involved are completely different, thus offering the interested reader more depth and insight into the problem. Theorem 3.2 in [63] gives conditions under which the measure μ will vanish on sets of appropriate Hausdorff dimension. This result is established in Lemma 2.25 of the present paper, where the optimal condition is expressed in terms of capacity. Section 4 in [63] begins with several results of the divergence theorem in its general form as a linear functional on the space of Lipschitz functions on $\partial\Omega$ as in [3], and then Šilhavý proceeds to obtain the result, theorem 4.4, which is the counterpart to theorem 2 in Chen and Torres [21] and is related to Theorem 5.2 of this paper, which contains several results of descending generality that also yield theorem 2 in [21]. The methods of the present paper are substantially different and use exclusively the methods of geometric measure theory to obtain Theorem 5.2.

Furthermore, our normal trace of a divergence-measure field on the boundary of a set of finite perimeter is derived as the limit of the normal traces on the smooth boundaries of approximate sets. These approximate sets essentially belong to the measure-theoretical interior of the set of finite perimeter with respect to the divergence measure. Such a geometric interpretation has important applications in conservation laws. The present paper also contains results that are of interest in themselves, such as a completely self-contained proof of the divergence theorem in the case of a C^1 boundary (Theorem 3.3), the fact that the normal trace of a

divergence-measure field is the classic dot product when the field is continuous (see Theorem 7.2), and the fundamental theorem for almost one-sided approximation of sets of finite perimeter (Theorem 4.10). Even though the second part of theorem 5.2 of [21] is essentially the same as Theorem 9.4 of this paper, our results in Sections 9 and 10 highlight the importance of the normal traces of divergence-measure fields, in conservation laws, defined on *every set of finite perimeter* (versus *almost every set*). Also, see [61, 62] for related papers.

2 Radon Measures, Sets of Finite Perimeter, and Divergence-Measure Fields

In this section we first recall some properties of Radon measures, sets of finite perimeter, and related BV functions (cf. [2, 29, 33, 40, 72]). We then introduce the notion of oriented surfaces and develop some basic properties of divergence-measure fields. For the sake of completeness, we start with some basic notions and definitions.

First, denote by \mathcal{H}^M the M -dimensional Hausdorff measure in \mathbb{R}^N for $M \leq N$, and by \mathcal{L}^N the Lebesgue measure in \mathbb{R}^N (recall that $\mathcal{L}^N = \mathcal{H}^N$). For any \mathcal{L}^N -measurable set $E \subset \mathbb{R}^N$, we denote $|E|$ as the \mathcal{L}^N -Lebesgue measure of the set E and ∂E as its topological boundary. Also, we denote $B(x, r)$ as the open ball of radius r and center at x . The symmetric difference of sets is denoted by

$$A \Delta B := (A \setminus B) \cup (B \setminus A).$$

Let $\Omega \subset \mathbb{R}^N$ be open. We denote by $E \Subset \Omega$ that the closure of E is compact and contained in Ω . Let $C_c(\Omega)$ be the space of compactly supported continuous functions on Ω with $\|\varphi\|_{0;\Omega} := \sup\{|\varphi(y)| : y \in \Omega\}$.

DEFINITION 2.1 A *Radon measure* on Ω is a signed regular Borel measure whose total variation on each compact set $K \Subset \Omega$ is finite, i.e., $\|\mu\|(K) < \infty$. The space of Radon measures supported on an open set Ω is denoted by $\mathcal{M}(\Omega)$. If $\mu \in \mathcal{M}(\Omega)$ does not take negative values, then we refer to such μ as a nonnegative Radon measure.

Any Radon measure μ can be decomposed into the difference of two nonnegative Radon measures $\mu = \mu^+ - \mu^-$; the total variation of μ is $\|\mu\| = \mu^+ + \mu^-$. Equivalently, if μ is a Radon measure on Ω , the total variation of μ on any bounded open set $W \subset \Omega$ is equal to

$$\begin{aligned} \|\mu\|(W) &= \sup \left\{ \int_{\Omega} \varphi \, d\mu : \varphi \in C_c(W), \|\varphi\|_{0;\Omega} \leq 1 \right\} \\ (2.1) \qquad &= \sup \left\{ \sum_{i=0}^{\infty} |\mu(B_i)| \right\}, \end{aligned}$$

where the second supremum is taken over all pairwise disjoint Borel sets B_i with $W = \bigcup_{i=1}^{\infty} B_i$. Since the space of Radon measures can be identified with the dual

of $C_c(\Omega)$, we may consider a Radon measure μ as a linear functional on $C_c(\Omega)$, written as

$$(2.2) \quad \mu(\varphi) := \int_{\Omega} \varphi \, d\mu \quad \text{for each } \varphi \in C_c(\Omega).$$

We recall the familiar weak-star topology on $\mathcal{M}(\Omega)$, which, when restricted to a sequence $\{\mu_k\}$, yields

$$\mu_k \xrightarrow{*} \mu \quad \text{in } \mathcal{M}(\Omega);$$

that is, μ_k converges to μ in the weak-star topology if and only if

$$(2.3) \quad \mu_k(\varphi) \rightarrow \mu(\varphi) \quad \text{for each } \varphi \in C_c(\Omega).$$

The space $L^p(\Omega, \mu)$, $1 \leq p \leq \infty$, denotes all the functions f with the property that $|f|^p$ is μ -integrable. The conjugate of p is $p' := p/(p - 1)$. The L^p -norm of f on a set E with integration taken with respect to a measure μ is denoted by $\|f\|_{p;E,\mu}$. In the event μ is Lebesgue measure, we will simply write $\|f\|_{p;E}$.

THEOREM 2.2 (Uniform Boundedness Principle) *Let X be a Banach space. If T_k is a sequence of linear functionals on X that converges weak-star to T , then*

$$\limsup_{k \rightarrow \infty} \|T_k\| < \infty.$$

Next, we quote a familiar result concerning weak-star convergence.

LEMMA 2.3 *Let μ be a Radon measure on Ω , and let μ_k be a sequence of Radon measures converging weak-star to μ . Then we have the following:*

(i) *If $A \subset \Omega$ is any open set and μ_k are nonnegative Radon measures,*

$$\mu(A) \leq \liminf_{k \rightarrow \infty} \mu_k(A).$$

(ii) *If $K \subset \Omega$ is any compact set and μ_k are nonnegative Radon measures,*

$$\mu(K) \geq \limsup_{k \rightarrow \infty} \mu_k(K).$$

(iii) *If $\|\mu_k\| \xrightarrow{*} \sigma$, then $\|\mu\| \leq \sigma$. In addition, if the μ -measurable set $E \Subset \Omega$ satisfies $\sigma(\partial E) = 0$, then*

$$\mu(E) = \lim_{k \rightarrow \infty} \mu_k(E).$$

More generally, if f is a bounded Borel function with compact support in Ω such that the set of its discontinuity points is σ -negligible, then

$$\lim_{k \rightarrow \infty} \int_{\Omega} f \, d\mu_k = \int_{\Omega} f \, d\mu.$$

DEFINITION 2.4 For every $\alpha \in [0, 1]$ and every \mathcal{L}^N -measurable set $E \subset \mathbb{R}^N$, define

$$(2.4) \quad E^\alpha := \{y \in \mathbb{R}^N : D(E, y) = \alpha\},$$

where

$$(2.5) \quad D(E, y) := \lim_{r \rightarrow 0} \frac{|E \cap B(y, r)|}{|B(y, r)|}.$$

Then E^α is the set of all points where E has density α . We define the *measure-theoretic boundary* of E , $\partial^m E$, as

$$(2.6) \quad \partial^m E := \mathbb{R}^N \setminus (E^0 \cup E^1).$$

DEFINITION 2.5 A function $f : \Omega \rightarrow \mathbb{R}$ is called a *function of bounded variation* if each partial derivative of f is a Radon measure with finite total variation in Ω . Notationally, we write $f \in \text{BV}(\Omega)$. Let $E \Subset \Omega$ be an \mathcal{L}^N -measurable subset. We say that E is a *set of finite perimeter* if $\chi_E \in \text{BV}(\Omega)$. Consequently, if E is a set of finite perimeter, then $\nabla \chi_E$ is a (vector-valued) Radon measure whose total variation, denoted by $\|\nabla \chi_E\|$, is finite.

DEFINITION 2.6 Let $E \Subset \Omega$ be a set of finite perimeter. The *reduced boundary* of E , denoted as $\partial^* E$, is the set of all points $y \in \Omega$ such that

- (i) $\|\nabla \chi_E\|(B(y, r)) > 0$ for all $r > 0$, and
- (ii) the limit

$$\nu_E(y) := \lim_{r \rightarrow 0} \frac{\nabla \chi_E(B(y, r))}{\|\nabla \chi_E\|(B(y, r))}$$

exists and $|\nu_E(y)| = 1$.

The set $\partial^* E$ is also called the *perimeter* of E and its $(N - 1)$ -Hausdorff measure is denoted by $P(E) := \mathcal{H}^{N-1}(\partial^* E)$. The following result was proved by De Giorgi [24] (see also [2, theorem 3.59] and [72, theorem 5.7.3]).

THEOREM 2.7 *Let E be a set of finite perimeter. The reduced boundary of E , $\partial^* E$, is an $(N - 1)$ -rectifiable set, which means that there exists a countable family of C^1 manifolds M_k of dimension $N - 1$ and a set \mathcal{N} of \mathcal{H}^{N-1} -measure zero such that*

$$(2.7) \quad \partial^* E \subset \left(\bigcup_{k=1}^{\infty} M_k \right) \cup \mathcal{N}.$$

Moreover, the generalized gradient of χ_E enjoys the following basic relationship with \mathcal{H}^{N-1} :

$$(2.8) \quad \|\nabla \chi_E\| = \mathcal{H}^{N-1} \llcorner \partial^* E$$

and, for \mathcal{H}^{N-1} -a.e. $y \in \partial^* E$,

$$\lim_{r \rightarrow 0} \frac{\|\nabla \chi_E\|(B(y, r))}{\alpha(N-1)r^{N-1}} = 1,$$

where $\alpha(N-1)$ is the Lebesgue measure of the unit ball in \mathbb{R}^{N-1} .

The unit vector $\mathbf{v}_E(y)$ is called the *measure-theoretic interior unit normal* to E at y (we sometimes write \mathbf{v} instead of \mathbf{v}_E for notational simplicity). In view of the following, we see that \mathbf{v} is aptly named because \mathbf{v} is the interior unit normal to E provided that E (in the limit and in measure) lies in the appropriate half-space determined by the hyperplane orthogonal to \mathbf{v} ; that is, \mathbf{v} is the interior unit normal to E at x provided that

$$D(\{y : (y-x) \cdot \mathbf{v} > 0, y \notin E\} \cup \{y : (y-x) \cdot \mathbf{v} < 0, y \in E\}, y) = 0.$$

The following result is due to Federer (see also [72, lemma 5.9.5] and [2, theorem 3.61]).

THEOREM 2.8 *If $E \Subset \Omega$ is a set of finite perimeter, then*

$$(2.9) \quad \partial^* E \subset E^{1/2} \subset \partial^m E, \quad \mathcal{H}^{N-1}(\Omega \setminus (E^0 \cup \partial^* E \cup E^1)) = 0.$$

In particular, E has density either 0 or $\frac{1}{2}$ or 1 at \mathcal{H}^{N-1} -a.e. $x \in \Omega$, and \mathcal{H}^{N-1} -a.e. $x \in \partial^m E$ belongs to $\partial^ E$.*

Remark 2.9. In view of Definition 2.5, (2.8), and (2.9), it is clear that, if $E \Subset \Omega$ is a set of finite perimeter, then $\mathcal{H}^{N-1}(\partial^m E) < \infty$. Conversely, it was proved by Federer (see [33, theorem 4.5.11]) that, if $\mathcal{H}^{N-1}(\partial^m E) < \infty$, then E is a set of finite perimeter.

We will refer to the sets E^0 and E^1 as the *measure-theoretic exterior and interior* of E . We note that, in general, the sets E^0 and E^1 do not coincide with the topological exterior and interior of the set E . The sets E^0 and E^1 also motivate the definition of measure-theoretic boundary. We note that (2.9) implies that, for any set $E \Subset \Omega$ of finite perimeter,

$$\Omega = E^1 \cup \partial^* E \cup E^0 \cup \mathcal{N}$$

where $\mathcal{H}^{N-1}(\mathcal{N}) = 0$. If we define a set E to be “open” if E is both measurable and $D(E, x) = 1$ for all $x \in E$, then this concept of openness defines a topology, called the *density topology*. It is an interesting exercise to prove that the arbitrary union of open sets is also open; the crux of the problem is to prove that the arbitrary union is, in fact, measurable. This topology is significant because it is the smallest topology (the one with the smallest number of open sets) for which the approximately continuous functions are continuous [41].

The following result, which is easily verified (although tedious), will be needed in what follows.

LEMMA 2.10 *If $A, B \in \Omega$ are sets of finite perimeter, then*

$$\partial^m(A \cap B) \subseteq (\partial^m A \cap B) \cup (A \cap \partial^m B) \cup (\partial^m A \cap \partial^m B).$$

DEFINITION 2.11 Let $\rho \in C_c^\infty(\mathbb{R}^N)$ be a standard symmetric mollifying kernel; that is, ρ is a nonnegative function with support in the unit ball and satisfies $\|\rho\|_{1;\mathbb{R}^N} = 1$. With $u \in L^1(\mathbb{R}^N)$, we set $u_\varepsilon := u * \rho_\varepsilon$, where the sequence $\rho_\varepsilon(y) := \frac{1}{\varepsilon^N} \rho(\frac{y}{\varepsilon})$ forms a mollifier.

Recall the following properties of mollifications (cf. [72]).

LEMMA 2.12

- (i) *If $u \in L^1_{\text{loc}}(\mathbb{R}^N)$, then, for every $\varepsilon > 0$, $u_\varepsilon \in C^\infty(\mathbb{R}^N)$ and $D^\alpha(\rho_\varepsilon * u) = (D^\alpha \rho_\varepsilon) * u$ for each multi-index α .*
- (ii) *With $k := \frac{1}{\varepsilon_k}$ and $\varepsilon_k \rightarrow 0$, $u_k(x) \rightarrow u(x)$ whenever x is a Lebesgue point of u . In particular, if u is continuous, then u_ε converges uniformly to u on compact subsets of \mathbb{R}^N .*

Remark 2.13. Functions in the spaces $\text{BV}(\mathbb{R}^N)$ and $W^{1,p}(\mathbb{R}^N)$, $1 \leq p \leq \infty$, have precise representatives; that is, if $u \in \text{BV}(\mathbb{R}^N)$, then there is a function $u^* \in \text{BV}(\mathbb{R}^N)$ such that u and u^* are equal a.e. and that the mollification sequence of u , u_k , converges to u^* at all points except those that belong to an exceptional set E with $\mathcal{H}^{N-1}(E) = 0$. However, this is not the same as saying that u has a Lebesgue point, which is slightly stronger.

A similar statement is true for functions in the Sobolev space $W^{1,p}(\mathbb{R}^N)$, $1 < p \leq \infty$, except that the exceptional set E has γ_p -capacity zero; see Definition 2.21 below. As we will see, the γ_1 -capacity vanishes precisely on sets of \mathcal{H}^{N-1} -measure zero. Thus, we can say that functions in the spaces BV and $W^{1,p}$ have precise representatives that are defined, respectively, γ_1 and γ_p almost everywhere.

DEFINITION 2.14 If $u = \chi_E$ for a set of finite perimeter E , we denote u_E as the corresponding precise representative u^* .

Indeed, when u is taken as χ_E , Lemma 2.12 can be considerably strengthened:

LEMMA 2.15 *If u_k is the mollification of χ_E for a set of finite perimeter E , then the following hold:*

- (i) $u_k \in C^\infty(\mathbb{R}^N)$.
- (ii) *There is a set \mathcal{N} with $\mathcal{H}^{N-1}(\mathcal{N}) = 0$ such that, for all $y \notin \mathcal{N}$, $u_k(y) \rightarrow u_E(y)$ as $k \rightarrow \infty$ and*

$$u_E(y) = \begin{cases} 1, & y \in E^1, \\ \frac{1}{2}, & y \in \partial^* E, \\ 0, & y \in E^0. \end{cases}$$

- (iii) $\nabla u_k \xrightarrow{*} \nabla u_E$ in $\mathcal{M}(\mathbb{R}^N)$.

- (iv) If U is an open set with $\|\nabla u_E\|(\partial U) = 0$, then $\|\nabla u_k\|(U) \rightarrow \|\nabla u_E\|(U)$ as $k \rightarrow \infty$.
- (v) $\nabla \chi_E = \nabla u_E$.

PROOF: Only (iii) requires a proof, since (i), (ii), and (iv) are results from standard BV theory and (v) is immediate from the definitions and the fact that $u_E = \chi_E$ almost everywhere. As for (iii), since $u_k \rightarrow u_E$ in $L^1(\mathbb{R}^N)$, then $u_k \rightarrow u_E$ when considered as distributions, which implies that $\nabla u_k \rightarrow \nabla u_E$ as distributions and consequently as measures since $\nabla u_k, \nabla u_E \in \mathcal{M}(\mathbb{R}^N)$. \square

The next result affirms the notion that the mollification is generally a norm-reducing operation.

LEMMA 2.16 *Let E be a set of finite perimeter, and let u_k denote the mollification of χ_E . Then*

$$\|\nabla u_k\|_1 \leq \|\nabla \chi_E\|.$$

PROOF: For any $f \in \text{BV}(\mathbb{R}^N)$, consider the convolutions

$$f_\varepsilon(y) := \int_{\mathbb{R}^N} \rho_\varepsilon(y-x) f(x) dx.$$

Using $\nabla f_\varepsilon = \rho_\varepsilon * (\nabla f)$ and $f_\varepsilon \in C^\infty(\mathbb{R}^N)$, we obtain

$$\nabla f_\varepsilon(y) = \int_{\mathbb{R}^N} \rho_\varepsilon(y-x) dm(x)$$

where $m := \nabla f$ is the measure. Thus, we have

$$|\nabla f_\varepsilon(y)| \leq \int_{\mathbb{R}^N} \rho_\varepsilon(y-x) d\|m\|(x).$$

In particular, when $f = \chi_E$ and $f_{\varepsilon_k} = u_k$ with $\varepsilon_k = \frac{1}{k}$, then $m = \nabla \chi_E$ and

$$|\nabla u_k(y)| \leq \int_{\mathbb{R}^N} \rho_{\varepsilon_k}(y-x) d\|m\|(x) \quad \text{for all } y \in \mathbb{R}^N.$$

That is,

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla u_k(y)| dy &\leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \rho_{\varepsilon_k}(y-x) d\|m\|(x) dy \\ &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \rho_{\varepsilon_k}(y-x) dy d\|m\|(x) \leq \|m\|(\mathbb{R}^N). \end{aligned}$$

\square

We recall that the BV space in fact represents equivalence classes of functions so that, when a function in a class is changed on a set of \mathcal{L}^N -measure zero, it remains in this class. The same is true for sets of finite perimeter because, by definition, the characteristic function χ_E of a set of finite perimeter E is a function of bounded variation. Thus, it follows that E may be altered by a set of \mathcal{L}^N -measure zero and still determine the same essential boundary $\partial^m E$. Throughout, we will choose a preferred representative for E and thereby adopt the following convention.

DEFINITION 2.17 $E := \{y : D(E, y) = 1\} \cup \partial^m E$.

DEFINITION 2.18 A vector field $\mathbf{F} \in L^p(\Omega; \mathbb{R}^N)$, $1 \leq p \leq \infty$, is called a *divergence-measure field*, written as $\mathbf{F} \in \mathcal{DM}^p(\Omega)$, if $\mu := \operatorname{div} \mathbf{F}$ is a Radon measure with finite total variation on Ω in the sense of distributions. Thus, for $\varphi \in C_c^\infty(\Omega)$, we have

$$\mu(\varphi) := (\operatorname{div} \mathbf{F})(\varphi) = - \int_{\Omega} \mathbf{F} \cdot \nabla \varphi \, dy.$$

The total variation of μ is a nonnegative measure which, for any open set $W \subset \Omega$, is defined as

$$\begin{aligned} \|\mu\|(W) &:= \sup\{\mu(\varphi) : \|\varphi\|_{0;\Omega} \leq 1, \varphi \in C_c^\infty(W)\} \\ &= \sup\left\{ \int_{\Omega} \mathbf{F} \cdot \nabla \varphi \, dy : \|\varphi\|_{0;\Omega} \leq 1, \varphi \in C_c^\infty(W) \right\}. \end{aligned}$$

A vector field $\mathbf{F} \in \mathcal{DM}_{\text{loc}}^p(\Omega)$ means that, for any $W \Subset \Omega$, $\mathbf{F} \in \mathcal{DM}^p(W)$.

DEFINITION 2.19 Let $\mathbf{F} \in \mathcal{DM}^p(\Omega)$, $1 \leq p \leq \infty$. For an arbitrary measurable set $E \Subset \Omega$, the trace of the normal component of \mathbf{F} on ∂E is a functional defined by

$$(2.10) \quad (T\mathbf{F})_{\partial E}(\varphi) = \int_E \nabla \varphi \cdot \mathbf{F} \, dy + \int_E \varphi \, d\mu$$

for any test function $\varphi \in C_c^\infty(\Omega)$. Clearly, $(T\mathbf{F})_{\partial E}$ is a distribution defined on Ω . Note that this definition assumes only that the set E is measurable. Later, we will provide an alternative definition when E is a set of finite perimeter (see Theorem 5.3).

PROPOSITION 2.20 *Let $E \Subset \Omega$ be an open set. Then $\operatorname{spt}((T\mathbf{F})_{\partial E}) \subset \partial E$. That is, if ψ and φ are test functions in $\mathcal{D}(\Omega)$ with $\psi = \varphi$ on ∂E , then $(T\mathbf{F})_{\partial E}(\psi) = (T\mathbf{F})_{\partial E}(\varphi)$.*

PROOF: If the support were not contained in ∂E , there would be a point $x_0 \notin \partial E$ with $x_0 \in \operatorname{spt}((T\mathbf{F})_{\partial E}) \cap E$. This implies that, for each open set U containing x_0 , there exists a test function $\varphi \in C_c^\infty(U \cap E)$ such that $(T\mathbf{F})_{\partial E}(\varphi) \neq 0$. Choose

U so that $U \subset \mathbb{R}^N \setminus \partial E$. Let \mathbf{F}_ε denote the mollification of \mathbf{F} (see Lemma 2.12). Then, since $\text{spt}(\mathbf{F}_\varepsilon\varphi) \Subset E$,

$$\begin{aligned} 0 &= \int_E \text{div}(\mathbf{F}_\varepsilon\varphi) \, dy = (T\mathbf{F}_\varepsilon)_{\partial E}(\varphi) \\ &= \int_E \mathbf{F}_\varepsilon \cdot \nabla\varphi + \int_E \varphi \text{div} \mathbf{F}_\varepsilon \, dy \\ &\rightarrow \int_E \mathbf{F} \cdot \nabla\varphi \, dy + \int_E \varphi \, d\mu = (T\mathbf{F})_{\partial E}(\varphi) \neq 0, \end{aligned}$$

where we have used $\partial E \cap \text{spt}(\varphi) = \emptyset$ in the limit. Thus, we arrive at our desired contradiction. \square

DEFINITION 2.21 For $1 \leq p \leq N$, the p -capacity of an arbitrary set $A \Subset \mathbb{R}^N$ is defined as

$$(2.11) \quad \gamma_p(A) := \inf \left\{ \int_\Omega |\nabla\varphi|^p \, dy \right\},$$

where the infimum is taken over all test functions $\varphi \in C_c^\infty(\Omega)$ that are identically 1 in a neighborhood of A . It is well known (cf. [34]) that $\gamma_p(A) = 0$ for $1 < p < N$ implies that $\mathcal{H}^{N-p+\varepsilon}(A) = 0$ for each $\varepsilon > 0$ and that, conversely, if $\mathcal{H}^{N-p}(A) < \infty$, then $\gamma_p(A) = 0$. In view of Remark 2.13 and Lemma 2.25, it is easy to verify that the class of competing functions in (2.11) can be enlarged to the Sobolev space $W^{1,p}(\Omega)$.

Remark 2.22. The case of $p = 1$ requires special consideration. In 1957, Fleming conjectured that $\gamma_1(A) = 0$ if and only if $\mathcal{H}^{N-1}(A) = 0$. This was settled in the affirmative by Gustin [44], who proved the boxing inequality, from which Fleming’s conjecture easily follows (cf. [35]).

The next results are basic (cf. [69, 70, 72]).

PROPOSITION 2.23 Let γ_p be the p -capacity defined in (2.11). Then

(i) If $E_k \subset \mathbb{R}^N$ is a sequence of arbitrary sets, then

$$\gamma_p(\liminf_{k \rightarrow \infty} E_k) \leq \liminf_{k \rightarrow \infty} \gamma_p(E_k).$$

(ii) If $E_1 \subset E_2 \subset \dots$ are arbitrary sets, then

$$\gamma_p\left(\bigcup_{k=1}^\infty E_k\right) = \lim_{k \rightarrow \infty} \gamma_p(E_k).$$

(iii) If $K_1 \supset K_2 \supset \dots$ are compact sets, then

$$\gamma_p\left(\bigcap_{k=1}^\infty K_k\right) = \lim_{k \rightarrow \infty} \gamma_p(K_k).$$

(iv) If $\{E_k\}$ is a sequence of Borel sets, then

$$\gamma_p\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \lim_{k \rightarrow \infty} \gamma_p(E_k).$$

(v) If $A \subset \mathbb{R}^N$ is a Suslin set, then

$$\sup\{\gamma_p(K) : K^{\text{compact}} \subset A\} = \inf\{\gamma_p(U) : U^{\text{open}} \supset A\}.$$

Any set function γ satisfying conditions (i)–(iv) is called a true capacity in the sense of Choquet and a set A satisfying condition (v) is said to be γ -capacitable.

Remark 2.24. One of the main reasons for studying the capacity is its important role in the development of Sobolev theory. It was first shown in [34] that every function $u \in W^{1,p}(\Omega)$ has a Lebesgue point γ_p -a.e. In particular, in view of Remark 2.22, this implies that a function $u \in W^{1,1}(\Omega)$ has a Lebesgue point everywhere except for an exceptional set E with $\mathcal{H}^{N-1}(E) = 0$. In case $u \in \text{BV}(\Omega)$, we have a slightly weaker statement than the corresponding one for $u \in W^{1,p}(\Omega)$:

$$\lim_{r \rightarrow 0} \int B(x,r) u(y) dy = u^*(x) \quad \text{for } \mathcal{H}^{N-1}\text{-a.e. } x \in \Omega.$$

It turns out that the Sobolev space is the *perfect functional completion* of the space $C_c^\infty(\Omega)$ relative to the p -capacity. See [4] where the concept of perfect functional completion was initiated and developed.

LEMMA 2.25 *If $\mathbf{F} \in \mathcal{DM}_{\text{loc}}^p(\Omega)$, $\frac{N}{N-1} \leq p \leq \infty$, then $\|\text{div } \mathbf{F}\| \ll \gamma_{p'}$; that is, if $B \subset \Omega$ is a Borel set satisfying $\gamma_{p'}(B) = 0$, $p' := p/(p-1)$, then $\|\text{div } \mathbf{F}\|(B) = 0$. In particular, when $p = \infty$ (i.e., $p' = 1$), Remark 2.22 implies that $\|\text{div } \mathbf{F}\| \ll \mathcal{H}^{N-1}$.*

PROOF: Because of the inner regularity of $\|\text{div } \mathbf{F}\|$ and condition (v) of Proposition 2.23, it suffices to show that $\mu(K) = 0$ for any compact set $K \subset B$, where $\mu := \text{div } \mathbf{F}$. Since $\gamma_{p'}(K) = 0$, then there exists a sequence of test functions $\varphi_k \in C_c^\infty(\Omega)$ (see, for example, lemma 2.2 in [6]) such that

- (i) $0 \leq \varphi_k \leq 1$ and $\varphi_k = 1$ on K ,
- (ii) $\|\nabla \varphi_k\|_{p';\Omega} \rightarrow 0$ as $k \rightarrow \infty$,
- (iii) $\varphi_k(y) \rightarrow 0$ as $k \rightarrow \infty$ for all $y \in \Omega$ except those in some set $A \subset \Omega$ with $\gamma_{p'}(A) = 0$, and
- (iv) φ_k is supported in an open set O_k with $O_1 \supset O_2 \supset \dots \supset K$ and $\bigcap_k O_k = K$.

Since $\text{div } \mathbf{F} = \mu$, we have

$$\mu(K) + \int_{\Omega \setminus K} \varphi_k d\mu = - \int_{\Omega} \mathbf{F} \cdot \nabla \varphi_k dy \leq \|\mathbf{F}\|_{p;\Omega} \|\nabla \varphi_k\|_{p';\Omega}.$$

Thus, using (i), (ii), and (iv), we conclude

$$|\mu(K)| \leq \|\mu\|(O_k \setminus K) + \|\mathbf{F}\|_{p;\Omega} \|\nabla\varphi_k\|_{p';\Omega} \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

□

COROLLARY 2.26 *If $\mathbf{F} \in \mathcal{DM}_{\text{loc}}^p(\Omega)$, $\frac{N}{N-1} < p < \infty$, and if $\mathcal{H}^{N-p'}(B) < \infty$ for $B \subset \Omega$, then $\gamma_{p'}(B) = 0$ and hence $\|\operatorname{div} \mathbf{F}\|(B) = 0$.*

Remark 2.27. If $\mathbf{F} \in \mathcal{DM}^p(\Omega)$, $\frac{N}{N-1} < p \leq \infty$, with $\operatorname{div} \mathbf{F} = \mu$, then, in view of the fact that $\varphi \in W^{1,p'}(\Omega)$ is defined $\mathcal{H}^{N-p'}$ -a.e. and therefore μ -a.e., it follows that the integral

$$\int_{\Omega} \varphi \, d\mu$$

is defined and is meaningful.

DEFINITION 2.28 A compact set $K \subset \mathbb{R}^N$ is called a *p-removable set* for $\mathbf{F} \in L^p(\mathbb{R}^N)$, $1 \leq p \leq \infty$, provided that $\mathbf{F} \in \mathcal{DM}^p(\mathbb{R}^N)$ whenever $\mathbf{F} \in \mathcal{DM}^p(\mathbb{R}^N \setminus K)$.

THEOREM 2.29 *If $K \subset \mathbb{R}^N$ is compact with $\mathcal{H}^{N-p'}(K) = 0$, $1 \leq p' < N$, then K is a p-removable set for $\mathbf{F} \in \mathcal{DM}^p$.*

PROOF: It suffices to show that, for each test function $\psi \in C_c^\infty(\mathbb{R}^N)$,

$$(2.12) \quad \mu(\psi) = - \int_{\mathbb{R}^N} \mathbf{F} \cdot \nabla \psi \, dy.$$

Let φ_k be the sequence defined in the proof of Lemma 2.25 that satisfies (i)–(iv). We approximate ψ by a sequence

$$(2.13) \quad \psi_k := \psi(1 - \varphi_k) \in C_c^\infty(\mathbb{R}^N \setminus K).$$

Since $\mathbf{F} \in \mathcal{DM}^p(\mathbb{R}^N \setminus K)$, we have

$$(2.14) \quad \mu(\psi_k) = - \int_{\mathbb{R}^N} \mathbf{F} \cdot \nabla \psi_k \, dy.$$

From the fact that $\varphi_k \rightarrow 0$, $\gamma_{p'}$ -a.e. (and therefore μ -a.e., by Lemma 2.25 and Corollary 2.26), we obtain that $\psi_k \rightarrow \psi$, μ -a.e., and therefore Lebesgue's dominated convergence theorem yields

$$\mu(\psi_k) := \int_{\mathbb{R}^N} \psi_k \, d\mu \rightarrow \int_{\mathbb{R}^N} \psi \, d\mu := \mu(\psi).$$

On the other hand, the Hölder inequality and (ii)–(iii) in the proof of Lemma 2.25 imply

$$\int_{\mathbb{R}^N} \mathbf{F} \cdot \nabla \psi_k \, dy \rightarrow \int_{\mathbb{R}^N} \mathbf{F} \cdot \nabla \psi \, dy,$$

and thus, we conclude with our desired result (2.12). \square

EXAMPLE 2.30 (Chen and Frid [16]) Denote U the open unit square in \mathbb{R}^2 that has one of its sides contained in the line segment

$$L := \{y = (y_1, y_2) : y_1 = y_2\} \cap \partial U.$$

Define a field $\mathbf{F} : \mathbb{R}^2 \setminus L \rightarrow \mathbb{R}^2$ by

$$\mathbf{F}(y) = \mathbf{F}(y_1, y_2) = \left(\sin \left(\frac{1}{y_1 - y_2} \right), -\sin \left(\frac{1}{y_1 - y_2} \right) \right).$$

Clearly, $\mathbf{F} \in L^\infty(\mathbb{R}^2)$, and a simple calculation reveals that $\operatorname{div} \mathbf{F} = 0$ in $\mathbb{R}^2 \setminus L$. Then \mathbf{F} belongs to $\mathcal{DM}^\infty(\mathbb{R}^2)$, but the field is singular on one side L of ∂U , and therefore \mathbf{F} is undefined on ∂U ; it has no trace on ∂U in the classical sense. Note also that the points of L are all essential singularities of \mathbf{F} because the following limit does not exist:

$$\lim_{y \rightarrow x} \mathbf{F}(y) \quad \text{for } y \in \mathbb{R}^2 \setminus L, \, x \in L,$$

and therefore the normal trace of \mathbf{F} on ∂U is given by

$$\lim_{t \rightarrow 0} \int_{\partial U_t} \mathbf{F}(y) \cdot \nu(y) d\mathcal{H}^1(y) = \lim_{t \rightarrow 0} \int_{U_t} \operatorname{div} \mathbf{F} \, dy = 0,$$

where $U_t := \{y \in U : \operatorname{dist}(y, \partial U) > t\}$. Thus, we have shown the following:

- (1) \mathbf{F} is an element of $\mathcal{DM}^\infty(\mathbb{R}^2)$, while each component function of \mathbf{F} is not in $\operatorname{BV}(\mathbb{R}^2)$;
- (2) \mathbf{F} has an essential singularity at each point of L and therefore cannot be defined on L ;
- (3) as we will see later, Theorem 5.3, \mathbf{F} has a weak normal trace on L that is sufficient for the Gauss–Green theorem to hold.

For more properties of the spaces \mathcal{DM}^p of divergence-measure vector fields, see Chen and Frid [16, 19].

The following theorem provides a product rule for the case $p = \infty$. For the sake of completeness, we will include its proof, which is slightly different from that given in [16]. We denote by $\{g_k\}$ the sequence of $C_c^\infty(\mathbb{R}^N)$ mollifications with the property that $g_k \rightarrow g$ in $L^1(\mathbb{R}^N)$ and such that $\|\nabla g_k\| \rightarrow \|\nabla g\|$ (cf. [72], p. 500).

THEOREM 2.31 (Chen and Frid [16]) *Let $\mathbf{F} \in \mathcal{DM}^\infty(\mathbb{R}^N)$ and $g \in \text{BV}(\mathbb{R}^N)$ be bounded with compact support. Then*

$$(2.15) \quad \text{div}(g\mathbf{F}) = g^* \text{div } \mathbf{F} + \overline{\mathbf{F} \cdot \nabla g},$$

where $\overline{\mathbf{F} \cdot \nabla g}$ denotes the weak-star limit of the measures $\mathbf{F} \cdot \nabla g_k$, and g^* is the precise representative of g .

PROOF: Let \mathbf{F}_ε be the mollification of \mathbf{F} and set $\mu := \text{div } \mathbf{F}$. Since \mathbf{F}_ε are smooth, the classical product rule yields

$$(2.16) \quad \text{div}(g_k \mathbf{F}_\varepsilon) = g_k \text{div } \mathbf{F}_\varepsilon + \mathbf{F}_\varepsilon \cdot \nabla g_k.$$

First, we note that $\text{div } \mathbf{F}_\varepsilon = (\text{div } \mathbf{F})_\varepsilon = \mu_\varepsilon \xrightarrow{*} \mu$ in $\mathcal{M}(\mathbb{R}^N)$ as $\varepsilon \rightarrow 0$. Since $g_k \mathbf{F}_\varepsilon \rightarrow g_k \mathbf{F}$ in $L^1_{\text{loc}}(\mathbb{R}^N)$ as $\varepsilon \rightarrow 0$, we obtain from (2.16) that, in the sense of distributions,

$$(2.17) \quad \mathbf{F} \cdot \nabla g_k = \text{div}(g_k \mathbf{F}) - \mu(g_k).$$

Owing to the fact that $\mathbf{F} \in L^\infty$, we see that $\mathbf{F} \cdot \nabla g_k$ is a bounded sequence in $L^1(\mathbb{R}^N)$ and hence there is a subsequence such that $\mathbf{F} \cdot \nabla g_k$ converges weak-star to some measure, denoted by $\overline{\mathbf{F} \cdot \nabla g}$. Letting $k \rightarrow \infty$ in (2.17) yields

$$(2.18) \quad \overline{\mathbf{F} \cdot \nabla g} = \text{div}(g\mathbf{F}) - \mu(g^*). \quad \square$$

The next result, Federer’s co-area and area formulas (see [33, cor. 3.2.20]), will be of critical importance to us in what follows.

THEOREM 2.32 (Federer [32]) *Suppose that Y and X are Riemannian manifolds of dimension N and k , respectively, with $N \geq k$. If $f : Y \rightarrow X$ is a Lipschitz map, then*

$$(2.19) \quad \int_Y g(y) Jf(y) d\mathcal{H}^N(y) = \int_X \left\{ \int_{f^{-1}(x)} g(y) d\mathcal{H}^{N-k}(y) \right\} d\mathcal{H}^k(x)$$

whenever $g : Y \rightarrow \mathbb{R}$ is \mathcal{H}^N -integrable. Here, $Jf(y)$ denotes the k -dimensional Jacobian of f at y , namely, the norm of the differential $df(y)$ of f at y . Alternatively, it is the square root of the sum of the squares of the determinants of the $k \times k$ minors of the differential of f at y .

Furthermore, if $k \geq N$, the following area formula holds:

$$(2.20) \quad \int_Y Jf(y) d\mathcal{H}^N(y) = \int_X N(x) d\mathcal{H}^k(x) = \int_{f(Y)} N(x) d\mathcal{H}^k(x),$$

where $N(x)$ denotes the (possibly infinite) number of points in $f^{-1}(x)$.

In the event that $u \in \text{BV}(\mathbb{R}^N)$, there is another version of the co-area formula due to Fleming and Rishel [36].

THEOREM 2.33 *If $u \in \text{BV}(\mathbb{R}^N)$, then*

$$\|\nabla u\|(\mathbb{R}^N) = \int_{-\infty}^{\infty} P(\{u > t\})dt.$$

Conversely, if $u \in L^1(\mathbb{R}^N)$ with $\int_{-\infty}^{\infty} P(\{u > t\})dt < \infty$, then $u \in \text{BV}(\mathbb{R}^N)$.

LEMMA 2.34 *Let $u : \mathbb{R}^N \rightarrow \mathbb{R}$ be a Lipschitz function, and let $A \subset \mathbb{R}^N$ be a set of measure zero. Then*

$$\mathcal{H}^{N-1}(u^{-1}(s) \cap A) = 0 \quad \text{for almost all } s.$$

PROOF: This is immediate from the co-area formula:

$$0 = \int_A |\nabla u(y)|dy = \int_{\mathbb{R}} \mathcal{H}^{N-1}(A \cap u^{-1}(s))ds. \quad \square$$

One of the fundamental results of geometric measure theory is that any set of finite perimeter possesses a measure-theoretic interior unit normal \mathbf{v} that is suitably general to ensure the validity of the Gauss-Green theorem.

THEOREM 2.35 (De Giorgi and Federer [24, 25, 30, 31]) *If E has finite perimeter, then*

$$\int_E \text{div } \mathbf{F} \, dy = - \int_{\partial^* E} \mathbf{F}(y) \cdot \mathbf{v}(y) d\mathcal{H}^{N-1}(y)$$

whenever $\mathbf{F} : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is Lipschitz.

The De Giorgi–Federer result shows that integration by parts holds on a very large and rich family of sets, but only for fields \mathbf{F} that are Lipschitz. As we explained in the introduction, the Gauss-Green formula for BV vector fields over sets of finite perimeter was treated in [10, 66]. We contrast their results with the following result by Fuglede [39].

THEOREM 2.36 (Fuglede [39]) *Let $\mathbf{F} \in L^p(\mathbb{R}^N; \mathbb{R}^N)$, $1 \leq p < \infty$, with $\text{div } \mathbf{F} \in L^p(\mathbb{R}^N)$. Then there exists a function $g : \mathbb{R}^N \rightarrow \mathbb{R}$ with $g \in L^p$ such that*

$$(2.21) \quad \int_E \text{div } \mathbf{F} = - \int_{\partial^* E} \mathbf{F}(y) \cdot \mathbf{v}(y) d\mathcal{H}^{N-1}(y)$$

for all sets of finite perimeter E except possibly those for which

$$\int_{\partial^* E} g(y) d\mathcal{H}^{N-1}(y) = \infty.$$

The following, which is a direct consequence of Fuglede’s result, will be of use to us. Suppose that $u : \mathbb{R}^N \rightarrow \mathbb{R}$ is Lipschitz. For $s < t$, consider the “annulus” $A_{s;t} := \{x : s < u(x) \leq t\}$ determined by u . Then, by appealing to the co-area

formula, we see that $\mathcal{A}_{s;t}$ is a set of finite perimeter for almost all $s < t$. Moreover, again appealing to the co-area formula, we see that, for almost all $s < t$,

$$(2.22) \quad \int_{\mathcal{A}_{s;t}} \operatorname{div} \mathbf{F} = - \int_{\partial^* \mathcal{A}_{s;t}} \mathbf{F}(y) \cdot \mathbf{v}(y) d\mathcal{H}^{N-1}(y).$$

One of the main objectives of this paper is to demonstrate that, even when $\mathbf{F} \in \mathcal{DM}^\infty(\mathbb{R}^N)$, we can extend Fuglede’s result by showing that (2.21) and (2.22) hold for all sets of finite perimeter, not merely for “almost all” sets in the sense of Fuglede [39]. Although we don’t employ Fuglede’s theorem directly, his result provided the motivation and insight for the development of our method.

The Gauss-Green formula for bounded divergence-measure fields over sets of finite perimeter was obtained in Chen and Torres [21]. The product rule from Lemma 2.31 was used to prove that

$$(2.23) \quad (\operatorname{div} \mathbf{F})(E^1) := \int_{E^1} \operatorname{div} \mathbf{F} = - \int_{\partial^* E} \overline{2\chi_E \mathbf{F} \cdot \nabla u_E},$$

where $\overline{2\chi_E \mathbf{F} \cdot \nabla u_E}$ is the weak-star limit of the measures $2\chi_E \mathbf{F} \cdot \nabla u_k$. Another objective of this paper is to obtain the trace measure as the limit of the normal trace over smooth boundaries that approximate $\partial^* E$.

3 The Normal Trace and the Gauss-Green Formula for \mathcal{DM}^∞ Fields over Smoothly Bounded Sets

In this section we obtain the normal trace and the corresponding Gauss-Green formula for a bounded divergence-measure field over any smoothly bounded set. Although Lemma 3.1 below is sufficient to obtain our main result, the reason why we give a proof of Theorem 3.3, which is much stronger than what our development requires, is twofold: First, it gives a self-contained treatment of the Gauss-Green formula for C^1 domains; second, it reveals the general outline of our method used to obtain the main result, Theorem 5.2. It also underscores the fact that the normal trace is indeed an interior normal trace in the sense that our definition is determined by the behavior of \mathbf{F} in the interior of U (see Definition 3.4). This method consists, roughly speaking, in approximating the boundary of the given set by a family of suitable surfaces for which the Gauss-Green theorem holds and then obtaining the desired trace as the weak limit of the normal trace over the approximating surfaces.

LEMMA 3.1 *Let $\mathbf{F} \in \mathcal{DM}_{\text{loc}}^\infty(\mathbb{R}^N)$ whose distributional divergence is a measure μ , and let \mathbf{F}_ε be a mollification of \mathbf{F} . Then the classical divergence theorem holds whenever $E \Subset \mathbb{R}^N$ is a set of finite perimeter, namely,*

$$(3.1) \quad \int_E \operatorname{div} \mathbf{F}_\varepsilon = - \int_{\partial^* E} \mathbf{F}_\varepsilon(y) \cdot \mathbf{v}(y) d\mathcal{H}^{N-1}(y).$$

If, in addition, we assume the two conditions

- (i) $\mathbf{F}_\varepsilon \rightarrow \mathbf{F}$ \mathcal{H}^{N-1} -a.e. on $\partial^* E$ and
- (ii) $\mu(\partial E) = 0$,

then

$$(3.2) \quad \mu(E) = - \int_{\partial^* E} \mathbf{F}(y) \cdot \mathbf{v}(y) d\mathcal{H}^{N-1}(y).$$

PROOF: First, the classical Gauss-Green formula (3.1) holds because \mathbf{F}_ε is smooth (in particular, Lipschitz). Furthermore, since $\mu_\varepsilon := \operatorname{div} \mathbf{F}_\varepsilon \xrightarrow{*} \operatorname{div} \mathbf{F} = \mu$ in $\mathcal{M}(\mathbb{R}^N)$ and since $\mathbf{F}_\varepsilon \llcorner \partial^* E$ is uniformly bounded, we obtain from (3.1) that

$$\begin{aligned} \mu_\varepsilon(E) &:= \int_E \operatorname{div} \mathbf{F}_\varepsilon = - \int_{\partial^* E} \mathbf{F}_\varepsilon(y) \cdot \mathbf{v}(y) d\mathcal{H}^{N-1}(y) \\ &\rightarrow - \int_{\partial^* E} \mathbf{F}(y) \cdot \mathbf{v}(y) d\mathcal{H}^{N-1}(y), \\ \mu_\varepsilon(E) &\rightarrow \mu(E) \quad (\text{by assumption (ii)}). \end{aligned}$$

This establishes our result. □

The importance of this result is that, with assumptions (i) and (ii), we obtain the Gauss-Green theorem for all sets of finite perimeter whenever \mathbf{F} is a bounded, measurable vector field with $\operatorname{div} \mathbf{F} = \mu$. As stated earlier, our main objective is to obtain the same result without assuming (i) and (ii) by defining a suitable notion of normal trace for \mathbf{F} on $\partial^* E$.

DEFINITION 3.2 Given a compact C^1 manifold M , we define the *exterior* determined by M to be that (connected) component \mathcal{U} of $\mathbb{R}^N \setminus M$ that is unbounded. The *interior* determined by M , U , is defined to be everything else in the complement of M ; namely,

$$U = \bigcup_{k=1}^{\infty} B_k \quad \text{with } B_k \subset \mathbb{R}^N \setminus M \quad \text{is a bounded component.}$$

Thus,

$$\mathbb{R}^N \setminus M = \mathcal{U} \cup \left(\bigcup_{k=1}^{\infty} B_k \right) = \mathcal{U} \cup U.$$

THEOREM 3.3 Let $U \subset \mathbb{R}^N$ be the interior determined by a compact C^1 manifold M of dimension $N - 1$ with $\mathcal{H}^{N-1}(M) < \infty$. Then, for any $\mathbf{F} \in \mathcal{DM}_{\text{loc}}^\infty(\mathbb{R}^N)$,

there exist a signed measure σ supported on $\partial U = M$ with $\sigma \ll \mathcal{H}^{N-1} \llcorner \partial U$ and a function $\mathcal{F}_i \cdot \mathbf{v} : \partial U \rightarrow \mathbb{R}$ such that, for any $\varphi \in C_c^1(\mathbb{R}^N)$,

$$(3.3) \quad \begin{aligned} \int_U \operatorname{div}(\varphi \mathbf{F}) &= \int_U \varphi \operatorname{div} \mathbf{F} + \int_U \mathbf{F} \cdot \nabla \varphi \\ &= - \int_{\partial U} \varphi \, d\sigma = - \int_{\partial U} \varphi (\mathcal{F}_i \cdot \mathbf{v})(y) \, d\mathcal{H}^{N-1}(y), \end{aligned}$$

and

$$\|\mathcal{F}_i \cdot \mathbf{v}\|_\infty \leq C \|\mathbf{F}\|_\infty,$$

where C is a constant depending only on N and U .

DEFINITION 3.4 With $\mathbf{v}(y)$ denoting the interior unit normal to M at y , we may regard $\mathcal{F}_i \cdot \mathbf{v}$ as the interior normal trace of \mathbf{F} on ∂U and thus write

$$(\mathcal{F}_i \cdot \mathbf{v})(y) = \mathbf{F}(y) \cdot \mathbf{v}(y).$$

Hence, with this convention, it is convenient to abuse the notation and write (3.3) as

$$(3.4) \quad \int_U \operatorname{div}(\varphi \mathbf{F}) = - \int_{\partial U} \varphi \mathbf{F}(y) \cdot \mathbf{v}(y) \, d\mathcal{H}^{N-1}(y),$$

while bearing in mind that, since \mathbf{F} is merely a measurable field and thus defined only up to a Lebesgue null set, it may not even be defined on ∂U . We use the term “interior normal trace” to suggest that $\mathcal{F}_i \cdot \mathbf{v}$ is determined by the behavior of \mathbf{F} in the interior determined by the manifold M . The proof will reveal that, in a similar way, it is possible to define the concept of “exterior normal trace.” This will be discussed more fully below in Theorem 5.2.

First, we adapt a result of Whitney [67] to our situation in which the open set U is the interior determined by a compact C^1 manifold M of dimension $N - 1$ with $\mathcal{H}^{N-1}(M) < \infty$. Whitney’s result states that an $(N - 1)$ -dimensional manifold of class C^1 has a C^1 vector field that is almost normal.

THEOREM 3.5 [67, theorem 10A, p. 121] *Suppose that $M := \partial U$ is an $(N - 1)$ -dimensional compact C^1 manifold in \mathbb{R}^N and let $\alpha > 0$. Then there exist a unit C^1 vector field, $\Lambda^* : M \rightarrow \mathbb{R}^N$, and a number $0 < \delta < 1$ (that depends on α) with the following properties:*

- (i) *If $\Lambda^*(p) = v$, then $|\pi_p(v)| \leq \alpha|v|$, where $\pi_p : \mathbb{R}^N \rightarrow T_p(M)$ denotes the orthogonal projection onto the tangent plane of M at p . Thus, $\Lambda^*(p)$ is close to $\mathbf{v}(p)$ when $\alpha > 0$ is small, and*

$$S_p^* := \{q \in \mathbb{R}^N : q = t\Lambda^*(p), 0 < t < \delta\} \subset U.$$

We think of the vectors $Q_p^ := \delta\Lambda^*(p)$ as quasi-normals and observe that $|Q_p^*| = \delta$.*

- (ii) As p ranges over M , the segments S_p^* fill out a neighborhood U_δ^* of M in a one-to-one way. That is,

$$U_\delta^* = \bigcup_{p \in M} S_p^*,$$

where $S_{p_1}^* \cap S_{p_2}^* = \emptyset$ whenever $p_1, p_2 \in M$ with $p_1 \neq p_2$.

- (iii) The mapping $\pi^* : U_\delta^* \rightarrow M$ defined by

$$\pi^*(q) := p \quad \text{if } q \in S_p^*$$

is of class C^1 and has the property that

$$\psi^*(q) := |\pi^*(q) - q| \leq 2 \operatorname{dist}(q, M) \quad \text{for } q \in U_\delta^*.$$

PROOF OF THEOREM 3.3: The proof is divided into ten parts.

Step 1. We begin with some preliminaries:

- (1) The mapping $\pi^* : U_\delta^* \rightarrow \partial U = M$ may be considered as the projection of U_δ^* onto ∂U along the quasi-normal Q_p^* , and the number $\psi^*(q)$ is the distance from q to M , measured along S_p^* where $\pi^*(q) = p$.
- (2) The open set U_t is defined, for all $0 < t < \delta$, as

$$U_t := U \setminus \{q \in U_\delta^* : \psi^*(q) < t\}.$$

All of the open sets U_t , $0 < t < \delta$, are nested and contained in U with $\lim_{t \rightarrow 0} U_t \uparrow U$.

- (3) Using the fact that ψ^* is continuous, we have

$$\partial U_t \subset (\psi^*)^{-1}(t) \quad \text{for all } t,$$

with equality holding whenever t is not a critical value of ψ^* .

- (4) We define $M_t := \partial U_t$. It will be shown that M_t is a C^1 manifold in Step 3.

Step 2. The open sets U_t are sets of finite perimeter for almost all $t \in (0, \delta)$.

From (iii) in Theorem 3.5, we know that π^* is of class C^1 on U_δ^* ; thus, so is $\psi^* \llcorner (U_\delta^* \setminus M)$. Since ψ^* is C^1 on the open set U_δ^* , it is therefore only locally Lipschitz. We may employ the co-area formula in Theorem 2.33 to conclude that, for any compact set A of finite perimeter, with $u := \psi^* \llcorner A \in \operatorname{BV}(\mathbb{R}^N)$,

$$\begin{aligned} \int_0^\infty \mathcal{H}^{N-1}(\partial U_t \cap A) dt &= \int_0^\infty \mathcal{H}^{N-1}(u^{-1}(t) \cap A) dt \\ &= \int_A |\nabla u| dx \leq \operatorname{Lip}(u) \mathcal{L}^N(U_\delta^*) < \infty. \end{aligned}$$

Step 3. For all $t \in (0, \delta)$, $\partial U_t := M_t$ is a manifold of class C^1 and there exists $C(\partial U, N)$ independent of t such that

$$(3.5) \quad \mathcal{H}^{N-1}(\partial U_t) = \mathcal{H}^{N-1}((\psi^*)^{-1}(t)) \leq C(\partial U, N) \mathcal{H}^{N-1}(\partial U).$$

The set $M_t := \partial U_t$ may be considered as a deformation of $M = \partial U$ along the vector field Q^* . To see this, consider the C^1 mapping $h_t : \partial U = M \rightarrow \partial U_t = M_t$ defined for $0 < t < \delta$ as

$$q := h_t(p) = t\Lambda^*(p) \in \partial U_t,$$

so that $h_0(p) = p$ and $\pi^*(q) = p$ with $|h_t(p) - p| = t$.

For each $0 < t < \delta$, the map $h_t : M \rightarrow M_t$ is clearly of class C^1 ; the Jacobian Jh_t depends only on t and $\|D\Lambda^*\|_\infty$. Therefore, since h is univalent, we may invoke (2.20) to conclude the following global bound whenever $A \subset M$ is an \mathcal{H}^{N-1} -measurable set:

$$\begin{aligned} \mathcal{H}^{N-1}(h_t(A)) &= \mathcal{H}^{N-1}(h_t(M \cap A)) \\ &= \int_A Jh_t d\mathcal{H}^{N-1}(x) \\ (3.6) \quad &\leq C(t, \|D\Lambda^*\|_\infty) \mathcal{H}^{N-1}(A \cap M) \\ &\leq C(\delta, \|D\Lambda^*\|_\infty) \mathcal{H}^{N-1}(A). \end{aligned}$$

Since $\pi^* \circ h_t = I$ (i.e., the identity), the chain rule implies that the differential of h_t is nonsingular everywhere on M . Thus, we see that h_t is a diffeomorphism and hence that M_t is an $(N - 1)$ -manifold of class C^1 . Then (3.5) is immediate from (3.6).

Observe also that there exists a set $\mathbb{N} \subset (0, \delta)$ of \mathcal{L}^1 -measure zero that includes those countable values of t for which $\|\operatorname{div} \mathbf{F}\|((\psi^*)^{-1}(t)) \neq 0$. Also, referring to Lemma 2.34, it is clear that the values of t for which condition (i) in Lemma 3.1 fails will be included in \mathbb{N} and, for the rest of the proof, we consider only values of t outside of the exceptional set \mathbb{N} .

Step 4. The signed measures defined by

$$\begin{aligned} \sigma_t(B) &:= \int_{B \cap \partial U_t} \mathbf{F}(y) \cdot \mathbf{v}(y) d\mathcal{H}^{N-1}(y) \quad \text{for each Borel set } B \subset \mathbb{R}^N, \\ (3.7) \quad &= \int_{B \cap u^{-1}(t)} \mathbf{F}(y) \cdot \mathbf{v}(y) d\mathcal{H}^{N-1}(y) \quad \text{for a.e. } t \text{ (by Theorem 2.32),} \end{aligned}$$

along with σ_t^+ and σ_t^- , where $\sigma_t = \sigma_t^+ - \sigma_t^-$, all converge, for a suitable subsequence $t_k \rightarrow 0$, weak-star to

$$(3.8) \quad (\sigma_{t_k}^+, \sigma_{t_k}^-, \sigma_{t_k}) \xrightarrow{*} (\sigma^+, \sigma^-, \sigma) \quad \text{in } \mathcal{M}(\mathbb{R}^N).$$

Since $\mathbf{F} \in L^\infty$ and $\mathcal{H}^{N-1}(\partial U) < \infty$, we see from (3.5) that $\mathcal{H}^{N-1}(\partial U_t) \leq C$ for some $C > 0$ independent of t , which yields that the measures σ_t , $t > 0$, form

a bounded set in $\mathcal{M}(\mathbb{R}^N)$. Hence there exist a sequence $\{t_k\} \rightarrow 0$ and Radon measures σ, σ^+ , and σ^- with $\sigma = \sigma^+ - \sigma^-$ such that

$$(3.9) \quad (\sigma_{t_k}^+, \sigma_{t_k}^-, \sigma_{t_k}) \xrightarrow{*} (\sigma^+, \sigma^-, \sigma) \quad \text{in } \mathcal{M}(\mathbb{R}^N).$$

It should be kept in mind that the convergence in (3.9) means, for example, that

$$(3.10) \quad \sigma_{t_k}(\varphi) \rightarrow \sigma(\varphi) \quad \text{for each test function } \varphi \in C_c(\mathbb{R}^N).$$

Step 5. The supports of the measures σ^+ , σ^- , and σ are all contained in ∂U .

To prove the claim for σ^+ , we assume the contrary, and let $x \in \text{spt}(\sigma^+) \setminus \partial U$ and choose $B(x, r)$ such that $B(x, r) \cap \partial U = \emptyset$. Since $x \in \text{spt}(\sigma^+)$, there exists $\varphi \in C(B(x, r))$ such that $\int \varphi d\sigma^+ := \sigma^+(\varphi) \neq 0$. Then, since φ is continuous, we find that $\sigma_{t_k}^+(\varphi) \rightarrow \sigma^+(\varphi) \neq 0$. This implies that $\sigma_{t_k}^+(\varphi) \neq 0$ for all small $t_k > 0$, which leads to a contradiction since $U_{t_k} \subset U$, $\text{spt}(\sigma_{t_k}^+) \subset \partial U_{t_k}$, and $\partial U_{t_k} \cap B(x, r) = \emptyset$. The proof of the claim for σ^- and σ can be done in the same way.

Step 6. With τ representing any of the three measures σ^+ , σ^- , and σ , we have

$$(3.11) \quad \lim_{t_k \rightarrow 0} \tau_{t_k}(\partial U_{t_k}) \rightarrow \tau(M) = \tau(\partial U).$$

First, let $\tau := \sigma^+$. Observe that $\text{spt } \tau_{t_k} \subset \partial U_{t_k}$ implies

$$(3.12) \quad \lim_{t_k \rightarrow 0} \tau_{t_k}(\partial U_{t_k}) \rightarrow \tau(\partial U).$$

By Lemma 2.3 and Step 5,

$$(3.13) \quad \liminf_{t_k \rightarrow 0} \tau_{t_k}(\partial U_{t_k}) = \liminf_{k \rightarrow \infty} \tau_{t_k}(\mathbb{R}^N) \geq \tau(\mathbb{R}^N) = \tau(\partial U).$$

Now choose a compact set $K \supset \partial U \cup \partial U_{t_k}$. Then, again by Lemma 2.3 and Step 5, we obtain

$$(3.14) \quad \begin{aligned} \limsup_{t_k \rightarrow 0} \tau_{t_k}(\partial U_{t_k}) &= \limsup_{t_k \rightarrow 0} \tau_{t_k}(K) \\ &\leq \tau(K) = \tau(\partial U) \quad \text{since } K \text{ contains } \partial U. \end{aligned}$$

Thus, we obtain our desired result, (3.11), by (3.13) and (3.14). By taking τ to be the other positive measure $\tau = \sigma^-$, we obtain (3.11) for σ^- , and consequently (3.11) holds for σ as well.

Step 7. The measure σ is well-defined.

This can be seen as follows: Let $U_{t'_k}$ be another sequence of open sets to which Lemma 3.1 applies. Then, assuming that $t_k > t'_k$ for all k , we have

$$\begin{aligned} \mu(U_{t'_k} \setminus U_{t_k}) &= \int_{U_{t'_k}} \operatorname{div} \mathbf{F} - \int_{U_{t_k}} \operatorname{div} \mathbf{F} \\ &= - \int_{\partial U_{t'_k}} \mathbf{F}(y) \cdot \mathbf{v}(y) d\mathcal{H}^{N-1}(y) + \int_{\partial U_{t_k}} \mathbf{F}(y) \cdot \mathbf{v}(y) d\mathcal{H}^{N-1}(y) \\ &= -\sigma_{t'_k}(\partial U_{t'_k}) + \sigma_{t_k}(\partial U_{t_k}). \end{aligned}$$

Since $U_{t'_k} \setminus U_{t_k} \subset U \setminus U_{t_k}$ is a monotone decreasing sequence of sets with $U \setminus U_{t_k} \rightarrow \emptyset$, it follows that $\|\mu\|(U_{t'_k} \setminus U_{t_k}) \rightarrow 0$ and therefore that $\sigma_{t'_k}(\partial U_{t'_k}) - \sigma_{t_k}(\partial U_{t_k}) \rightarrow 0$, which shows that σ is well-defined.

Step 8. $\sigma \ll \mathcal{H}^{N-1} \llcorner \partial U$.

Let $A \subset \partial U$ with $\mathcal{H}^{N-1}(A) = 0$. From (2.8), we have that $\|\nabla \chi_U\| = \mathcal{H}^{N-1} \llcorner \partial U$ and hence $\|\nabla \chi_U\|(A) = 0$. From general measure theory, we have

$$(3.15) \quad 0 = \|\nabla \chi_E\|(A) = \inf\{\|\nabla \chi_E\|(G) : A \subset G, G \text{ open}\}$$

and thus there exists an open set $G \subset \mathbb{R}^N$, $G \supset A$, such that $\mathcal{H}^{N-1}(G \cap \partial U) < \varepsilon$. Moreover, using (3.6), we obtain

$$\begin{aligned} \|\sigma_{t_k}\|(G) &\leq \int_{G \cap \partial U_{t_k}} |\mathbf{F}(y) \cdot \mathbf{v}(y)| d\mathcal{H}^{N-1}(y) \\ &\leq \|\mathbf{F}\|_\infty \mathcal{H}^{N-1}(G \cap \partial U_{t_k}) \\ &\leq C(N, \partial U) \|\mathbf{F}\|_\infty \mathcal{H}^{N-1}(h_{t_k}^{-1}(G \cap \partial U_{t_k})), \end{aligned}$$

where $\lim_{t_k \rightarrow 0} \mathcal{H}^{N-1}(h_{t_k}^{-1}(G \cap \partial U_{t_k})) = \mathcal{H}^{N-1}(G \cap \partial U)$. Then,

$$\begin{aligned} \|\sigma\|(A) \leq \|\sigma\|(G) &\leq \liminf_{t_k \rightarrow 0} \|\sigma_{t_k}\|(G) = C \|\mathbf{F}\|_\infty \mathcal{H}^{N-1}(G \cap \partial U) \\ &\leq \varepsilon C \|\mathbf{F}\|_\infty. \end{aligned}$$

Since ε is arbitrary, we conclude $\|\sigma\|(A) = 0$, as desired.

Step 9. We apply Lemma 3.1 to the divergence-measure field $\varphi \mathbf{F}$ and take limits to obtain (3.3).

From Theorem 2.31, it follows that, if $\varphi \in C_c^1(\mathbb{R}^N)$, then $\varphi \mathbf{F}$ is also a divergence-measure field and

$$(3.16) \quad \operatorname{div}(\varphi \mathbf{F}) = \varphi \operatorname{div} \mathbf{F} + \mathbf{F} \cdot \nabla \varphi.$$

An application of Lemma 3.1 shows that

$$(3.17) \quad \int_{U_{t_k}} \operatorname{div}(\varphi \mathbf{F}) = - \int_{\partial U_{t_k}} \varphi \mathbf{F}(y) \cdot \mathbf{v}(y) d\mathcal{H}^{N-1}(y) \quad \text{for all } t_k \notin \mathbb{N},$$

where $\mu_\varphi := \operatorname{div}(\varphi \mathbf{F})$ is a Radon measure. Therefore, since the sets U_{t_k} are increasing, we have

$$(3.18) \quad \lim_{t_k \rightarrow 0} \mu_\varphi(U_{t_k}) \rightarrow \mu_\varphi(U).$$

Thus, using (3.10) and (3.18) and letting $k \rightarrow \infty$ in (3.17) yields

$$(3.19) \quad \int_U \operatorname{div}(\varphi \mathbf{F}) = - \int_{\partial U} \varphi d\sigma.$$

Step 10. The Radon-Nikodym derivative of σ with respect to $\mathcal{H}^{N-1} \llcorner \partial U$, $T(y)$, is by definition the normal trace of \mathbf{F} , denoted by $\mathcal{F}_i \cdot \mathbf{v}(y)$, and enjoys the bound

$$|\mathcal{F}_i \cdot \mathbf{v}(y)| \leq C(\partial U, N).$$

Since $\sigma \ll \mathcal{H}^{N-1} \llcorner \partial U$, the Radon-Nikodym theorem implies that there exists an \mathcal{H}^{N-1} -integrable function $T : \partial U \rightarrow \mathbb{R}$ such that (3.19) can be written as

$$\int_U \operatorname{div}(\varphi \mathbf{F}) = - \int_{\partial U} \varphi T(y) d\mathcal{H}^{N-1}(y).$$

Note that T is the Radon-Nikodym derivative of σ with respect to $\mathcal{H}^{N-1} \llcorner \partial U$ whose value at \mathcal{H}^{N-1} -a.e. $y \in \partial U$ can be determined by Besicovitch's differentiation theorem [7]:

$$|T(y)| = \lim_{r \rightarrow 0} \left| \frac{\sigma(B(y, r))}{\mathcal{H}^{N-1}(\partial U \cap B(y, r))} \right|.$$

Since the balls $B(y, r_j)$ can be chosen such that $\|\sigma\|(\partial B(y, r_j)) = 0$, we have

$$\begin{aligned} |T(y)| &= \lim_{r_j \rightarrow 0} \lim_{t_k \rightarrow 0} \left| \frac{\sigma_{t_k}(B(y, r_j))}{\mathcal{H}^{N-1}(\partial U \cap B(y, r_j))} \right| \\ &= \lim_{r_j \rightarrow 0} \lim_{t_k \rightarrow 0} \left| \frac{\int_{\partial U_{t_k} \cap B(y, r_j)} \mathbf{F}(y) \cdot \mathbf{v}(y) d\mathcal{H}^{N-1}(y)}{\mathcal{H}^{N-1}(\partial U \cap B(y, r_j))} \right| \\ &\leq \|\mathbf{F}\|_\infty \lim_{r_j \rightarrow 0} \lim_{t_k \rightarrow 0} \frac{\mathcal{H}^{N-1}(\partial U_{t_k} \cap B(y, r_j))}{\mathcal{H}^{N-1}(\partial U \cap B(y, r_j))} \\ &\leq C(\partial U, N) \|\mathbf{F}\|_\infty \lim_{r_j \rightarrow 0} \frac{\mathcal{H}^{N-1}(\partial U \cap B(y, r_j))}{\mathcal{H}^{N-1}(\partial U \cap B(y, r_j))} \quad (\text{by (3.6)}) \\ &= C(\partial U, N) \|\mathbf{F}\|_\infty. \end{aligned}$$

□

Remark 3.6. In particular, when U is of class C^2 , the interior normals to ∂U themselves do not intersect in a sufficiently small neighborhood of ∂U . Therefore, in the above proof, one may directly use the interior normals to ∂U as Λ_p^* .

4 Almost One-Sided Smooth Approximation of Sets of Finite Perimeter

We now proceed to establish a fundamental approximation theorem for a set of finite perimeter by a family of sets with smooth boundary essentially from the measure-theoretic interior of the set with respect to any Radon measure that is absolutely continuous with respect to \mathcal{H}^{N-1} . That is, we prove that, for any Radon measure μ on \mathbb{R}^N such that $\mu \ll \mathcal{H}^{N-1}$, the superlevel sets of the mollifications of the characteristic functions of sets of finite perimeter provide an approximation by smooth sets that are $\|\mu\|$ -almost contained in the measure-theoretic interior of E . This approximation and Lemma 3.1, after passage to a limit, lead to our main result, Theorem 5.2.

LEMMA 4.1 *Let μ be a Radon measure on \mathbb{R}^N such that $\mu \ll \mathcal{H}^{N-1}$. Let E be a set of finite perimeter, and let u_k be the mollification of χ_E . Then, for any $t \in (0, 1)$ and $A_{k;t} := \{y : u_k(y) > t\}$, there exist $\varepsilon = \varepsilon(t)$ and $k^* = k^*(\varepsilon, t)$ such that*

- (i) $\|\mu\|(A_{k;t} \setminus E) < \varepsilon$ if $t \in (0, \frac{1}{2})$ and $k \geq k^*$;
- (ii) $\|\mu\|(A_{k;t} \setminus E^1) < \varepsilon$ if $t \in (\frac{1}{2}, 1)$ and $k \geq k^*$;
- (iii) $\|\mu\|(E^1 \setminus A_{k;t}) < \varepsilon$ if $t \in (\frac{1}{2}, 1)$ and $k \geq k^*$;
- (iv) $\|\mu\|(E \setminus A_{k;t}) < \varepsilon$ if $t \in (0, \frac{1}{2})$ and $k \geq k^*$.

PROOF: We first show (ii). With $t \in (\frac{1}{2}, 1)$, choose $0 < \varepsilon < t - \frac{1}{2}$. Since u_k is the mollification of χ_E , we know that $u_k(y) \rightarrow u_E(y)$ for \mathcal{H}^{N-1} -a.e. y and therefore the same is true for $\|\mu\|$ as well. By Egorov's theorem, for any $\varepsilon > 0$, there exist $k^* = k^*(\varepsilon, t)$ and an open set U_ε such that $\|\mu\|(U_\varepsilon) < \varepsilon$ and that $|u_k(y) - u_E(y)| < \varepsilon$ for all $y \notin U_\varepsilon$ and for all $k \geq k^*$. On $A_{k;t} \setminus U_\varepsilon$, we have

$$t < u_k(y).$$

Since $u_k(y) < u_E(y) + \varepsilon$ on $\mathbb{R}^N \setminus U_\varepsilon$, we have

$$\frac{1}{2} < t - \varepsilon < u_E(y) \implies u_E(y) = 1 \implies y \in E^1.$$

This yields

$$A_{k;t} \setminus U_\varepsilon \subset E^1 \implies A_{k;t} \setminus E^1 \subset U_\varepsilon.$$

Since $\|\mu\|(U_\varepsilon) < \varepsilon$, our desired result (ii) follows.

For the proof of (i), given $t \in (0, \frac{1}{2})$, we choose $0 < \varepsilon < t$ and proceed as above.

We next show (iv). With $t \in (0, \frac{1}{2})$, choose $0 < \varepsilon < \frac{1}{2} - t$. For all large k , we have $|u_k(y) - u_E(y)| < \varepsilon$ for all $y \notin U_\varepsilon$. Thus, on $E \setminus U_\varepsilon$,

$$\begin{aligned} \frac{1}{2} - u_k(y) &\leq u_E(y) - u_k(y) < \varepsilon \\ \implies u_k(y) &> \frac{1}{2} - \varepsilon > t \quad \text{for all } y \in E \setminus U_\varepsilon, \end{aligned}$$

which implies $E \setminus U_\varepsilon \subset A_{k;t}$; therefore, $E \setminus A_{k;t} \subset U_\varepsilon$ and thus $\|\mu\|(E \setminus A_{k;t}) < \varepsilon$.

For the proof of (iii), given $t \in (\frac{1}{2}, 1)$, we choose $0 < \varepsilon < 1 - t$ and proceed as above. \square

COROLLARY 4.2 *For each $t \in (0, \frac{1}{2})$ and $\varepsilon < \min\{t, \frac{1}{2} - t\}$, there exists $k^* = k^*(\varepsilon, t) > 0$ such that*

$$(4.1) \quad \|\mu\|(A_{k;t} \Delta E) < \varepsilon \quad \text{whenever } k \geq k^*.$$

For each $t \in (\frac{1}{2}, 1)$ and $\varepsilon < \min\{t - \frac{1}{2}, 1 - t\}$, there exists $k^ = k^*(\varepsilon, t) > 0$ such that*

$$(4.2) \quad \|\mu\|(A_{k;t} \Delta E^1) < \varepsilon \quad \text{whenever } k \geq k^*.$$

Remark 4.3. In the previous results, we have used the open superlevel sets $A_{k;t} := \{y : u_k(y) > t\}$. However, we could have used the closed superlevel sets $\bar{A}_{k;t} := \{y : u_k(y) \geq t\}$ to obtain the same results. We also note that, for an arbitrary Radon measure ω , we have

$$(4.3) \quad \omega(\bar{A}_{k;t}) - \omega(A_{k;t}) = \omega(\partial A_{k;t}) = 0$$

for all but countably many t , for the reason that the family of sets $\{\partial A_{k;t} : t \in \mathbb{R}\}$ is pairwise disjoint and any Radon measure ω can assign positive values to only a countable number of such a family.

COROLLARY 4.4 *For almost every $t > 0$, there exist $\varepsilon(t)$ and $k^* = k^*(\varepsilon, t) > 0$ such that*

- (i) $\|\mu\|(A_{k;t} \Delta E) = \|\mu\|(\bar{A}_{k;t} \Delta E) < \varepsilon$ if $t \in (0, \frac{1}{2})$ and $k \geq k^*$;
- (ii) $\|\mu\|(A_{k;t} \Delta E^1) = \|\mu\|(\bar{A}_{k;t} \Delta E^1) < \varepsilon$ if $t \in (\frac{1}{2}, 1)$ and $k \geq k^*$;
- (iii) $\|\mu\|(\partial A_{k;t} \Delta E) = \|\mu\|(u_k^{-1}(t) \Delta E) < \varepsilon$ if $t \in (0, \frac{1}{2})$ and $k \geq k^*$;
- (iv) $\|\mu\|(\partial A_{k,t} \Delta E^1) = \|\mu\|(u_k^{-1}(t) \Delta E^1) < \varepsilon$ if $t \in (\frac{1}{2}, 1)$ and $k \geq k^*$.

For the case $t = \frac{1}{2}$, only (i) and (iii) in Lemma 4.1 remain valid. To see this, we first show the following:

LEMMA 4.5 *Let μ be a Radon measure on \mathbb{R}^N such that $\mu \ll \mathcal{H}^{N-1}$. Let E be a set of finite perimeter, and let u_k be the mollification of χ_E . Then, for $t = \frac{1}{2}$ and $\varepsilon > 0$, there exists $k^* = k^*(\varepsilon)$ such that*

$$(4.4) \quad \|\mu\|(E^1 \setminus A_{k;1/2}) < \varepsilon \quad \text{and} \quad \|\mu\|(A_{k;1/2} \setminus E) < \varepsilon.$$

PROOF: Since $u_k(y) \rightarrow u_E(y)$ for \mathcal{H}^{N-1} -a.e. y , the dominated convergence theorem implies that $u_k \rightarrow u_E$ in $L^1(\mathbb{R}^N, \|\mu\|)$. Thus, given any $\varepsilon > 0$, there exists $k^* = k^*(\varepsilon)$ such that, when $k \geq k^*$, we have

$$(4.5) \quad \begin{aligned} \frac{\varepsilon}{2} &\geq \int_{\mathbb{R}^N} |u_E - u_k| d\|\mu\| \\ &\geq \int_{E^1 \setminus A_{k;1/2}} (u_E - u_k) d\|\mu\| \geq \left(1 - \frac{1}{2}\right) \|\mu\|(E^1 \setminus A_{k;1/2}), \end{aligned}$$

which implies

$$(4.6) \quad \|\mu\|(E^1 \setminus A_{k;1/2}) \leq \varepsilon.$$

In the same way, we compute

$$(4.7) \quad \frac{\varepsilon}{2} \geq \int_{A_{k;1/2} \setminus E} |u_E - u_k| d\|\mu\| \geq \left(\frac{1}{2} - 0\right) \|\mu\|(A_{k;1/2} \setminus E),$$

which implies

$$(4.8) \quad \|\mu\|(A_{k;1/2} \setminus E) \leq \varepsilon.$$

□

The following remark shows that, with $t = \frac{1}{2}$ and $\mu = \mathcal{H}^{N-1} \llcorner \partial^* E \geq 0$, (ii) and (iv) in Lemma 4.1 do not hold.

Remark 4.6. If we define $E := \{y \in \mathbb{R}^N : |y| \leq 1\}$, then $u_k^{-1}(\frac{1}{2}) \subset \mathbb{R}^N \setminus E$ for all k . Therefore, it is clear that

$$(4.9) \quad \mathcal{H}^{N-1}((A_{k;1/2} \setminus E^1) \cap \partial^* E) = \mathcal{H}^{N-1}(\partial^* E) \not\rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

If we now define $E := \{y \in \mathbb{R}^N : |y| \geq 1\}$, then $u_k^{-1}(\frac{1}{2}) \subset E$ for all k . Thus, we have

$$(4.10) \quad \mathcal{H}^{N-1}((E \setminus A_{k;1/2}) \cap \partial^* E) = \mathcal{H}^{N-1}(\partial^* E) \not\rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

LEMMA 4.7 *There exists $C < \infty$ such that, for all positive integers k and almost all $t \in (0, 1)$,*

$$(4.11) \quad \mathcal{H}^{N-1}(u_k^{-1}(t)) \leq C.$$

PROOF: From Corollaries 4.2 and 4.4, it follows that, for almost all $t \in (0, 1)$, the sequence of smoothly bounded sets $A_{k;t} = \{u_k > t\}$ satisfies $\chi_{A_{k;t}} \rightarrow \chi_{E^1}$ μ -a.e. if $t \in (\frac{1}{2}, 1)$, or $\chi_{A_{k;t}} \rightarrow \chi_E$ μ -a.e. if $t \in (0, \frac{1}{2})$. Since $\mu = \mathcal{H}^{N-1} \llcorner \partial^* E \ll \mathcal{H}^{N-1}$, it follows that $\chi_{A_{k;t}} \rightarrow \chi_E$ everywhere except for a set of Lebesgue measure zero.

We let \mathcal{V} denote the Banach space $C^1(K)$ of vector fields ψ on K endowed with the norm

$$\|\psi\| := \sup_{y \in K} \left(|\psi(y)| + \sum_{i=1}^N |\nabla \psi_i(y)| \right),$$

where K is a compact set such that $E \Subset K$. Let

$$T_E(\psi) := \int_E \psi(y) dy$$

and, for almost every $t \in (0, 1)$, let

$$T_{k;t}(\psi) := \int_{\{y: u_k(y) > t\}} \psi(y) dy = \int_{A_{k;t}} \psi(y) dy \quad \text{for } \psi \in \mathcal{V}.$$

Then, for $\psi \in \mathcal{V}$, we define the linear operators

$$\partial T_E(\psi) := T_E(\operatorname{div} \psi) = \int_E \operatorname{div} \psi \, dy = - \int_{\partial^* E} \psi \cdot \nu \, d\mathcal{H}^{N-1}$$

and

$$\partial T_{k;t}(\psi) := T_{k;t}(\operatorname{div} \psi) = \int_{A_{k;t}} \operatorname{div} \psi \, dy = - \int_{\partial A_{k;t}} \psi \cdot \nu \, d\mathcal{H}^{N-1},$$

where ν is the interior unit normal.

Since $u_k^{-1}(t)$ is a C^∞ manifold for almost every t , then

$$\|\partial T_{k;t}\| := \sup_{\|\psi\| \leq 1} |\partial T_{k;t}(\psi)| = \mathcal{H}^{N-1}(u_k^{-1}(t)).$$

Indeed, with $\psi := \nu_k / |\nu_k|$ defined on the manifold $u_k^{-1}(t)$, the norm-preserving extension of ψ to all of \mathbb{R}^N by Whitney's extension theorem yields the inequality

$$\|\partial T_{k;t}\| := \sup_{\|\psi\| \leq 1} |\partial T_{k;t}(\psi)| \geq \mathcal{H}^{N-1}(u_k^{-1}(t)).$$

The opposite inequality is obvious.

Moreover, we find by the dominated convergence theorem that

$$\lim_{k \rightarrow \infty} \partial T_{k;t}(\psi) \rightarrow \partial T_E(\psi) \quad \text{for } \psi \in \mathcal{V},$$

and therefore

$$\sup_k \{|\partial T_{k;t}(\psi)|\} < \infty \quad \text{for } \psi \in \mathcal{V}.$$

By the uniform boundedness principle (Theorem 2.2), we see that, since $\partial T_{k;t}$ is a linear functional on \mathcal{V} whose weak limit ∂T_E is independent of t , we have

$$\sup_k \mathcal{H}^{N-1}(u_k^{-1}(t)) = \sup_k \|\partial T_{k;t}\| \leq C < \infty,$$

where $C > 0$ is independent of t , which gives our desired result. \square

The above argument simply rephrases the following basic fact from the theory of currents. We know that, since E has finite perimeter, T_E is an integral current. Moreover, the currents $T_{k;t}$ converge to T_E weakly, and so do their boundaries, $\partial T_{k;t} \rightarrow \partial T_E$; that is,

$$\int_{u_k^{-1}(t)} \sigma d\mathcal{H}^{N-1} = \int_{\partial A_{k;t}} \sigma d\mathcal{H}^{N-1} \rightarrow \int_{\partial^* E} \sigma d\mathcal{H}^{N-1}$$

for each smooth differential $(N - 1)$ -form σ . Appealing to Theorem 2.2 yields our result.

LEMMA 4.8 *Let $u : \Omega \rightarrow \mathbb{R}$ be a Lipschitz function and $E \Subset \Omega$ a set of finite perimeter. Then*

$$\mathcal{H}^{N-1}(\partial^* E \cap u^{-1}(t)) = 0 \quad \text{for almost all } t.$$

PROOF: This result follows directly from Lemma 2.34, since we know that $\mathcal{H}^{N-1}(\partial^* E) < \infty$ for any bounded set of finite perimeter $E \Subset \Omega$. \square

LEMMA 4.9 *For almost every $t \in (\frac{1}{2}, 1)$, we have*

$$(4.12) \quad \mathcal{H}^{N-1}(\partial^* E \cap u_k^{-1}(t)) = 0$$

and

$$(4.13) \quad \lim_{k \rightarrow \infty} \mathcal{H}^{N-1}(\partial^* E \cap A_{k;t}) = 0.$$

PROOF: This can be seen as follows. If we use Corollary 4.2 with $\mu = \mathcal{H}^{N-1} \llcorner \partial^* E$, we obtain

$$\lim_{k \rightarrow \infty} \mu(A_{k;t} \setminus E^1) = \lim_{k \rightarrow \infty} \mathcal{H}^{N-1}(A_{k;t} \cap \partial^* E) = 0.$$

Clearly, (4.12) follows from Lemma 4.8 (see also Remark 4.3). \square

Now we can establish the main theorem of this section.

THEOREM 4.10 (Approximation Theorem) *For almost every $t \in (\frac{1}{2}, 1)$, we have*

$$\lim_{k \rightarrow \infty} \mathcal{H}^{N-1}((E^0 \cup \partial^* E) \cap u_k^{-1}(t)) = 0.$$

PROOF: Since the Lebesgue measure is absolutely continuous with respect to \mathcal{H}^{N-1} , then using (4.2) in Corollary 4.2 with $s > \frac{1}{2}$ leads to

$$|A_{k;s} \Delta E^1| \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Therefore, if we define

$$R_{k;s} := A_{k;s} \setminus E^1,$$

it follows that

$$(4.14) \quad |R_{k;s}| \rightarrow 0 \quad \text{provided that } s > \frac{1}{2}.$$

Remark 4.3 indicates that we have the option of defining $A_{k;t} := \{y : u_k(y) \geq t\}$ without altering the development. With this option in force, we have $u_k^{-1}(s) \subset A_{k;s}$, and consequently, by the co-area formula (Theorem 2.32),

$$\begin{aligned} \int_{R_{k;s}} |\nabla u_k| dy &= \int_0^1 \mathcal{H}^{N-1}(u_k^{-1}(t) \cap R_{k;s}) dt \\ &= \int_s^1 \mathcal{H}^{N-1}(u_k^{-1}(t) \cap (E^0 \cup \partial^* E)) dt. \end{aligned}$$

Since $\nabla u_k \xrightarrow{*} \nabla \chi_E$ and $\|\nabla u_k\|_1 \leq \|\nabla \chi_E\|$ (Lemma 2.16), it follows from Vitali's convergence theorem for $s > \frac{1}{2}$ that

$$\int_{R_{k;s}} |\nabla u_k| dy \rightarrow 0.$$

Thus, for a subsequence if necessary, we can conclude that, for a.e. $t > s$,

$$\mathcal{H}^{N-1}(u_k^{-1}(t) \cap (E^0 \cup \partial^* E)) \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

The dependence on the subsequence is illusory. The reason is that, if there were a subsequence such that, for a.e. t ,

$$\mathcal{H}^{N-1}(u_k^{-1}(t) \cap (E^0 \cup \partial^* E)) \rightarrow \alpha \neq 0 \quad \text{as } k \rightarrow \infty,$$

one could then appeal to our previous argument to conclude that, for some further subsequence and for a.e. t ,

$$\mathcal{H}^{N-1}(u_k^{-1}(t) \cap (E^0 \cup \partial^* E)) \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

which is contrary to our assertion that $\alpha \neq 0$.

Since $s > \frac{1}{2}$ is fixed arbitrarily at the beginning of the proof, we conclude that, for a.e. $t > \frac{1}{2}$,

$$\mathcal{H}^{N-1}(u_k^{-1}(t) \cap (E^0 \cup \partial^* E)) \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

□

5 Main Theorem

In this section we establish our main result of this paper, Theorem 5.2. Let $\mathbf{F} \in \mathcal{DM}_{\text{loc}}^\infty(\Omega)$. We define, for almost every t , a measure $\sigma_{k;t}$ for all Borel sets $B \Subset \Omega$ by

$$(5.1) \quad \sigma_{k;t}(B) := \int_{B \cap \partial A_{k;t}} \mathbf{F}(y) \cdot \mathbf{v}(y) d\mathcal{H}^{N-1}(y),$$

where $\mathbf{F}(y) \cdot \mathbf{v}(y)$ denotes the classical dot product of \mathbf{F} with the unit normal \mathbf{v} .

We begin with a lemma that will lead to several of the assertions in Theorem 5.2.

LEMMA 5.1 *If $E \Subset \Omega$ is an arbitrary set of finite perimeter, then we have*

$$(5.2) \quad \int_E \mathbf{F} \cdot \nabla u_k \, dy = \int_0^1 \int_{E \cap u_k^{-1}(t)} \mathbf{F} \cdot \mathbf{v}_k \, d\mathcal{H}^{N-1} \, dt$$

for any $\mathbf{F} \in L_{\text{loc}}^\infty(\Omega)$, where u_k denotes the mollification of χ_E as introduced in Definition 2.11 and Lemma 2.15.

PROOF: Let \mathcal{N} be the set on which $\nabla u_k = 0$. Then

$$\begin{aligned} \int_E \mathbf{F} \cdot \nabla u_k \, dy &= \int_{E \setminus \mathcal{N}} \mathbf{F} \cdot \nabla u_k \, dy + \int_{\mathcal{N}} \mathbf{F} \cdot \nabla u_k \, dy \\ &= \int_{E \setminus \mathcal{N}} |\nabla u_k| \frac{\mathbf{F} \cdot \nabla u_k}{|\nabla u_k|} \, dy + 0 \\ &= \int_E |\nabla u_k| g \, dy, \end{aligned}$$

where $g = \frac{\chi_{E \setminus \mathcal{N}} \mathbf{F} \cdot \nabla u_k}{|\nabla u_k|}$. Then, by the co-area formula, we have

$$\begin{aligned} \int_E \mathbf{F} \cdot \nabla u_k \, dy &= \int_0^1 \int_{u_k^{-1}(t) \cap (E \setminus \mathcal{N})} g \, d\mathcal{H}^{N-1} \, dt \\ &= \int_0^1 \int_{u_k^{-1}(t) \cap (E \setminus \mathcal{N})} \mathbf{F} \cdot \mathbf{v}_k \, d\mathcal{H}^{N-1} \, dt \\ &= \int_0^1 \int_{u_k^{-1}(t) \cap E} \mathbf{F} \cdot \mathbf{v}_k \, d\mathcal{H}^{N-1} \, dt, \end{aligned}$$

where we have used $\mathbf{v}_k(y) = \frac{\nabla u_k(y)}{|\nabla u_k(y)|}$ for $y \in u_k^{-1}(t) \cap (E \setminus \mathcal{N})$. □

With the help of Lemma 3.1 and the results in Section 4, we now establish our main theorem.

THEOREM 5.2 (Main Theorem) *Let $\Omega \subset \mathbb{R}^N$ be an open set. Suppose that $\mathbf{F} \in \mathcal{DM}_{\text{loc}}^\infty(\Omega)$ with $\text{div } \mathbf{F} = \mu \in \mathcal{M}(\Omega)$. Let $E \Subset \Omega$ be a set of finite perimeter. Then*

- (i) *For almost every $s \in (\frac{1}{2}, 1)$, there exist a signed measure σ_i (independent of s) and a family of sets $A_{k;s}$ with smooth boundaries such that*
 - (a) $\|\mu\|(A_{k;s} \Delta E^1) \rightarrow 0$;
 - (b) *the measure σ_i is the weak-star limit of the measures $\sigma_{k;s}$;*
 - (c) σ_i *is carried by $\partial^* E$ in the sense that $\|\sigma_i\|(\Omega \setminus \partial^* E) = 0$;*
 - (d) $\|\sigma_i\| \ll \mathcal{H}^{N-1} \llcorner \partial^* E$;

- (e) $\lim_{k \rightarrow \infty} \mathcal{H}^{N-1}(\partial A_{k;s} \cap (E^0 \cup \partial^* E)) = 0$;
- (f) $\lim_{k \rightarrow \infty} \|\sigma_{k;s}\|(E^0 \cup \partial^* E) = 0$;
- (g) *the density of σ_i , denoted as $\mathcal{F}_i \cdot \mathbf{v}$, is called the interior normal trace relative to E of \mathbf{F} on $\partial^* E$ and satisfies*

$$(5.3) \quad \int_{E^1} \operatorname{div} \mathbf{F} =: \mu(E^1) = -\sigma_i(\partial^* E) = - \int_{\partial^* E} (\mathcal{F}_i \cdot \mathbf{v})(y) d\mathcal{H}^{N-1}(y);$$

- (h) *if $(2\mathbf{F} \cdot \nabla u_k)\chi_E$ is considered as a sequence of measures, then this sequence converges weak-star to the measure $(\mathcal{F}_i \cdot \mathbf{v})\mathcal{H}^{N-1} \llcorner \partial^* E$, i.e.,*

$$(2\mathbf{F} \cdot \nabla u_k)\chi_E \xrightarrow{*} (\mathcal{F}_i \cdot \mathbf{v})\mathcal{H}^{N-1} \llcorner \partial^* E \quad \text{in } \mathcal{M}(\Omega);$$

- (i) $\|\sigma_i\| = \|\mathcal{F}_i \cdot \mathbf{v}\|_{\infty; \partial^* E, \mathcal{H}^{N-1}} \leq \|\mathbf{F}\|_{\infty; E}$.
- (ii) *For almost every $s \in (0, \frac{1}{2})$, there exist a signed measure σ_e (independent of s) and a family of sets $A_{k;s}$ with smooth boundaries such that*
 - (a) $\|\mu\|(A_{k;s} \Delta E) \rightarrow 0$;
 - (b) *the measure σ_e is the weak-star limit of $\sigma_{k;s}$;*
 - (c) σ_e *is carried by $\partial^* E$ in the sense that $\|\sigma_e\|(\Omega \setminus \partial^* E) = 0$;*
 - (d) $\|\sigma_e\| \ll \mathcal{H}^{N-1} \llcorner \partial^* E$;
 - (e) $\lim_{k \rightarrow \infty} \mathcal{H}^{N-1}(\partial A_{k;s} \cap E) = \lim_{k \rightarrow \infty} \mathcal{H}^{N-1}(u_k^{-1}(s) \cap E) = 0$;
 - (f) $\lim_{k \rightarrow \infty} \|\sigma_{k;s}\|(\Omega \setminus E^0) = \lim_{k \rightarrow \infty} \|\sigma_{k;s}\|(E) = 0$;
 - (g) *the density of σ_e , denoted as $\mathcal{F}_e \cdot \mathbf{v}$, is called the exterior normal trace relative to E of \mathbf{F} on $\partial^* E$ and satisfies*

$$(5.4) \quad \int_E \operatorname{div} \mathbf{F} =: \mu(E) = -\sigma_e(\partial^* E) = - \int_{\partial^* E} (\mathcal{F}_e \cdot \mathbf{v})(y) d\mathcal{H}^{N-1}(y);$$

- (h) *if $(2\mathbf{F} \cdot \nabla u_k)\chi_{E^0}$ is considered as a sequence of measures, then this sequence converges weak-star to the measure $(\mathcal{F}_e \cdot \mathbf{v})\mathcal{H}^{N-1} \llcorner \partial^* E$, i.e.,*

$$(2\mathbf{F} \cdot \nabla u_k)\chi_{E^0} \xrightarrow{*} (\mathcal{F}_e \cdot \mathbf{v})\mathcal{H}^{N-1} \llcorner \partial^* E \quad \text{in } \mathcal{M}(\Omega);$$

- (i) $\|\sigma_e\| = \|\mathcal{F}_e \cdot \mathbf{v}\|_{\infty; \partial^* E, \mathcal{H}^{N-1}} \leq \|\mathbf{F}\|_{\infty; \Omega \setminus E}$.

PROOF: We will prove only part (i), since the proof of part (ii) is virtually identical. For notational simplicity, we will use the notation σ rather than σ_i in the proof of part (i). Throughout the proof, we will consider only those values of $s \in (\frac{1}{2}, 1)$ for which $u_k^{-1}(s)$ is a smooth manifold for all k , Lemma 3.1 holds, and the results in Section 4 are valid for all the mollifications u_k of χ_E , thus omitting at most a set \mathcal{S} of measure zero. For the rest of the proof, fix $s \notin \mathcal{S}$.

We start with (a). We consider the sets $A_{k;s}$ as in Lemma 4.1. The desired result follows directly from Corollary 4.2.

(b) Since \mathbf{F} is bounded, Lemma 4.7 implies that there exists a constant C such that

$$(5.5) \quad \|\sigma_{k;s}\|(\Omega) \leq C,$$

which yields, as in (3.8), the existence of a signed measure σ_s such that

$$(5.6) \quad \sigma_{k;s} \xrightarrow{*} \sigma_s \quad \text{in } \mathcal{M}(\Omega).$$

Utilizing (4.2), we also obtain that $\mu(A_{k;s}) \rightarrow \mu(E^1)$. Since Lemma 3.1 yields $\mu(A_{k;s}) = -\sigma_{k;s}(\Omega)$, we obtain, after letting $k \rightarrow \infty$, that

$$(5.7) \quad \mu(E^1) = -\sigma_s(\Omega).$$

Since the left side of equation (5.7) is independent of s , we show next that σ_s is also independent of s (and independent of the sequence in the weak-star convergence (5.6)). To see this, we fix any $\phi \in C_c^1(\Omega)$ and note that, since \mathbf{F} is a divergence-measure field, the product rule in Lemma 2.31 implies that $\phi\mathbf{F}$ is also a divergence-measure field. Proceeding as above with $\phi\mathbf{F}$ instead of \mathbf{F} , we obtain

$$(5.8) \quad \int_{E^1} \operatorname{div}(\phi\mathbf{F}) = - \int_{\Omega} \phi d\sigma_s$$

for any $\phi \in C_c^1(\Omega)$. Therefore, for any two measures σ_s and $\sigma_{s'}$ with limits as in (5.6), we have that $\int_{\Omega} \phi d\sigma_s = \int_{\Omega} \phi d\sigma_{s'}$ for any $\phi \in C_c^1(\Omega)$ and thus we conclude that $\sigma_s = \sigma_{s'}$.

(c) Let $A \subset \Omega \setminus \partial^*E$ be an arbitrary Borel set. Referring to (2.8), we see that

$$\|\nabla\chi_E\|(A) = 0.$$

On the other hand, we know

$$(5.9) \quad \begin{aligned} 0 &= \|\nabla\chi_E\|(A) \\ &= \inf\{\|\nabla\chi_E\|(U) : A \subset U, U \text{ open}\} \\ &= \inf\{\|\nabla\chi_E\|(U) : A \subset U, U \text{ open}, \|\nabla\chi_E\|(\partial U) = 0\}. \end{aligned}$$

In order to prove that $\|\sigma\|(A) = 0$, we proceed by contradiction by assuming $\|\sigma\|(A) > 0$. From (5.9), there is an open set $U \supset A$ such that $\|\nabla\chi_E\|(\partial U) = 0$ and

$$(5.10) \quad \|\nabla\chi_E\|(U) < \frac{\|\sigma\|(A)}{2\|\mathbf{F}\|_{\infty}}.$$

From Lemma 5.1, we have

$$(5.11) \quad \int_U |\mathbf{F} \cdot \nabla u_k| dy = \int_0^1 \int_{U \cap u_k^{-1}(t)} |\mathbf{F} \cdot \mathbf{v}_k| d\mathcal{H}^{N-1} dt.$$

Since U is open and $\sigma_{k;t} \xrightarrow{*} \sigma$ in $\mathcal{M}(\Omega)$,

$$\begin{aligned} \|\sigma\|(A) &\leq 2 \int_{\frac{1}{2}}^1 \|\sigma\|(U) dt \leq 2 \int_{\frac{1}{2}}^1 \liminf_{k \rightarrow \infty} \|\sigma_{k;t}\|(U) dt \\ &\leq 2 \int_0^1 \liminf_{k \rightarrow \infty} \|\sigma_{k;t}\|(U) dt \\ &\leq 2 \liminf_{k \rightarrow \infty} \int_0^1 \|\sigma_{k;t}\|(U) dt \end{aligned}$$

by Fatou's lemma. Therefore, we have

$$\begin{aligned} \|\sigma\|(A) &\leq 2 \liminf_{k \rightarrow \infty} \int_0^1 \int_{u_k^{-1}(t) \cap U} |\mathbf{F}(y) \cdot \mathbf{v}(y)| d\mathcal{H}^{N-1}(y) dt \\ &= 2 \liminf_{k \rightarrow \infty} \int_U |\mathbf{F} \cdot \nabla u_k| dy \\ &\leq 2 \|\mathbf{F}\|_\infty \lim_{k \rightarrow \infty} \int_U |\nabla u_k| dy \\ &= 2 \|\mathbf{F}\|_\infty \|\nabla \chi_E\|(U) \\ &< \|\sigma\|(A), \end{aligned}$$

where we have used Lemma 2.15(iv) and the fact that $\|\nabla \chi_E\|(\partial U) = 0$. This yields a contradiction and thus establishes our result.

(d) Let $A \subset \partial^* E$ be a Borel set with $\mathcal{H}^{N-1}(A) = 0$. Then, appealing to (2.8), we find that $\|\nabla \chi_E\|(A) = 0$. From this, the proof can proceed precisely as in (c) to yield our desired conclusion.

(e) This is the result of Theorem 4.10.

(f) In view of the definition

$$\sigma_{k;s}(B) := \int_{\partial A_{k;s} \cap B} \mathbf{F}(y) \cdot \mathbf{v}(y) d\mathcal{H}^{N-1}(y)$$

and the fact that \mathbf{F} is bounded, the result follows immediately from (e).

(g) From (a), we have the existence of smoothly bounded sets such that

$$(5.12) \quad \|\mu\|(A_{k;s} \Delta E^1) \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

where $s > \frac{1}{2}$ is fixed as in the beginning of the proof. From Lemma 3.1, we know that our desired result holds for the sets $A_{k;s}$:

$$(5.13) \quad \mu(A_{k;s}) := \int_{A_{k;s}} \operatorname{div} \mathbf{F} = - \int_{\partial A_{k;s}} \mathbf{F}(y) \cdot \mathbf{v}(y) d\mathcal{H}^{N-1}(y).$$

We note that, with our notation in force, we may write (5.13) as

$$(5.14) \quad \mu(A_{k;s}) = -\sigma_{k;s}(\Omega) = -\sigma_{k;s}(\partial A_{k;s}).$$

Since

$$(5.15) \quad \mu(A_{k;s}) \rightarrow \mu(E^1) \quad \text{and} \quad \sigma_{k;s}(\Omega) \rightarrow \sigma(\Omega) \quad \text{as } k \rightarrow \infty,$$

we obtain

$$\mu(E^1) = -\sigma(\partial^* E).$$

Because $\|\sigma\| \ll \mathcal{H}^{N-1} \llcorner \partial^* E$, we know that there exists $\mathcal{F}_i \cdot \mathbf{v} \in L^1(\partial^* E)$ such that

$$\sigma(B) = \int_{B \cap \partial^* E} (\mathcal{F}_i \cdot \mathbf{v})(y) d\mathcal{H}^{N-1}(y),$$

which gives (5.3).

(h) From Lemma 5.1, we obtain

$$\begin{aligned} \lim_{k \rightarrow \infty} \int_E \mathbf{F} \cdot \nabla u_k \, dy &= \lim_{k \rightarrow \infty} \int_0^1 \int_{u_k^{-1}(t) \cap E} \mathbf{F}(y) \cdot \mathbf{v}_k(y) d\mathcal{H}^{N-1}(y) dt \\ &= \lim_{k \rightarrow \infty} \int_0^1 \sigma_{k;t}(E) dt. \end{aligned}$$

Thus,

$$\begin{aligned} \overline{\chi_E \mathbf{F} \cdot \nabla u_E}(\Omega) &:= \lim_{k \rightarrow \infty} \int_{\Omega} \chi_E \mathbf{F} \cdot \nabla u_k \, dy \\ &= \lim_{k \rightarrow \infty} \int_E \mathbf{F} \cdot \nabla u_k \, dy \\ &= \lim_{k \rightarrow \infty} \int_{\frac{1}{2}}^1 \sigma_{k;t}(E) dt + \lim_{k \rightarrow \infty} \int_{\frac{1}{2}}^1 \sigma_{k;t}(E^0) dt \quad (\text{by (f) above}) \\ &= \lim_{k \rightarrow \infty} \int_{\frac{1}{2}}^1 \sigma_{k;t}(\Omega) dt \\ &= \frac{1}{2} \sigma(\Omega). \end{aligned}$$

Let φ be a function in $C_c^1(\Omega)$. Since $\varphi \mathbf{F}$ is also a bounded divergence-measure field, we can proceed as above with the vector field $\varphi \mathbf{F}$ instead of \mathbf{F} to conclude that

$$(5.16) \quad \int_{\Omega} \varphi d \overline{\chi_E \mathbf{F} \cdot \nabla u_E} = \frac{1}{2} \int_{\Omega} \varphi d\sigma,$$

which implies that $\sigma = 2 \overline{\chi_E \mathbf{F} \cdot \nabla u_E}$.

(i) We have that, for \mathcal{H}^{N-1} -a.e. $y \in \partial^* E$,

$$(5.17) \quad \mathcal{F}_i \cdot \mathbf{v}(y) = \lim_{r \rightarrow 0} \frac{\sigma(B(y, r))}{\|\nabla \chi_E\|(B(y, r))},$$

where we can choose the balls $B(y, r)$ such that

$$\|\nabla \chi_E\|(\partial B(y, r)) = \|\sigma\|(\partial B(y, r)) = 0.$$

Using a similar argument as in (c), we obtain

$$\begin{aligned} & \|\sigma\|(B(y, r)) \\ &= 2 \int_{\frac{1}{2}}^1 \|\sigma\|(B(y, r)) dt \\ &= 2 \lim_{k \rightarrow \infty} \int_{\frac{1}{2}}^1 \|\sigma_{k;t}\|(B(y, r)) dt \\ &= 2 \lim_{k \rightarrow \infty} \int_{\frac{1}{2}}^1 (\|\sigma_{k;t}\|(B(y, r) \cap E^1) + \|\sigma_{k;t}\|(B(y, r) \cap (E^0 \cup \partial^* E))) dt \\ &= 2 \lim_{k \rightarrow \infty} \int_{\frac{1}{2}}^1 \|\sigma_{k;t}\|(B(y, r) \cap E^1) dt \\ &= 2 \lim_{k \rightarrow \infty} \int_{\frac{1}{2}}^1 \int_{u_k^{-1}(t) \cap B(y, r) \cap E^1} |\mathbf{F} \cdot \mathbf{v}| d\mathcal{H}^{N-1} dt \\ &\leq 2 \|\mathbf{F}\|_{\infty; E^1} \lim_{k \rightarrow \infty} \int_{\frac{1}{2}}^1 \int_{u_k^{-1}(t) \cap B(y, r)} d\mathcal{H}^{N-1} dt, \end{aligned}$$

where, in proceeding from the third equality to the fourth, we have used the fact that $\|\sigma_{k;t}\|(E^0 \cup \partial^* E) \rightarrow 0$ as $k \rightarrow \infty$ for a.e. $t > \frac{1}{2}$.

Therefore, from (5.17), we obtain

$$\begin{aligned} |\mathcal{F}_i \cdot \mathbf{v}(y)| &\leq \lim_{r \rightarrow 0} \frac{\|\sigma\|(B(y, r))}{\|\nabla \chi_E\|(B(y, r))} \\ &\leq 2 \|\mathbf{F}\|_{\infty; E^1} \lim_{r \rightarrow 0} \lim_{k \rightarrow \infty} \frac{\int_{1/2}^1 \int_{u_k^{-1}(t) \cap B(y, r)} d\mathcal{H}^{N-1} dt}{\int_{B(y, r)} |\nabla u_k|} = \end{aligned}$$

$$\begin{aligned}
 &= 2\|\mathbf{F}\|_{\infty;E^1} \lim_{r \rightarrow 0} \lim_{k \rightarrow \infty} \frac{\int_{1/2}^1 \int_{u_k^{-1}(t) \cap B(y,r)} d\mathcal{H}^{N-1} dt}{\int_0^1 \int_{u_k^{-1}(t) \cap B(x,r)} d\mathcal{H}^{N-1} dt} \\
 &= \|\mathbf{F}\|_{\infty;E^1} = \|\mathbf{F}\|_{\infty;E}. \quad \square
 \end{aligned}$$

As a direct result, we obtain the Gauss-Green theorem for divergence-measure fields over sets of finite perimeter. This also shows that our definition of the normal trace is in agreement with that given in the sense of distributions, Definition 2.19.

THEOREM 5.3 (Gauss-Green Theorem) *Let $\Omega \subset \mathbb{R}^N$ be an open set. Let $\mathbf{F} \in \mathcal{DM}_{loc}^\infty(\Omega)$ and let $E \Subset \Omega$ be a bounded set of finite perimeter. Then,*

$$(5.18) \quad \int_{E^1} \varphi \operatorname{div} \mathbf{F} + \int_{E^1} \mathbf{F} \cdot \nabla \varphi = - \int_{\partial^* E} \varphi (\mathcal{F}_i \cdot \nu) d\mathcal{H}^{N-1}$$

for all $\varphi \in C_c^\infty(\Omega)$, where $\mathcal{F}_i \cdot \nu$ is the interior normal trace of \mathbf{F} relative to E on $\partial^* E$.

PROOF: From Theorem 2.31, it follows that $\varphi \mathbf{F}$ is a bounded divergence-measure field and

$$(5.19) \quad \operatorname{div}(\varphi \mathbf{F}) = \varphi \operatorname{div} \mathbf{F} + \mathbf{F} \cdot \nabla \varphi.$$

Following the proof of Theorem 5.2 applied to $\varphi \mathbf{F}$ (instead of \mathbf{F}), we obtain

$$(5.20) \quad \int_{E^1} \operatorname{div}(\varphi \mathbf{F}) = - \int_{\partial^* E} \varphi (\mathcal{F}_i \cdot \nu) d\mathcal{H}^{N-1},$$

which, due to (5.19), gives the desired result. □

We conclude this section with the following remark.

Remark 5.4. Theorem 5.3 implies that, when E is an open set of finite perimeter, our trace $\mathcal{F}_i \cdot \nu$ agrees with the one defined in (2.10).

6 The Divergence Measure of Jump Sets via the Normal Trace

In Theorem 5.2, we have defined the interior and exterior normal traces of $\mathbf{F} \in \mathcal{DM}_{loc}^\infty(\Omega)$, $\mathcal{F}_i \cdot \nu$ and $\mathcal{F}_e \cdot \nu$, over the boundary of a set of finite perimeter $E \Subset \Omega$. In order to obtain the interior normal trace of \mathbf{F} on $\partial^* \tilde{E}$, where $\tilde{E} := E^0 \cup \partial^m E$, we reproduce the proof of Theorem 5.2 and apply it to \tilde{E} . Therefore, the trace measure, denoted by σ_- , is obtained by using the level sets $B_{k;s} = \{v_k > s\}$ for some $s \in (\frac{1}{2}, 1)$, where v_k is the mollification of $\chi_{\tilde{E}}$. We note that, for all $y \in \Omega$,

$$\rho_\varepsilon * \chi_E(y) + \rho_\varepsilon * \chi_{\tilde{E}}(y) = 1,$$

and therefore

$$v_k^{-1}(s) = u_k^{-1}(1 - s),$$

where $1 - s \in (0, \frac{1}{2})$. Since $-\mathbf{v}$ is the interior unit normal to \tilde{E} , we have

$$\begin{aligned}
 \sigma_-(\Omega) &= - \lim_{k \rightarrow \infty} \int_{\partial B_{k;s}} \mathbf{F} \cdot \mathbf{v} \, d\mathcal{H}^{N-1} = - \lim_{k \rightarrow \infty} \int_{\partial A_{k;1-s}} \mathbf{F} \cdot \mathbf{v} \, d\mathcal{H}^{N-1} \\
 (6.1) \qquad &= - \lim_{k \rightarrow \infty} \sigma_{k;1-s}(\mathbb{R}^N) = -\sigma_e(\Omega).
 \end{aligned}$$

The following observation now becomes evident.

COROLLARY 6.1 *The interior trace of \mathbf{F} relative to \tilde{E} on $\partial^* E$ is the same as minus the exterior trace of \mathbf{F} relative to E on $\partial^* E$.*

In order to establish the relation between σ_i and σ_e , we subtract (5.4) from (5.3) and obtain the following formula for $\mu = \operatorname{div} \mathbf{F}$.

$$\text{COROLLARY 6.2} \quad \mu(\partial^* E) = \int_{\partial^* E} (\mathcal{F}_i \cdot \mathbf{v} - \mathcal{F}_e \cdot \mathbf{v})(y) \, d\mathcal{H}^{N-1}(y).$$

We offer the following simple example to illustrate our result. This example also dramatically demonstrates the difference between the classical derivative and the weak (distributional) derivative.

EXAMPLE 6.3 Consider the most elementary situation: $N = 1$, $\Omega := (-1, 2)$, $E := [0, 1]$, and f is a nondecreasing function defined on $(-1, 2)$ that is continuous everywhere except at $y = 0, 1$, at which points we assume that f is right-continuous.

(i) Case $\frac{1}{2} < s < 1$. Since f is in BV, we know that $f' = \mu$ for some measure μ . Then, according to Theorem 5.2,

$$\mu(E^1) = \mu((0, 1)) := \int_{0+}^{1-} f' = \mathcal{F}_i \cdot \mathbf{v}(1) - \mathcal{F}_i \cdot \mathbf{v}(0),$$

where $\mathcal{F}_i \cdot \mathbf{v}(1) = \lim_{y \rightarrow 1-} f(y)$ and $\mathcal{F}_i \cdot \mathbf{v}(0) = f(0+)$. Indeed, the sets $A_{k;s}$, with fixed $s \in (\frac{1}{2}, 1)$, form a nested family of open intervals contained in $[0, 1]$. The measures σ_k correspond to f evaluated on the point masses located at y_k ; thus, as in (5.15), $f(y_k)$ converges to a limit, $\mathcal{F}_i \cdot \mathbf{v}(1)$.

(ii) Case $0 < s < \frac{1}{2}$. Then the sets $A_{k;s}$, $s \in (0, \frac{1}{2})$, form a nested family of open intervals containing $[0, 1]$. Similar to the above, we have

$$\mu(E) = \mu([0, 1]) := \int_{0-}^{1+} f' = \mathcal{F}_e \cdot \mathbf{v}(1) - \mathcal{F}_e \cdot \mathbf{v}(0),$$

the measures σ_k correspond to f evaluated on the point masses located at y_k , and thus $\mathcal{F}_e \cdot \mathbf{v}(1) = f(1+)$ and $\mathcal{F}_e \cdot \mathbf{v}(0) = \lim_{y \rightarrow 0-} f(y)$.

7 Consistency of the Normal Trace with the Classical Trace

We now proceed to show the consistency of our normal trace with the classical trace when \mathbf{F} is continuous. First we have the following lemma:

LEMMA 7.1 *Let $\mu = \operatorname{div} \mathbf{F}$ for $\mathbf{F} \in \mathcal{DM}_{\text{loc}}^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N; \mathbb{R}^N)$. Then*

$$\|\mu\|(G) = 0$$

for any set G that can be written as the graph of a Lipschitz function f .

PROOF: First we have

$$G := \{(y', f(y')) : y' \in W \subset \mathbb{R}^{N-1}\}.$$

By regularity of μ , it suffices to show that $\mu(K) = 0$ for any compact set $K \subset G$. Given such $K \subset G$, let $U_k \subset \mathbb{R}^N$ be a sequence of open sets satisfying

$$(7.1) \quad \|\mu\|(U_k) \rightarrow \|\mu\|(K).$$

Fix any set U_k . We note by Besicovitch's theorem that U_k can be written up to a set of $\|\mu\|$ -measure zero as a countable union of disjoint open parallelepipeds I_i^k (the fact that we can use parallelepipeds instead of balls follows from Morse [53]). Thus, we have

$$(7.2) \quad \bigcup_{i=1}^\infty I_i^k \subset U_k \quad \text{and} \quad \|\mu\|\left(U_k \setminus \bigcup_{i=1}^\infty I_i^k\right) = 0.$$

Denote U_k simply as U and I_i^k as I_i . We fix an i and note that, for t small enough, the graphs $T_t := \{(y', f(y') + t) : y' \in W \subset \mathbb{R}^{N-1}\}$ and $B_t := \{(y', f(y') - t) : y' \in W \subset \mathbb{R}^{N-1}\}$ are contained in I_i . Let R_t be the region inside I_i , bounded above and below by T_t and B_t , respectively. We define

$$\alpha_t = \int_{\partial R_t \setminus (T_t \cup B_t)} \mathbf{F}(y) \cdot \mathbf{v}(y) d\mathcal{H}^{N-1}(y) \quad \text{for a.e. } t,$$

where $\mathbf{v}(y)$ is the interior unit normal to R_t on $\partial R_t \setminus (T_t \cup B_t)$. Since Lemma 3.1 applies to R_t for a.e. t , we arrive at

$$\begin{aligned} \mu(R_t) &= \int_{R_t} \operatorname{div} \mathbf{F} \\ &= - \int_{B_t} \mathbf{F}(y) \cdot \mathbf{v}(y) d\mathcal{H}^{N-1}(y) - \int_{T_t} \mathbf{F}(y) \cdot \mathbf{v}(y) d\mathcal{H}^{N-1}(y) - \alpha_t = \end{aligned}$$

$$\begin{aligned}
&= \int_{T_t} \mathbf{F}(y', y_n - 2t) \cdot \mathbf{v}(y) d\mathcal{H}^{N-1}(y) \\
&\quad - \int_{T_t} \mathbf{F}(y', y_n) \cdot \mathbf{v}(y) d\mathcal{H}^{N-1}(y) - \alpha_t \\
&= \int_{T_t} (\mathbf{F}(y', y_n - 2t) - \mathbf{F}(y', y_n)) \cdot \mathbf{v}(y) d\mathcal{H}^{N-1}(y) - \alpha_t.
\end{aligned}$$

Since \mathbf{F} is continuous and $\alpha_t \rightarrow 0$ as $t \rightarrow 0$, we find that there exists $t_0(\varepsilon, \mathbf{F}, G) > 0$ such that

$$\mu(R_t) \leq \varepsilon \quad \text{for all } t \leq t_0(\varepsilon, \mathbf{F}, G).$$

Then we have

$$\mu(I_i \cap G) = \lim_{t \rightarrow 0} \mu(R_t) \leq \varepsilon,$$

which implies $\mu(I_i \cap G) = 0$ since ε is arbitrary. Therefore, using (7.1) and (7.2), we obtain

$$\mu(K) = \lim_{k \rightarrow \infty} \sum \mu(I_i^k \cap G) = 0.$$

□

THEOREM 7.2 *If $\mathbf{F} \in \mathcal{DM}_{\text{loc}}^\infty(\Omega)$ is continuous and $E \Subset \Omega$ is a set of finite perimeter, then $\sigma_i = (\mathcal{F}_i \cdot \mathbf{v})\mathcal{H}^{N-1} \llcorner \partial^* E = \overline{\mathbf{F} \cdot \nabla u_E}$, where $\overline{\mathbf{F} \cdot \nabla u_E}$ is the weak-star limit of the measures $\mathbf{F} \cdot \nabla u_k$. Moreover, the normal trace $\mathcal{F}_i \cdot \mathbf{v}$ is in fact the classical dot product $\mathbf{F} \cdot \mathbf{v}$, where \mathbf{v} is the measure-theoretical interior unit normal to E on $\partial^* E$.*

PROOF: We recall that, by definition, $E = E^1 \cup \partial^* E$. Denote $\tilde{E} = E^0 \cup \partial^* E$. Then we have

$$\begin{aligned}
\overline{\mathbf{F} \cdot \nabla u_E} &= \lim_{k \rightarrow \infty} \int_{\mathbb{R}^N} \mathbf{F} \cdot \nabla u_k \, dy \\
&= \lim_{k \rightarrow \infty} \int_{\mathbb{R}^N} \chi_E \mathbf{F} \cdot \nabla u_k \, dy + \lim_{k \rightarrow \infty} \int_{\mathbb{R}^N} \chi_{\tilde{E}} \mathbf{F} \cdot \nabla u_k \, dy.
\end{aligned}$$

If v_k denotes the convolution $\chi_{\tilde{E}} * \rho_{1/k}$, since $u_k + v_k = 1$, we obtain

$$\begin{aligned}
\overline{\mathbf{F} \cdot \nabla u_E} &= \lim_{k \rightarrow \infty} \int_{\mathbb{R}^N} \chi_E \mathbf{F} \cdot \nabla u_k \, dy - \lim_{k \rightarrow \infty} \int_{\mathbb{R}^N} \chi_{\tilde{E}} \mathbf{F} \cdot \nabla v_k \, dy \\
&= \frac{\sigma_i}{2} + \frac{\sigma_e}{2} = \frac{1}{2}(\sigma_i + \sigma_i - \mu(\partial^* E)) = \sigma_i - \frac{1}{2} \mu(\partial^* E),
\end{aligned}$$

where we have used Theorem 5.2(h) and Corollary 6.2.

Since $\partial^* E$ is an $(N - 1)$ -rectifiable set (see (2.7)), it follows from Lemma 7.1 that

$$\|\mu\|(\partial^* E) = 0,$$

that is,

$$\overline{\mathbf{F} \cdot \nabla u_E} = \sigma_i.$$

Thus, for \mathcal{H}^{N-1} -a.e. $y \in \partial^* E$,

$$(\mathcal{F}_i \cdot \nu)(y) = \lim_{r \rightarrow 0} \frac{\overline{\mathbf{F} \cdot \nabla u_E}(B(y, r))}{\|\nabla \chi_E\|(B(y, r))} = \lim_{r \rightarrow 0} \lim_{k \rightarrow \infty} \frac{\int_{B(y, r)} \mathbf{F} \cdot \nabla u_k \, dx}{\int_{B(y, r)} d\|\nabla \chi_E\|}.$$

Since $\nabla u_k \rightarrow \nabla \chi_E$ weak-star and \mathbf{F} is continuous, and noting that r_j can be chosen such that $\|\nabla \chi_E\|(\partial B(y, r_j)) = 0$, we obtain

$$\begin{aligned} (\mathcal{F}_i \cdot \nu)(y) &= \lim_{j \rightarrow \infty} \frac{\int_{B(y, r_j)} \mathbf{F} \cdot \nabla \chi_E}{\int_{B(y, r_j)} d\|\nabla \chi_E\|} \\ &= \lim_{j \rightarrow \infty} \frac{\int_{B(y, r_j)} \mathbf{F}(x) \cdot \nu(x) \, d\|\nabla \chi_E\|(x)}{\int_{B(y, r_j)} d\|\nabla \chi_E\|(x)} \\ &= \mathbf{F}(y) \cdot \nu(y), \end{aligned}$$

by differentiation of measures. □

The following corollary gives more information of the trace σ_i and the level sets $u_k^{-1}(s)$ when $s \rightarrow \frac{1}{2}+$.

COROLLARY 7.3 *The trace measure σ_i given in Theorem 5.2 satisfies*

$$\begin{aligned} \sigma_i(\mathbb{R}^N) &= 2 \lim_{k \rightarrow \infty} \int_E \mathbf{F} \cdot \nabla u_k \, dy = 2 \lim_{k \rightarrow \infty} \lim_{s \rightarrow 1/2+} \int_{A_{k;s}} \mathbf{F} \cdot \nabla u_k \, dy \\ &= 2 \lim_{k \rightarrow \infty} \int_{A_{k;1/2}} \mathbf{F} \cdot \nabla u_k \, dy. \end{aligned}$$

PROOF: Theorem 5.2 (h) shows

$$(7.3) \quad \sigma_i(\mathbb{R}^N) = 2 \lim_{k \rightarrow \infty} \int_E \mathbf{F} \cdot \nabla u_k \, dy.$$

Using Lemma 5.1, we find

$$\begin{aligned} \sigma_i(\mathbb{R}^N) &= 2 \lim_{s \rightarrow 1/2+} \int_s^1 \sigma_i(\mathbb{R}^N) \, dt = 2 \lim_{s \rightarrow 1/2+} \lim_{k \rightarrow \infty} \int_s^1 \int_{u_k^{-1}(t)} \mathbf{F} \cdot \nu \, d\mathcal{H}^{N-1} \, dt \\ &= 2 \lim_{s \rightarrow 1/2+} \lim_{k \rightarrow \infty} \int_{A_{k;s}} \mathbf{F} \cdot \nabla u_k \, dy. \end{aligned}$$

One can easily verify that the limits $s \rightarrow \frac{1}{2}+$ and $k \rightarrow \infty$ can be interchanged. Noting that $\bigcup_{s>1/2} A_{k;s} = A_{k;1/2}$, we conclude

$$\sigma_i(\mathbb{R}^N) = 2 \lim_{k \rightarrow \infty} \lim_{s \rightarrow 1/2+} \int_{A_{k;s}} \mathbf{F} \cdot \nabla u_k \, dy = 2 \lim_{k \rightarrow \infty} \int_{A_{k;1/2}} \mathbf{F} \cdot \nabla u_k \, dy.$$

□

8 One-Sided Approximation of Sets of Finite Perimeter

It is well-known that a set of finite perimeter E cannot be approximated by smooth sets that lie completely in the interior of E . For example, consider the open unit disk with a single radius removed, and let U be the resulting open set. Then the Hausdorff measure of the boundary of U is 2π plus the measure of the radius, while the Hausdorff measure of the reduced boundary is 2π . Thus, if U_k is an approximating open subset of U , then its boundary will be close to that of U and so its Hausdorff measure will be close to 2π plus 1. Adding more radii, say m of them, will force the approximating set to have boundaries whose Hausdorff measure is close to 2π plus m . In general, if we let K denote any compact subset without interior and of infinite Hausdorff measure, then the approximating sets will have boundaries whose measures will necessarily tend to infinity.

On the other hand, we have seen (Theorem 3.3) that one-sided approximation is possible for open sets of class C^1 .

We have the following:

PROPOSITION 8.1 *Let $U \subset \mathbb{R}^N$ be an open set with $\mathcal{H}^{N-1}(\partial U) < \infty$. Then there exists a sequence of bounded open sets $U_k \subset \overline{U_k} \subset U$ such that*

- (i) $|U_k| = |\overline{U_k}|$;
- (ii) $|U_k| \rightarrow |U|$;
- (iii) $\mathcal{H}^{N-1}(\partial U_k) \rightarrow \mathcal{H}^{N-1}(\partial U)$.

PROOF: By definition, for each integer k , there exists a covering of ∂U by balls

$$\partial U \subset \bigcup B_i(r_i),$$

each with radius r_i , such that

$$\sum_{i=1}^{\infty} \mathcal{H}^{N-1}(\partial B_i(r_i)) = \sum_{i=1}^{\infty} \omega_{N-1} r_i^{N-1} < \mathcal{H}^{N-1}(\partial U) + \frac{1}{k},$$

where ω_{N-1} is the \mathcal{H}^{N-1} measure of the boundary of the unit ball in \mathbb{R}^N . Since ∂U is compact, the covering may be taken as a finite covering, say by m of them, $B_1(r_1), \dots, B_m(r_m)$. Then the open set $V_k := \bigcup_{i=1}^m B_i(r_i)$ has the property that

$$\partial V_k \subset \bigcup_{i=1}^m \partial B_i(r_i)$$

and therefore that

$$\mathcal{H}^{N-1}(\partial V_k) \leq \mathcal{H}^{N-1}\left(\bigcup_{i=1}^m \partial B_i(r_i)\right) \leq \sum_{i=1}^{\infty} \omega_{N-1} r_i^{N-1} < \mathcal{H}^{N-1}(\partial U) + \frac{1}{k}.$$

Thus, the open sets $U_k := U \setminus \overline{V}_k \subset U$ satisfy our desired result. Note that they are not smooth. \square

Given an arbitrary set of finite perimeter E , we know from Section 4 that E can be approximated by sets with smooth boundaries essentially from the measure-theoretic interior of E , that is, a one-sided approximation can “almost” be achieved (see Theorem 5.2(e)). On the other hand, the next result shows that, if E is sufficiently regular, there does, in fact, exist a one-sided approximation. The condition of regularity we impose is similar to Lewis’s *uniformly flat* condition in potential theory [49].

THEOREM 8.2 *Suppose that E is a bounded set of finite perimeter with the property that, for all $y \in \partial E$, there are positive constants c_0 and r_0 such that*

$$(8.1) \quad \frac{|E^0 \cap B(y, r)|}{|B(y, r)|} \geq c_0 \quad \text{for all } r \leq r_0.$$

Then there exists $t \in (0, 1)$ such that

$$(8.2) \quad A_{k;t} \Subset E \quad \text{for large } k.$$

PROOF: Choose a mollifying kernel ρ such that $\rho = 1$ on $B(0, \frac{1}{2})$. If $y \in \partial E$, we have

$$\begin{aligned} v_k(y) &:= \chi_{\mathbb{R}^N \setminus E} * \rho_{\varepsilon_k}(y) = \frac{1}{\varepsilon_k^N} \int_{B(y, \varepsilon_k)} \chi_{\mathbb{R}^N \setminus E}(x) \rho\left(\frac{x-y}{\varepsilon_k}\right) dx \\ &\geq \frac{1}{\varepsilon_k^N} \int_{B(y, \varepsilon_k/2)} \chi_{\mathbb{R}^N \setminus E}(x) dx \\ &= \frac{|(\mathbb{R}^N \setminus E) \cap B(y, \varepsilon_k/2)|}{\varepsilon_k^N} \\ &= \frac{|E^0 \cap B(y, \varepsilon_k/2)|}{\varepsilon_k^N} \geq \frac{c_0}{2^N} := \tilde{c}_0, \end{aligned}$$

where $0 < \tilde{c}_0 < 1$ depends only on the dimension N and is independent of the point y . Note that $u_k(y) + v_k(y) = 1$ for all $y \in \mathbb{R}^N$. Therefore, for all $y \in \partial E$,

$$u_k(y) = 1 - v_k(y) \leq 1 - \tilde{c}_0.$$

Thus, taking $1 - \tilde{c}_0 < t < 1$, we see that $A_{k;t} \cap \partial E = \emptyset$. Consequently, each connected component of the open set $A_{k;t}$ lies either in the interior of E or in its exterior and thus must lie in its interior. \square

COROLLARY 8.3 *Let E be a bounded set of finite perimeter with uniform Lipschitz boundary. Then there exists $T \in (0, 1)$ such that $A_{k;T} \Subset E$.*

PROOF: Since E has a uniform Lipschitz boundary, for each $x \in \partial E$, there is a finite cone C_x with vertex x that completely lies in the complement of E . Each cone C_x is assumed to be congruent to a fixed cone C . This implies that the hypothesis of Theorem 8.2 is satisfied. Therefore, there exists $0 < T < 1$ such that $u_k(y) < T$ for all k and all $y \in \partial E$. \square

DEFINITION 8.4 An open set $U \subset \mathbb{R}^N$ is called an *extension domain* for $\mathbf{F} \in \mathcal{DM}^\infty(U)$ if there exists a field $\mathbf{F}^* \in \mathcal{DM}^\infty(\mathbb{R}^N)$ such that $\mathbf{F} = \mathbf{F}^*$ on U .

THEOREM 8.5 *An open set U satisfying $\mathcal{H}^{N-1}(\partial U) < \infty$ is an extension domain for any $\mathbf{F} \in \mathcal{DM}^\infty(U)$. More generally, if $\mathbf{F} \in \mathcal{DM}_{\text{loc}}^\infty(\Omega)$, then any open set of finite perimeter $U \Subset \Omega$ is an extension domain for \mathbf{F} .*

PROOF: We define an extension of \mathbf{F} by

$$\mathbf{F}^*(y) := \chi_U(y)\mathbf{F}(y) \quad \text{for all } y \in \mathbb{R}^N.$$

According to Definition 2.18, it suffices to show that

$$\sup \left\{ \int_{\mathbb{R}^N} \mathbf{F}^* \cdot \nabla \varphi : |\varphi| \leq 1, \varphi \in C_c^\infty(\mathbb{R}^N) \right\} < \infty.$$

We consider first the case $\mathcal{H}^{N-1}(U) < \infty$. Let U_k be the sequence of approximate sets given in Proposition 8.1. Therefore, for any $\varphi \in C_c^\infty(\mathbb{R}^N)$ with $|\varphi| \leq 1$, we employ our general Gauss-Green theorem, Theorem 5.2, to obtain

$$\int_{U_k} \mathbf{F} \cdot \nabla \varphi \, dy + \int_{U_k} \varphi \operatorname{div} \mathbf{F} = - \int_{\partial U_k} \varphi \mathcal{F}_i \cdot \mathbf{v} \, d\mathcal{H}^{N-1}.$$

Thus,

$$\begin{aligned} \int_{U_k} \mathbf{F} \cdot \nabla \varphi \, dy &= - \int_{U_k} \varphi \operatorname{div} \mathbf{F} - \int_{\partial U_k} \varphi \mathcal{F}_i \cdot \mathbf{v} \, d\mathcal{H}^{N-1} \\ &\leq \|\operatorname{div} \mathbf{F}\|(U_k) + \|\mathbf{F}\|_\infty \mathcal{H}^{N-1}(\partial U_k) \\ &\leq \|\operatorname{div} \mathbf{F}\|(U) + \|\mathbf{F}\|_\infty \mathcal{H}^{N-1}(\partial U_k). \end{aligned}$$

Letting $k \rightarrow \infty$, we obtain

$$\int_U \mathbf{F} \cdot \nabla \varphi \, dy \leq \|\operatorname{div} \mathbf{F}\|(U) + \|\mathbf{F}\|_\infty \mathcal{H}^{N-1}(\partial U) < \infty.$$

Thus,

$$\int_{\mathbb{R}^N} \mathbf{F}^* \cdot \nabla \varphi \, dy = \int_U \mathbf{F} \cdot \nabla \varphi \, dy < \infty.$$

We now consider the case that $U \Subset \Omega$ is a set of finite perimeter and $\mathbf{F} \in \mathcal{DM}_{\text{loc}}^\infty(\Omega)$. Proceeding as above and using Theorem 5.2,

$$\begin{aligned} \int_{\mathbb{R}^N} \mathbf{F}^* \cdot \nabla \varphi \, dy &= \int_U \mathbf{F} \cdot \nabla \varphi \, dy \\ &= - \int_U \varphi \operatorname{div} \mathbf{F} - \int_{\partial^* U} \varphi \mathcal{F}_i \cdot \mathbf{v} \, d\mathcal{H}^{N-1} \\ &\leq \|\operatorname{div} \mathbf{F}\|(U) + \|\mathbf{F}\|_\infty \mathcal{H}^{N-1}(\partial^* U) < \infty. \end{aligned}$$

□

COROLLARY 8.6 *Let $U \subset \mathbb{R}^N$ be a bounded and open set with $\mathcal{H}^{N-1}(\partial U) < \infty$. Let $\mathbf{F}_1 \in \mathcal{DM}^\infty(U)$ and $\mathbf{F}_2 \in \mathcal{DM}^\infty(\mathbb{R}^N \setminus \overline{U})$. Then, with*

$$\mathbf{F}(y) := \begin{cases} \mathbf{F}_1(y), & y \in U, \\ \mathbf{F}_2(y), & y \in \mathbb{R}^N \setminus \overline{U}, \end{cases}$$

we have

$$\mathbf{F} \in \mathcal{DM}^\infty(\mathbb{R}^N).$$

PROOF: Applying the previous result to

$$\mathbf{F}_1^* := \chi_U(y) \mathbf{F}_1(y) \quad \text{for all } y \in \mathbb{R}^N$$

and

$$\mathbf{F}_2^* := \chi_{\mathbb{R}^N \setminus \overline{U}} \mathbf{F}_2(y) \quad \text{for all } y \in \mathbb{R}^N,$$

we see that

$$\mathbf{F} = \mathbf{F}_1^* + \mathbf{F}_2^*.$$

□

9 Cauchy Fluxes and Divergence-Measure Fields

The physical principle of balance law of the form

$$(9.1) \quad \int_{\partial E} f(y, \mathbf{v}(y)) \, d\mathcal{H}^{N-1}(y) + \int_E b(y) \, dy = 0$$

is basic in all of continuum physics. Here, $\mathbf{v}(y)$ is the interior unit normal to the boundary ∂E of E . In mechanics, f represents the surface force per unit area on ∂E , while in thermodynamics, f gives the heat flow per unit area across the boundary ∂E .

In 1823, Cauchy [12] (also see [13]) established the *stress theorem* that is probably the most important result in continuum mechanics: If $f(y, \nu(y))$, defined for each y in an open region Ω and every unit vector ν , is continuous in y and $b(y)$ is uniformly bounded on Ω , and if (9.1) is satisfied for every smooth region $E \Subset \Omega$, then $f(y, \nu)$ must be linear in ν . The *Cauchy postulate* states that the density flux f through a surface depends on the surface solely through the normal at that point. For instance, if $f(y, \nu)$ represents the heat flow, then the *stress theorem* states that there exists a vector field \mathbf{F} such that

$$f(y, \nu) = \mathbf{F}(y) \cdot \nu.$$

Since the time of Cauchy's stress result, many efforts have been made to generalize his ideas and remove some of his hypotheses. The first results in this direction were obtained by Noll [54] in 1959, who set up a basis for an axiomatic foundation for continuum thermodynamics. In particular, Noll showed that the Cauchy postulate may directly follow from the balance law. In [42], Gurtin and Martins introduced the concept of Cauchy flux and removed the continuity assumption on f . In [71], Ziemer proved Noll's theorem in the context of geometric measure theory, in which the Cauchy flux was first formulated at the level of generality with sets of finite perimeter in the absence of jump surfaces, "shock waves."

However, as we explain below, all the previous formulations of (9.1) do not allow the presence of "shock waves"; one of our main intentions in this paper is to develop a theory that allows the presence of "shock waves."

In this section we first introduce a class of Cauchy fluxes that allows the presence of the exceptional surfaces or "shock waves," and we then prove that such a Cauchy flux induces a bounded divergence-measure (vector) field \mathbf{F} so that the Cauchy flux over *every* oriented surface can be recovered through \mathbf{F} and the normal to the oriented surface. Before introducing this framework, we need the following definitions.

DEFINITION 9.1 An *oriented surface* in Ω is a pair (S, ν) so that $S \Subset \Omega$ is a Borel set and $\nu : \mathbb{R}^N \rightarrow \mathbb{S}^{N-1}$ is a Borel measurable unit vector field that satisfy the following property: There is a set $E \Subset \Omega$ of finite perimeter such that $S \subset \partial^* E$ and

$$\nu(y) = \nu_E(y) \chi_S(y),$$

where χ_S is the characteristic function of the set S and $\nu_E(y)$ is the interior measure-theoretic unit normal to E at y .

DEFINITION 9.2 Two oriented surfaces (S_j, ν_j) , $j = 1, 2$, are said to be *compatible* if there exists a set of finite perimeter E such that $S_j \subset \partial^* E$ and $\nu_j(y) = \nu_E(y) \chi_{S_j}(y)$, $j = 1, 2$. For simplicity, we will denote the pair (S, ν) simply as S , with the implicit understanding that S is oriented by the interior normal of some set E of finite perimeter. We define $-S = (S, -\nu)$, which is regarded as a different oriented surface.

DEFINITION 9.3 Let Ω be a bounded open set. A *Cauchy flux* is a functional \mathcal{F} that assigns to each oriented surface $S := (S, \nu) \in \Omega$ a real number and has the following properties:

- (i) $\mathcal{F}(S_1 \cup S_2) = \mathcal{F}(S_1) + \mathcal{F}(S_2)$ for any pair of compatible disjoint surfaces $S_1, S_2 \in \Omega$.
- (ii) There exists a nonnegative Radon measure σ in Ω such that

$$|\mathcal{F}(\partial^* E)| \leq \sigma(E^1)$$

for every set of finite perimeter $E \in \Omega$.

- (iii) There exists a constant C such that

$$|\mathcal{F}(S)| \leq C \mathcal{H}^{N-1}(S)$$

for every oriented surface $S \in \Omega$.

This general framework for Cauchy fluxes allows the presence of exceptional surfaces, “shock waves,” in the formulation of the axioms, on which the measure σ has support. On these exceptional surfaces, the Cauchy flux \mathcal{F} has a discontinuity, i.e., $\mathcal{F}(S) \neq -\mathcal{F}(-S)$. In fact, the exceptional surfaces are supported on the singular part of the measure σ in general. When σ reduces to the N -dimensional Lebesgue measure \mathcal{L}^N , the formulation reduces to Ziemer’s formulation in [71], and in this case σ vanishes on any \mathcal{H}^{N-1} -dimensional surface, which excludes shock waves.

The theory developed in this paper allows us to approximate the exceptional oriented surfaces or “shock waves” with smooth boundaries and rigorously pass to the limit to recover the flux on the exceptional oriented surfaces as the precise representative. This allows us to capture measure production density in the formulation of the balance law and entropy dissipation for entropy solutions of hyperbolic conservation laws. Once we know the flux across *every surface*, we proceed to obtain a rigorous derivation of nonlinear systems of balance laws with measure source terms from the physical principle of balance law in Section 10. The framework also allows the recovery of Cauchy entropy fluxes through the Lax entropy inequality for entropy solutions of hyperbolic conservation laws by capturing entropy dissipation; see Section 11.

The main theorem of this section is the following:

THEOREM 9.4 *Let \mathcal{F} be a Cauchy flux in Ω . Then there exists a unique divergence-measure field $\mathbf{F} \in \mathcal{DM}_{\text{loc}}^\infty(\Omega)$ such that*

$$(9.2) \quad \mathcal{F}(S) = - \int_S \mathcal{F}_i \cdot \nu \, d\mathcal{H}^{N-1}$$

for every oriented surface $(S, \nu) \in \Omega$, where $\mathcal{F}_i \cdot \nu$ is the normal trace of \mathbf{F} to the oriented surface.

When σ reduces to the N -dimensional Lebesgue measure \mathcal{L}^N , as in Ziemer's formulation, the vector field \mathbf{F} satisfies $\operatorname{div} \mathbf{F} \in L^\infty$ and $\mathcal{F}(S) = -\mathcal{F}(-S)$ for every surface S , which thus excludes shock waves where the Cauchy flux \mathcal{F} has a discontinuity, i.e., $\mathcal{F}(S) \neq -\mathcal{F}(-S)$.

In order to establish Theorem 9.4, we need Lemma 9.6, which was first shown in Degiovanni, Marzocchi, and Musesti [26]. Here we offer a simplified proof of this fact for completeness. In particular, Lemma 9.6 is a direct application of Theorem 9.5 (due to Fuglede) below. We also refer to Schuricht [60] for a different approach in formulating the axioms in Definition 9.3. Our Theorem 9.4 follows then by an approximation and Theorem 5.2.

The following theorem, due to Fuglede, is a generalization of Riesz's theorem, whose proof can be found in [38].

THEOREM 9.5 *Let μ be a nonnegative measure defined on a σ -field \mathcal{V} of subsets of a fixed set X and $X \in \mathcal{V}$. Let φ be an additive set function defined on a system of sets $\mathcal{U} \subset \mathcal{V}$ such that all finite unions of disjoint sets from \mathcal{U} , together with the empty set, form a field \mathcal{F} that generates \mathcal{V} . Assume that $\mu(A) < \infty$ for every $A \subset \mathcal{U}$. Then there exists a function $g(y) \in L^1(X, \mathcal{V}, \mu)$ with the property that*

$$\varphi(A) = \int_A g(y) d\mu \quad \text{for every } A \in \mathcal{U}$$

if and only if the following hold:

- (i) *For every $\varepsilon > 0$, there exists $\delta > 0$ such that $\sum_{i=1}^n |\varphi(A_i)| \leq \varepsilon$ for every finite system of disjoint sets A_1, \dots, A_n from \mathcal{U} where $\sum_{i=1}^n \mu(A_i) < \delta$.*
- (ii) *There is a finite constant C such that $\sum_{i=1}^n |\varphi(A_i)| \leq C$ for every finite system of disjoint sets A_1, \dots, A_n from \mathcal{U} .*

The function g is then essentially uniquely determined. Under the additional assumption that $\mu(X) < \infty$, condition (ii) is a consequence of condition (i).

Let $\{I\}$ be the collection of all closed cubes in \mathbb{R}^N of the form

$$I = [a_1, b_1] \times \cdots \times [a_N, b_N],$$

where $a_1, b_1, \dots, a_N, b_N$ are real numbers. For almost every $\tau_j \in [a_j, b_j]$, we define

$$I_{\tau_j} = \{y \in I : y_j = \tau_j\}.$$

We define the vectors e_1, \dots, e_N so that the j^{th} component of e_j is -1 and the other components are 0. We orient the surface I_{τ_j} with the vector e_j .

LEMMA 9.6 *Let \mathcal{F} be a Cauchy flux in Ω . Then there exists a divergence-measure field $\mathbf{F} \in \mathcal{DM}_{\text{loc}}^\infty(\Omega)$ such that, for every cube $I = [a_1, b_1] \times \cdots \times [a_N, b_N] \Subset \Omega$ and almost every $\tau_j \in [a_j, b_j]$,*

$$\mathcal{F}(I_{\tau_j}) = - \int_{I_{\tau_j}} \mathbf{F}(y) \cdot e_j d\mathcal{H}^{N-1}(y).$$

PROOF:

Step 1. We fix $j \in \{1, \dots, N\}$. For every cube $I \subset \Omega$, we define

$$\mu^j(I) = \int_{a_j}^{b_j} \mathcal{F}(I_{\tau_j}) d\tau_j.$$

We have

$$(9.3) \quad |\mu^j(I)| \leq \int_{a_j}^{b_j} |\mathcal{F}(I_{\tau_j})| d\tau_j \leq C \int_{a_j}^{b_j} \int_{I_{\tau_j}} d\mathcal{H}^{N-1} d\tau_j = C|I|.$$

Thus, from Theorem 9.5, there exists a function $f^j \in L^1(\Omega)$ such that

$$\mu^j(I) = \int_I f^j dy \quad \text{for every } I.$$

In fact, inequality (9.3) implies that $f^j \in L^\infty(\Omega)$ since, for \mathcal{L}^N -a.e. y ,

$$f^j(y) = \lim_{\substack{|I| \rightarrow 0 \\ y \in I}} \frac{\int_I f^j dx}{|I|} \leq C.$$

Fubini's theorem implies that

$$(9.4) \quad \mu^j(I) = \int_{a_j}^{b_j} \mathcal{F}(I_{\tau_j}) d\tau_j = \int_I f^j dy = \int_{a_j}^{b_j} \int_{I_{\tau_j}} f^j d\mathcal{H}^{N-1} d\tau_j.$$

Let $\tau_j \in [a_j, b_j]$, $\alpha_{k,j}$, and $\beta_{k,j}$ be sequences such that

$$\alpha_{k,j} \leq \tau_j \leq \beta_{k,j},$$

where $\alpha_{k,j}$ is an increasing sequence that converges to τ_j as $k \rightarrow \infty$, and $\beta_{k,j}$ is a decreasing sequence that converges to τ_j as $k \rightarrow \infty$. Thus, from (9.4), we obtain

$$(9.5) \quad \frac{1}{\alpha_{k,j} - \beta_{k,j}} \int_{a_j}^{b_j} \mathcal{F}(I_{\tau_j}) d\tau_j = \frac{1}{\alpha_{k,j} - \beta_{k,j}} \int_{a_j}^{b_j} \int_{I_{\tau_j}} f^j d\mathcal{H}^{N-1} d\tau_j.$$

We let $k \rightarrow \infty$ to obtain that, for a.e. τ_j ,

$$(9.6) \quad \mathcal{F}(I_{\tau_j}) = \int_{I_{\tau_j}} f^j d\mathcal{H}^{N-1}.$$

Define

$$\mathbf{F} := (f^1, \dots, f^N).$$

Then we find, for almost every τ_j , $j \in \{1, \dots, N\}$,

$$(9.7) \quad \mathcal{F}(I_{\tau_j}) = - \int_{I_{\tau_j}} \mathbf{F}(y) \cdot e_j d\mathcal{H}^{N-1}(y).$$

Step 2. We now prove that the divergence of \mathbf{F} , in the sense of distributions, is a measure. We define, for a.e. cube I ,

$$(9.8) \quad \eta(I) := - \int_{\partial I} \mathbf{F}(y) \cdot \mathbf{v}(y) d\mathcal{H}^{N-1}(y),$$

where \mathbf{v} is the interior unit normal to ∂I . From Step 1 and the definition of Cauchy fluxes, we have

$$(9.9) \quad |\eta(I)| = |\mathcal{F}(\partial I)| \leq \sigma(I^1) \leq \sigma(I)$$

for almost all closed cubes. Thus, we can again apply Theorem 9.5 to conclude that there exists a function $g \in L^1(\Omega; \sigma)$, uniquely defined in Ω up to a set of σ -measure zero, such that

$$(9.10) \quad \eta(I) = - \int_{\partial I} \mathbf{F}(y) \cdot \mathbf{v}(y) d\mathcal{H}^{N-1}(y) = \int_I g(y) d\sigma$$

for almost every closed cube $I \subset \Omega$.

Denote by $\tilde{\sigma}$ the measure given by $g d\sigma$ in Ω . We now prove

$$(9.11) \quad \operatorname{div} \mathbf{F} = \tilde{\sigma}$$

in the sense of distributions in any open set $U \Subset \Omega$.

Let $I \Subset U$ be any closed cube. Then, for any $\phi \in C^1$ with support contained in I ,

$$(9.12) \quad \int_U \mathbf{F} \cdot \nabla \phi dy = \lim_{\varepsilon \rightarrow 0} \int_U \mathbf{F}_\varepsilon \cdot \nabla \phi dy = - \lim_{\varepsilon \rightarrow 0} \int_U \phi \operatorname{div} \mathbf{F}_\varepsilon dy,$$

where $\mathbf{F}_\varepsilon = \mathbf{F} * \rho_\varepsilon$ and ρ is the standard mollifying kernel. We now prove that, for \mathcal{L}^N -a.e. $y \in U$,

$$\tilde{\sigma}_\varepsilon(y) = \operatorname{div} \mathbf{F}_\varepsilon(y),$$

where $\tilde{\sigma}_\varepsilon$ is the convolution of function ρ_ε with the measure $\tilde{\sigma}$; that is,

$$(9.13) \quad \tilde{\sigma}_\varepsilon(y) := (\rho_\varepsilon * \tilde{\sigma})(y) = \int_{\Omega} \rho_\varepsilon(y-x) d\tilde{\sigma}(x).$$

From (9.9)–(9.10), we find that, for $\varepsilon < \operatorname{dist}(\partial U, \partial \Omega)$,

$$\begin{aligned} \int_I \operatorname{div} \mathbf{F}_\varepsilon(y) dy &= - \int_{\partial I} \mathbf{F}_\varepsilon(y) \cdot \mathbf{v}(y) dy \\ &= - \int_{\partial I} \int_{\mathbb{R}^N} \mathbf{F}(y-x) \cdot \mathbf{v}(y) \rho_\varepsilon(x) dx dy \\ &= - \int_{\mathbb{R}^N} \int_{\partial I} \mathbf{F}(y-x) \cdot \mathbf{v}(y) \rho_\varepsilon(x) dy dx = \end{aligned}$$

$$\begin{aligned}
 &= - \int_{\mathbb{R}^N} \left(\int_{\partial I_x} \mathbf{F}(y) \cdot \mathbf{v}(y) dy \right) \rho_\varepsilon(x) dx \\
 &= \int_{\mathbb{R}^N} \tilde{\sigma}(I_x) \rho_\varepsilon(x) dx,
 \end{aligned}$$

where $I_x = \{y : a_i \leq y_i - x_i \leq b_i, i = 1, \dots, N\}$. We can consider the smooth function ρ_ε as a measure in \mathbb{R}^N , say λ_ε , by defining $\lambda_\varepsilon(A) = \int_A \rho_\varepsilon(x) dx$ for any Borel set A . We can also extend the measure $\tilde{\sigma}$ by zero outside Ω . Therefore, we find

$$(9.14) \quad \int_{\mathbb{R}^N} \tilde{\sigma}(I_x) \rho_\varepsilon(x) dx = (\tilde{\sigma} * \lambda_\varepsilon)(I) = (\lambda_\varepsilon * \tilde{\sigma})(I) = \int_{\mathbb{R}^N} \lambda_\varepsilon(I_x) d\tilde{\sigma}(x).$$

From (9.14) and using (9.13), we compute

$$\begin{aligned}
 \int_{\mathbb{R}^N} \lambda_\varepsilon(I_x) d\tilde{\sigma}(x) &= \int_{\Omega} \lambda_\varepsilon(I_x) d\tilde{\sigma}(x) = \int_{\Omega} \left(\int_{I_x} \rho_\varepsilon(y) dy \right) d\tilde{\sigma}(x) \\
 &= \int_{\Omega} \int_I \rho_\varepsilon(y - x) dy d\tilde{\sigma}(x) \\
 &= \int_I \left(\int_{\Omega} \rho_\varepsilon(y - x) d\tilde{\sigma}(x) \right) dy \\
 &= \int_I (\rho_\varepsilon * \tilde{\sigma})(y) dy \\
 &= \int_I \tilde{\sigma}_\varepsilon(y) dy.
 \end{aligned}$$

Therefore,

$$\int_I \operatorname{div} \mathbf{F}_\varepsilon(y) dy = \int_I \tilde{\sigma}_\varepsilon(y) dy.$$

Since the cube $I \Subset U$ is arbitrary, this shows that $\tilde{\sigma}_\varepsilon(y) = \operatorname{div} \mathbf{F}_\varepsilon(y)$ for \mathcal{L}^N -a.e. $y \in U$. Using this in (9.12), we obtain

$$\begin{aligned}
 \int_U \mathbf{F} \cdot \nabla \phi dy &= - \lim_{\varepsilon \rightarrow 0} \int_U \phi \operatorname{div} \mathbf{F}_\varepsilon dy \\
 (9.15) \quad &= - \lim_{\varepsilon \rightarrow 0} \int_U \phi \tilde{\sigma}_\varepsilon dy = - \int_U \phi(y) d\tilde{\sigma}(y),
 \end{aligned}$$

since the sequence of measures $\tilde{\sigma}_\varepsilon$ converges locally weak-star to $\tilde{\sigma}$ in Ω as $\varepsilon \rightarrow 0$.

□

PROOF OF THEOREM 9.4: Using Lemma 9.6, it follows that there exists an $\mathbf{F} \in \mathcal{DM}_{\text{loc}}^\infty(\Omega)$ such that, for any cube $I = [a_1, b_1] \times \cdots \times [a_N, b_N] \Subset \Omega$,

$$(9.16) \quad \mathcal{F}(I_{\tau_j}) = - \int_{I_{\tau_j}} \mathbf{F}(y) \cdot e_j d\mathcal{H}^{N-1}(y)$$

for almost every $\tau_j \in [a_j, b_j]$.

Let (S, ν) be an oriented surface. Then there exists a set of finite perimeter E such that $S \subset \partial^* E$. We approximate S with closed cubes such that

$$(9.17) \quad S = \bigcap_{i=1}^{\infty} J_i,$$

where each J_i is a finite union of closed cubes (which can be chosen so that (9.16) holds for $\tau_j = a_j$ and $\tau = b_j$) centered at S and $J_{i+1} \subset J_i$. Using Lemma 2.10, we have $\partial^*(J_i \cap E) \subseteq (\partial^* E \cap J_i) \cup (\partial^* J_i \cap E) \cup (\partial^* E \cap \partial^* J_i)$. From (9.17) and the fact that the cubes can also be chosen so that $\mathcal{H}^{N-1}(\partial^* E \cap \partial^* J_i) = 0$, we have

$$(9.18) \quad [\partial^*(J_i \cap E)] \Delta [S \cup (\partial^* J_i \cap E)] = \mathcal{N}_i,$$

where $\lim_{i \rightarrow \infty} \mathcal{H}^{N-1}(\mathcal{N}_i) = 0$. Since $\mathcal{F}(\mathcal{N}_i) \leq C \mathcal{H}^{N-1}(\mathcal{N}_i)$, we obtain

$$(9.19) \quad \lim_{i \rightarrow \infty} \mathcal{F}(\mathcal{N}_i) = 0.$$

The definition of Cauchy fluxes implies that

$$|\mathcal{F}(\partial^*(J_i \cap E))| \leq \sigma((J_i \cap E)^1).$$

The standard measure theory and (9.17) imply that

$$(9.20) \quad \lim_{i \rightarrow \infty} \sigma((J_i \cap E)^1) = \sigma\left(\bigcap J_i \cap E^1\right) = \sigma(S \cap E^1) = 0,$$

since $S \subset \partial^* E$. Therefore, we have

$$(9.21) \quad \lim_{i \rightarrow \infty} |\mathcal{F}(\partial^*(J_i \cap E))| = 0.$$

On the other hand, using Theorem 5.2, we have

$$(9.22) \quad \left| - \int_{\partial^*(J_i \cap E)} \mathcal{F}_i \cdot \nu d\mathcal{H}^{N-1} \right| = \left| \int_{(J_i \cap E)^1} \operatorname{div} \mathbf{F} \right| \leq \|\operatorname{div} \mathbf{F}\|((J_i \cap E)^1),$$

which yields (from (9.20) with $\|\operatorname{div} \mathbf{F}\|$ instead of σ):

$$(9.23) \quad \lim_{i \rightarrow \infty} \left| \int_{\partial^*(J_i \cap E)} \mathcal{F}_i \cdot \nu d\mathcal{H}^{N-1} \right| = 0.$$

Using (9.18), (9.21), and (9.16), we obtain

$$\begin{aligned}
 & \lim_{i \rightarrow \infty} |\mathcal{F}(\partial^*(J_i \cap E))| \\
 (9.24) \quad &= \lim_{i \rightarrow \infty} |\mathcal{F}(S) + \mathcal{F}(\partial^* J_i \cap E)| \\
 &= \lim_{i \rightarrow \infty} \left| \mathcal{F}(S) - \int_{\partial^* J_i \cap E} \mathbf{F}(y) \cdot \mathbf{v}(y) d\mathcal{H}^{N-1} \right| = 0.
 \end{aligned}$$

From (9.18), (9.19), and (9.23), we have

$$(9.25) \quad \lim_{i \rightarrow \infty} \left| - \int_{\partial^* J_i \cap E} \mathbf{F}(y) \cdot \mathbf{v}(y) d\mathcal{H}^{N-1} - \int_S \mathcal{F}_i \cdot \mathbf{v} d\mathcal{H}^{N-1} \right| = 0.$$

Combining (9.24) with (9.25) yields

$$\mathcal{F}(S) = - \int_S \mathcal{F}_i \cdot \mathbf{v} d\mathcal{H}^{N-1}.$$

Assume now that there exists another vector field $G = (g^1, \dots, g^N)$ such that (9.2) holds. Then, for fixed $j \in \{1, \dots, N\}$, we have

$$\begin{aligned}
 (9.26) \quad \int_I f^j dy &= \int_{a_j}^{b_j} \int_{I_{\tau_j}} f^j d\mathcal{H}^{N-1} d\tau_j \\
 &= \int_{a_j}^{b_j} \int_{I_{\tau_j}} g^j d\mathcal{H}^{N-1} d\tau_j = \int_I g^j dy
 \end{aligned}$$

for any cube I . This implies that

$$f^j(y) = g^j(y) \quad \text{for almost every } y.$$

□

Remark 9.7. In the proof of Theorem 9.4, working with $\tilde{E} := E^0 \cup \partial^* E$ and $-S = (S, \mathbf{v})$, we obtain (see Corollary 6.1) that

$$\mathcal{F}(-S) = - \int_S \tilde{\mathcal{F}}_i \cdot \mathbf{v} = \int_S \mathcal{F}_e \cdot \mathbf{v},$$

where $\tilde{\mathcal{F}}_i \cdot \mathbf{v}$ and $\mathcal{F}_e \cdot \mathbf{v}$ are the interior and exterior normal traces of \mathbf{F} relative to \tilde{E} and E , respectively. That is, the normal traces of $\mathbf{F} \in \mathcal{DM}_{\text{loc}}^\infty(\Omega)$ are the Cauchy densities over all oriented surfaces.

10 Mathematical Formulation of the Balance Law and Derivation of Systems of Balance Laws

In this section we first present the mathematical formulation for the physical principle of balance law (9.1). Then we apply the results established in Sections 3 through 9 to give a rigorous derivation of systems of balance laws with measure source terms. In particular, we give a derivation of hyperbolic systems of conservation laws (10.11).

A balance law on an open subset Ω of \mathbb{R}^N postulates that the *production* of a vector-valued “extensive” quantity in any bounded measurable subset $E \Subset \Omega$ with finite perimeter is balanced by the *Cauchy flux* of this quantity through the measure-theoretic boundary $\partial^m E$ of E (see Dafermos [22, 23]).

Like the Cauchy flux, the production is introduced through a functional \mathcal{P} , defined on any bounded measurable subset of finite perimeter $E \subset \Omega$, taking value in \mathbb{R}^k and satisfying the conditions

$$(10.1) \quad \mathcal{P}(E_1 \cup E_2) = \mathcal{P}(E_1) + \mathcal{P}(E_2) \quad \text{if } E_1 \cap E_2 = \emptyset,$$

$$(10.2) \quad |\mathcal{P}(E)| \leq \sigma(E).$$

Then the physical principle of balance law can be mathematically formulated as

$$(10.3) \quad \mathcal{F}(\partial^m E) = \mathcal{P}(E)$$

for any bounded measurable subset of finite perimeter $E \subset \Omega$.

Fuglede’s theorem, Theorem 9.5, indicates that conditions (10.1) and (10.2) imply that there is a production density $P \in \mathcal{M}(\Omega; \mathbb{R}^k)$ such that

$$(10.4) \quad \mathcal{P}(E) = \int_{E^1} P(y).$$

On the other hand, combining Theorem 5.2 with the argument in Section 9 yields that there exists $\mathbf{F} \in \mathcal{DM}_{\text{loc}}^\infty(\Omega; \mathbb{R}^{N \times k})$ such that

$$(10.5) \quad \mathcal{F}(\partial^m E) = - \int_{\partial^m E} (\mathcal{F}_i \cdot \nu) d\mathcal{H}^{N-1} = \int_{E^1} \text{div } \mathbf{F}(y)$$

for any set of finite perimeter $E \Subset \Omega$.

Then (10.3)–(10.5) yields the system of field equations

$$(10.6) \quad \text{div } \mathbf{F}(y) = P(y)$$

in the sense of measures on Ω .

We assume that the state of the medium is described by a state vector field u , taking value in an open subset U of \mathbb{R}^k , which determines both the flux density field \mathbf{F} and the production density field P at the point $y \in \Omega$ by the *constitutive equations*

$$(10.7) \quad \mathbf{F}(y) := \mathbf{F}(u(y), y), \quad P(y) := P(u(y), y),$$

where $F(u, y)$ and $P(u, y)$ are given smooth functions defined on $U \times \Omega$.

Combining (10.6) with (10.7) leads to the first-order, quasi-linear system of partial differential equations

$$(10.8) \quad \operatorname{div} F(u(y), y) = P(u(y), y),$$

which is called a *system of balance laws* (cf. [22]).

If $\mathcal{P} = 0$, the previous derivation yields

$$(10.9) \quad \operatorname{div} F(u(y), y) = 0,$$

which is called a *system of conservation laws*. When the medium is homogeneous

$$F(u, y) = F(u);$$

that is, F depends on y only through the state vector. Then system (10.9) becomes

$$(10.10) \quad \operatorname{div} F(u(y)) = 0.$$

In particular, when the coordinate system y is described by the time variable t and the space variable $x = (x_1, \dots, x_n)$,

$$y = (t, x_1, \dots, x_n) = (t, x), \quad N = n + 1,$$

and the flux density is written as

$$F(u) = (u, f_1(u), \dots, f_n(u)) = (u, f(u)),$$

then we have the following standard form for the system of conservation laws:

$$(10.11) \quad \partial_t u + \nabla_x \cdot f(u) = 0, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^k.$$

11 Entropy Solutions of Hyperbolic Conservation Laws

We now apply the results established in Sections 3 through 9 to the recovery of Cauchy entropy fluxes through the Lax entropy inequality for entropy solutions of hyperbolic conservation laws by capturing entropy dissipation. We focus on system (10.11), which is assumed to be hyperbolic.

DEFINITION 11.1 A function $\eta : \mathbb{R}^k \rightarrow \mathbb{R}$ is called an *entropy* of (10.11) if there exists $q : \mathbb{R}^k \rightarrow \mathbb{R}^n$ such that

$$(11.1) \quad \nabla q_j(u) = \nabla \eta(u) \nabla f_j(u), \quad j = 1, \dots, n.$$

Then the vector function $q(u)$ is called an *entropy flux* associated with the entropy $\eta(u)$, and the pair $(\eta(u), q(u))$ is called an *entropy pair*. The entropy pair $(\eta(u), q(u))$ is called a *convex entropy pair on the domain* $U \subset \mathbb{R}^k$ if the Hessian matrix $\nabla^2 \eta(u) \geq 0$ for any $u \in U$. The entropy pair $(\eta(u), q(u))$ is called a *strictly convex entropy pair on the domain* U if $\nabla^2 \eta(u) > 0$ for any $u \in U$.

Friedrichs and Lax [37] observed that most systems of conservation laws that result from continuum mechanics are endowed with a globally defined, strictly convex entropy. The available existence theories show that solutions of (10.11) generally fall within the following class of entropy solutions.

DEFINITION 11.2 A vector function $u = u(t, x) \in L_{\text{loc}}^\infty(\mathbb{R}_+ \times \mathbb{R}^n)$ is called an *entropy solution* of (10.11) if $u(t, x)$ satisfies the Lax entropy inequality

$$(11.2) \quad \partial_t \eta(u(t, x)) + \nabla_x \cdot q(u(t, x)) \leq 0$$

in the sense of distributions for any convex entropy pair $(\eta, q) : \mathbb{R}^k \rightarrow \mathbb{R} \times \mathbb{R}^n$.

Clearly, an entropy solution is a weak solution by choosing $\eta(u) = \pm u$ in (11.2).

One of the main issues in conservation laws is to study the behavior of entropy solutions in this class to explore to the fullest extent possible all questions relating to large-time behavior, uniqueness, stability, structure, and traces of entropy solutions, with neither specific reference to any particular method for constructing the solutions nor additional regularity assumptions. Because the distribution

$$\partial_t \eta(u(t, x)) + \nabla_x \cdot q(u(t, x))$$

is nonpositive, we conclude that it is in fact a Radon measure; that is, the field $(\eta(u(t, x)), q(u(t, x)))$ is a divergence-measure field. Thus there exists $\mu_\eta \in \mathcal{M}(\mathbb{R}_+ \times \mathbb{R}^n)$ with $\mu_\eta \leq 0$ such that

$$(11.3) \quad \text{div}_{(t,x)}(\eta(u(t, x)), q(u(t, x))) = \mu_\eta.$$

For any L^∞ entropy solution u , it was first indicated in Chen [14] that if the system is endowed with a strictly convex entropy, then, for any C^2 entropy pair (η, q) , there exists $\mu_\eta \in \mathcal{M}(\mathbb{R}_+ \times \mathbb{R}^n)$ such that

$$(11.4) \quad \text{div}_{(t,x)}(\eta(u(t, x)), q(u(t, x))) = \mu_\eta.$$

We introduce a functional on any oriented surface S ,

$$(11.5) \quad \mathcal{F}_\eta(S) = \int_S (\eta(u), q(u)) \cdot \nu \, d\mathcal{H}^n,$$

where $(\eta(u), q(u)) \cdot \nu$ is the normal trace in the sense of Theorem 5.2, since $(\eta(u), q(u)) \in \mathcal{DM}_{\text{loc}}^\infty(\mathbb{R}_+ \times \mathbb{R}^n)$. It is easy to check that the functional \mathcal{F}_η defined by (11.5) is a Cauchy flux in the sense of Definition 9.3.

DEFINITION 11.3 (Cauchy Entropy Fluxes) A functional \mathcal{F}_η defined by (11.5) is called a *Cauchy entropy flux with respect to the entropy η* .

In particular, when η is convex,

$$\mathcal{F}_\eta(S) \geq 0$$

for any oriented surface S . Furthermore, we can reformulate the balance law of entropy from the recovery of an entropy production by capturing entropy dissipation.

Moreover, it is clear that understanding more properties of divergence-measure fields can advance our understanding of the behavior of entropy solutions for hyperbolic conservation laws and other related nonlinear equations by selecting appropriate entropy pairs. As examples, we refer the reader to [15, 16, 17, 19] for the stability of Riemann solutions, which may contain rarefaction waves, contact discontinuities, and/or vacuum states, in the class of entropy solutions of the Euler equations for gas dynamics; to [15, 18] for the decay of periodic entropy solutions for hyperbolic conservation laws; to [20, 65] for the initial and boundary layer problems for hyperbolic conservation laws; to [16, 21] for the initial boundary value problems for hyperbolic conservation laws; and to [11, 52] for nonlinear degenerate parabolic-hyperbolic equations.

It is hoped that the theory of divergence-measure fields can be used to develop techniques in entropy methods, measure-theoretic analysis, partial differential equations, and related areas.

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Bibliography

- [1] Ambrosio, L.; Crippa, G.; Maniglia, S. Traces and fine properties of a BD class of vector fields and applications. *Ann. Fac. Sci. Toulouse Math. (6)* **14** (2005), no. 4, 527–561.
- [2] Ambrosio, L.; Fusco, N.; Pallara, D. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. Clarendon Press, Oxford University Press, New York, 2000.
- [3] Anzellotti, G. Pairings between measures and bounded functions and compensated compactness. *Ann. Mat. Pura Appl. (4)* **135** (1983), 293–318 (1984).
- [4] Aronszajn, N.; Smith, K. T. Functional spaces and functional completion. *Ann. Inst. Fourier Grenoble* **6** (1955–1956), 125–185.
- [5] Baiocchi, C.; Capelo, A. *Variational and quasivariational inequalities. Applications to free boundary problems*. Wiley-Interscience. Wiley, New York, 1984.
- [6] Baras, P.; Pierre, M. Singularités éliminables pour des équations semi-linéaires. *Ann. Inst. Fourier (Grenoble)* **34** (1984), no. 1, 185–206.
- [7] Besicovitch, A. S. A general form of the covering principle and relative differentiation of additive functions. *Proc. Cambridge Philos. Soc.* **41** (1945), 103–110.
- [8] Bourgain, J.; Brezis, H. On the equation $\operatorname{div} Y = f$ and application to control of phases. *J. Amer. Math. Soc.* **16** (2003), no. 2, 393–426 (electronic).
- [9] Brezzi, F.; Fortin, M. *Mixed and hybrid finite element methods*. Springer Series in Computational Mathematics, 15. Springer, New York, 1991.
- [10] Burago, Y. D.; Maz'ya, V. G. *Potential theory and function theory for irregular regions*. Seminars in Mathematics, V. A. Steklov Mathematical Institute, Leningrad, 3. Consultants Bureau, New York, 1969.

- [11] Bürger, R.; Frid, H.; Karlsen, K. H. On a free boundary problem for a strongly degenerate quasi-linear parabolic equation with an application to a model of pressure filtration. *SIAM J. Math. Anal.* **34** (2002), no. 3, 611–635 (electronic).
- [12] Cauchy, A. L. Recherches sur l'équilibre et le mouvement intérieur des corps solides ou fluides, élastiques ou non élastiques. *Bull. Soc. Philomathique* **10** (1823), no. 2, 9–13.
- [13] Cauchy, A. L. De la pression ou tension dans un corps solide. *Ex. de Math.* **2** (1827), no. 2, 42–56; *Oeuvres Complètes* **2** no. 7, 60–78. Available online at: http://portail.mathdoc.fr/cgi-bin/oetoc?id=OE_CAUCHY_2_7
- [14] Chen, G. Q. Hyperbolic systems of conservation laws with a symmetry. *Comm. Partial Differential Equations* **16** (1991), no. 8-9, 1461–1487.
- [15] Chen, G.-Q. Euler equations and related hyperbolic conservation laws. *Evolutionary equations*, vol. 2, 1–104. Handbook of Differential Equations. Elsevier/North-Holland, Amsterdam, 2005.
- [16] Chen, G.-Q.; Frid, H. Divergence-measure fields and hyperbolic conservation laws. *Arch. Ration. Mech. Anal.* **147** (1999), no. 2, 89–118.
- [17] Chen, G.-Q.; Frid, H. Large-time behavior of entropy solutions of conservation laws. *J. Differential Equations* **152** (1999), no. 2, 308–357.
- [18] Chen, G.-Q.; Frid, H. Decay of entropy solutions of nonlinear conservation laws. *Arch. Ration. Mech. Anal.* **146** (1999), no. 2, 95–127.
- [19] Chen, G.-Q.; Frid, H. Extended divergence-measure fields and the Euler equations for gas dynamics. *Comm. Math. Phys.* **236** (2003), no. 2, 251–280.
- [20] Chen, G.-Q.; Rascle, M. Initial layers and uniqueness of weak entropy solutions to hyperbolic conservation laws. *Arch. Ration. Mech. Anal.* **153** (2000), no. 3, 205–220.
- [21] Chen, G.-Q.; Torres, M. Divergence-measure fields, sets of finite perimeter, and conservation laws. *Arch. Ration. Mech. Anal.* **175** (2005), no. 2, 245–267.
- [22] Dafermos, C. M. Balance laws in continuum physics. *Advanced topics in theoretical fluid mechanics (Paseky nad Jizerou, 1997)*, 89–117. Pitman Research Notes in Mathematics Series, 392. Longman, Harlow, England, 1998.
- [23] Dafermos, C. M. *Hyperbolic conservation laws in continuum physics*. 2nd ed. Grundlehren der mathematischen Wissenschaften, 325. Springer, Berlin, 2005.
- [24] De Giorgi, E. *Complementi alla teoria della misura ($n - 1$)-dimensionale in uno spazio n -dimensionale*. Seminario di Matematica della Scuola Normale Superiore di Pisa, 1960–61. Editrice Tecnico Scientifica, Pisa, 1961.
- [25] De Giorgi, E. *Frontiere orientate di misura minima*. Seminario di Matematica della Scuola Normale Superiore di Pisa, 1960–61. Editrice Tecnico Scientifica, Pisa, 1961.
- [26] Degiovanni, M.; Marzocchi, A.; Musesti, A. Cauchy fluxes associated with tensor fields having divergence measure. *Arch. Ration. Mech. Anal.* **147** (1999), no. 3, 197–223.
- [27] De Pauw, T.; Pfeffer, W. F. The Gauss-Green theorem and removable sets for PDEs in divergence form. *Adv. Math.* **183** (2004), no. 1, 155–182.
- [28] De Pauw, T.; Pfeffer, W. F. Distributions for which $\operatorname{div} v = F$ has a continuous solution. *Comm. Pure Appl. Math.* **61** (2008), no. 2, 230–260.
- [29] Evans, L. C.; Gariepy, R. F. *Measure theory and fine properties of functions*. Studies in Advanced Mathematics. CRC, Boca Raton, Fla., 1992.
- [30] Federer, H. The Gauss-Green theorem. *Trans. Amer. Math. Soc.* **58** (1945), 44–76.
- [31] Federer, H. A note on the Gauss-Green theorem. *Proc. Amer. Math. Soc.* **9** (1958), 447–451.
- [32] Federer, H. Curvature measures. *Trans. Amer. Math. Soc.* **93** (1959), 418–491.
- [33] Federer, H. *Geometric measure theory*. Die Grundlehren der mathematischen Wissenschaften, 153. Springer, New York, 1969.

- [34] Federer, H.; Ziemer, W. P. The Lebesgue set of a function whose distribution derivatives are p -th power summable. *Indiana Univ. Math. J.* **22** (1972/73), 139–158.
- [35] Fleming, W. H. Functions whose partial derivatives are measures. *Illinois J. Math.* **4** (1960), 452–478.
- [36] Fleming, W. H.; Rishel, R. An integral formula for total gradient variation. *Arch. Math.* **11** (1960), 218–222.
- [37] Friedrichs, K. O.; Lax, P. D. Systems of conservation equations with a convex extension. *Proc. Nat. Acad. Sci. U.S.A.* **68** (1971), 1686–1688.
- [38] Fuglede, B. On a theorem of F. Riesz. *Math. Scand.* **3** (1955), 283–302 (1956).
- [39] Fuglede, B. *Extremal length and closed extensions of partial differential operators*. Jul. Gjellerups Boghandel, Copenhagen, 1960.
- [40] Giusti, E. *Minimal surfaces and functions of bounded variation*. Notes on Pure Mathematics, 10. Department of Pure Mathematics, Australian National University, Canberra, 1977.
- [41] Goffman, C.; Neugebauer, C. J.; Nishiura, T. Density topology and approximate continuity. *Duke Math. J.* **28** (1961), 497–505.
- [42] Gurtin, M. E.; Martins, L. C. Cauchy's theorem in classical physics. *Arch. Rational Mech. Anal.* **60** (1975/76), no. 4, 305–324.
- [43] Gurtin, M. E.; Williams, W. O. An axiomatic foundation for continuum thermodynamics. *Arch. Rational Mech. Anal.* **26** (1967), 83–117.
- [44] Gustin, W. Boxing inequalities. *J. Math. Mech.* **9** (1960), 229–239.
- [45] Heinonen, J. Nonsmooth calculus. *Bull. Amer. Math. Soc. (N.S.)* **44** (2007), no. 2, 163–232 (electronic).
- [46] Jurkat, W. B.; Nonnenmacher, D. J. F. The general form of Green's theorem. *Proc. Amer. Math. Soc.* **109** (1990), no. 4, 1003–1009.
- [47] Jurkat, W. B.; Nonnenmacher, D. J. F. A generalized n -dimensional Riemann integral and the divergence theorem with singularities. *Acta Sci. Math. (Szeged)* **59** (1994), no. 1-2, 241–256.
- [48] Jurkat, W. B.; Nonnenmacher, D. J. F. The fundamental theorem for the ν_1 -integral on more general sets and a corresponding divergence theorem with singularities. *Czechoslovak Math. J.* **45(120)** (1995), no. 1, 69–77.
- [49] Lewis, J. L. Uniformly fat sets. *Trans. Amer. Math. Soc.* **308** (1988), no. 1, 177–196.
- [50] Lim, A.; Rodrigues, B.; Xu, Z. A m -parallel crane scheduling problem with a non-crossing constraint. *Naval Res. Logist.* **54** (2007), no. 2, 115–127.
- [51] Lin, F.; Yang, X. *Geometric measure theory—an introduction*. Advanced Mathematics (Beijing/Boston), 1. Science Press, Beijing; International Press, Boston, 2002.
- [52] Mascia, C.; Porretta, A.; Terracina, A. Nonhomogeneous Dirichlet problems for degenerate parabolic-hyperbolic equations. *Arch. Ration. Mech. Anal.* **163** (2002), no. 2, 87–124.
- [53] Morse, A. P. Perfect blankets. *Trans. Amer. Math. Soc.* **61** (1947), 418–442.
- [54] Noll, W. The foundations of classical mechanics in the light of recent advances in continuum mechanics. *The axiomatic method. With special reference to geometry and physics. Proceedings of an International Symposium held at the Univ. of Calif., Berkeley, Dec. 26, 1957-Jan. 4, 1958*, 266–281. Studies in Logic and the Foundations of Mathematics. North-Holland, Amsterdam, 1959.
- [55] Noll, W. *Finite-dimensional spaces*. Vol. 1. *Algebra, geometry, and analysis*. Mechanics: Analysis, 10. Martinus Nijhoff, Dordrecht, 1987.
- [56] Nonnenmacher, D. J. F. Sets of finite perimeter and the Gauss-Green theorem with singularities. *J. London Math. Soc. (2)* **52** (1995), no. 2, 335–344.
- [57] Pfeffer, W. F. The Stokes theorem for the generalized Riemann integral. *Real Anal. Exchange* **26** (2000/01), no. 2, 623–636.

- [58] Pfeffer, W. F. The Gauss-Green theorem in the context of Lebesgue integration. *Bull. London Math. Soc.* **37** (2005), no. 1, 81–94.
- [59] Phuc, N. C.; Torres, M. Characterizations of the existence and removable singularities of divergence-measure vector fields. *Indiana Univ. Math. J.*, forthcoming. Available online at: <http://www.math.purdue.edu/~torres/site/Recent-papers>
- [60] Schuricht, F. A new mathematical foundation for contact interactions in continuum physics. *Arch. Rational Mech. Anal.* **184** (2007), no. 3, 495–551.
- [61] Šilhavý, M. The existence of the flux vector and the divergence theorem for general Cauchy fluxes. *Arch. Rational Mech. Anal.* **90** (1985), no. 3, 195–212.
- [62] Šilhavý, M. Cauchy's stress theorem and tensor fields with divergences in L^p . *Arch. Rational Mech. Anal.* **116** (1991), no. 3, 223–255.
- [63] Šilhavý, M. Divergence measure fields and Cauchy's stress theorem. *Rend. Sem. Mat. Univ. Padova* **113** (2005), 15–45.
- [64] Simon, L. *Lectures on geometric measure theory*. Proceedings of the Centre for Mathematical Analysis, Australian National University, 3. Australian National University, Centre for Mathematical Analysis, Canberra, 1983.
- [65] Vasseur, A. Strong traces for solutions of multidimensional scalar conservation laws. *Arch. Rational Mech. Anal.* **160** (2001), no. 3, 181–193.
- [66] Vol'pert, A. I. Spaces BV and quasilinear equations. *Mat. Sb. (N.S.)* **73 (115)** (1967), 255–302.
- [67] Whitney, H. *Geometric integration theory*. Princeton University, Princeton, N. J., 1957.
- [68] Ziemer, W. P. Extremal length and conformal capacity. *Trans. Amer. Math. Soc.* **126** (1967), 460–473.
- [69] Ziemer, W. P. Extremal length and p -capacity. *Michigan Math. J.* **16** (1969), 43–51.
- [70] Ziemer, W. P. Extremal length as a capacity. *Michigan Math. J.* **17** (1970), 117–128.
- [71] Ziemer, W. P. Cauchy flux and sets of finite perimeter. *Arch. Rational Mech. Anal.* **84** (1983), no. 3, 189–201.
- [72] Ziemer, W. P. *Weakly differentiable functions. Sobolev spaces and functions of bounded variation*. Graduate Texts in Mathematics, 120. Springer, New York, 1989.

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