# DIVERGENCE-MEASURE FIELDS: GAUSS-GREEN FORMULAS AND NORMAL TRACES

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Have you ever thought how most fields of science could provide their stunning scientific descriptions and results if the *integration by parts formula* would not exist? It is hard to imagine! Indeed, when we think about the scientific fields such as Electromagnetism, Fluid Dynamics, Solid Mechanics, and Relativity, integration by parts is an indispensable fundamental operation. Even though the integration by parts formula is commonly known as the *Gauss-Green formula* (or the *divergence theorem*, or *Ostrogradsky's theorem*), its discovery and rigorous mathematical proof are the result of the combined efforts of many great mathematicians, starting back the period when the calculus was invented by Newton and Leibniz in the 17th century.

The one-dimensional integration by parts formula for smooth functions was first discovered by Taylor (1715). The formula is a consequence of the Leibniz product rule and the Newton-Leibniz formula for the fundamental theorem of calculus.

The classical Gauss-Green formula for the multidimensional case is generally stated for  $C^1$  vector fields and domains with  $C^1$  boundaries. However, motivated by the physical solutions with discontinuity/singularity for Nonlinear Partial Differential Equations (PDEs) and Calculus of Variations, such as nonlinear hyperbolic conservation laws and Euler-Lagrange equations, the following fundamental issue arises:

Does the Gauss-Green formula still hold for vector fields with discontinuity/singularity (such as divergence-measure fields) and domains with rough boundaries?

The objective of this paper is to provide an answer of this issue and to present a short historical review of the contributions by great mathematicians spanning more than two centuries and which have made the discovery of the Gauss-Green formula possible.

### THE CLASSICAL GAUSS-GREEN FORMULA

The Gauss-Green formula was originally motivated in the analysis of fluids, electric fields, and other problems in the sciences. In particular, the implications of the Gauss-Green theorem include the mathematical formulation of balance laws in Continuum Mechanics, as well as the Maxwell's discovery of the laws of Electrodynamics while "The special theory of relativity owes its origins to Maxwell's equations" as indicated by Einstein<sup>1</sup> in 1949. The derivations of the Euler equations and the Navier-Stokes equations in Fluid Dynamics and the Maxwell's equations in Electrodynamics are based on the validity of the Gauss-Green formula and associated Stokes theorem. As an example, see Fig. 1 for the derivation of the Euler equation of mass in the smooth case.

<sup>&</sup>lt;sup>1</sup>Einstein, A.: Autobiographical Notes [1949]. In: *Albert Einstein: Philosopher-Scientist*, pp. 1—95, P. A. Schilpp (ed.), 1988.



FIGURE 1. Conservation of mass: The rate of change of the mass in an open set E,  $\frac{d}{dt} \int_E \rho(t, x) dx$ , is equal to the flux of mass across boundary  $\partial E$ ,  $\int_{\partial E} (\rho \mathbf{v}) \cdot \nu d\mathcal{H}^{n-1}$ , where  $\rho$  is the density and  $\mathbf{v}$  is the velocity field. The Gauss-Green formula yields the Euler equation for th conservation of mass:  $\rho_t + div(\rho \mathbf{v}) = 0$  in the smooth case.



FIGURE 2. Joseph-Louis Lagrange (25 January 1736 – 10 April 1813)

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FIGURE 3. Carl Gauss (30 April 1777 – 23 February 1855)

The formula that would be later known as the *divergence theorem* was first discovered by Lagrange<sup>2</sup> in 1762 (see Fig. 2), but he did not provide a proof of the result. The theorem was later rediscovered by Gauss<sup>3</sup> in 1813 (see Fig. 3) and Ostrogradsky<sup>4</sup> in 1828 (see Fig. 4). Ortrogradsky's method of proof was similar to the approach Gauss used. Independently, Green<sup>5</sup> (see Fig. 5) also rediscovered the *divergence theorem* in the two-dimensional case and published his result in 1828.

<sup>&</sup>lt;sup>2</sup>Lagrange, J.-L.: Nouvelles recherches sur la nature et la propagation du son, *Miscellanea Taurinensia* (also known as: *Mélanges de Turin*), 2: 11–172, 1762. He treated a special case of the divergence theorem and transformed triple integrals into double integrals via integration by parts.

<sup>&</sup>lt;sup>3</sup>Gauss, C. F.: Theoria attractionis corporum sphaeroidicorum ellipticorum homogeneorum methodo nova tractata, *Commentationes Societatis Regiae Scientiarium Gottingensis Recentiores*, 2: 355–378, 1813. In this paper, a special case of the theorem was considered.

<sup>&</sup>lt;sup>4</sup>Ostrogradsky, M. (presented on November 5, 1828; published in 1831): Première note sur la théorie de la chaleur (First note on the theory of heat), *Mémoires de l'Académie Impériale des Sciences de St. Pétersbourg*, Series 6, 1: 129–133, 1831. He stated and proved the divergence-theorem in its cartesian coordinate form.

<sup>&</sup>lt;sup>5</sup>Green, G.: An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism, Nottingham, England: T. Wheelhouse, 1828.



FIGURE 4. Mikhail Ostrogradsky (24 September 1801 – 1 January 1862)



FIGURE 5. George Green (14 July 1793 – 31 May 1841)

The divergence theorem in its vector form for the n-dimensional case  $(n \ge 2)$  can be stated as

$$\int_{U} \operatorname{div} \boldsymbol{F} \, \mathrm{d}y = -\int_{\partial U} \boldsymbol{F} \cdot \nu \, \mathrm{d}\mathcal{H}^{n-1},\tag{1}$$

where  $\mathbf{F}$  is a  $C^1$  vector field, U is a bounded open set with piecewise smooth boundary,  $\nu$  is the *inner* unit normal to U, and  $\mathscr{H}^{n-1}$  is the (n-1)-dimensional Hausdorff measure (that is an extension of the surface area measure for 2-dimensional surfaces to general (n-1)dimensional boundaries  $\partial U$ ). Formula (1) was later formulated, thanks to the development of *Vector Calculus*. The formulation of (1), where  $\mathbf{F}$  represents a physical vector quantity, is also the result of the efforts of many mathematicians including Gibbs, Heaviside, Poisson, Sarrus, Stokes, and Volterra; see [16] and the references therein. In conclusion, formula (1) is the result of more than two centuries of efforts by great mathematicians!

## GAUSS-GREEN FORMULAS AND TRACES FOR LIPSCHITZ VECTOR FIELDS ON SETS OF FINITE PERIMETER

We first go back to the issue arisen earlier of extending the Gauss-Green formula to very rough sets. The development of geometric measure theory in the middle of the 20th century opened the door to the extension of the classical Gauss-Green formula over *sets of finite perimeter* (whose boundaries can be very rough and contain cusps, corners, among others; see Figs. 5–6) for Lipschitz vector fields.

Indeed, we may consider the left side of (1) as a linear functional acting on vector fields  $\mathbf{F} \in C_c^1(\mathbb{R}^n)$ . If E is such that the functional:  $\mathbf{F} \to \int_E div \mathbf{F} dy$  is bounded on  $C_c(\mathbb{R}^n)$ , then the Riesz representation theorem implies that there exists a Radon measure  $\mu_E$  such that

$$\int_{E} \operatorname{div} \boldsymbol{F} \, \mathrm{d}y = \int_{\mathbb{R}^{n}} \boldsymbol{F} \cdot \mathrm{d}\mu_{E} \qquad \text{for all } \boldsymbol{F} \in C_{c}^{1}(\mathbb{R}^{n}),$$
(2)

and the set, E, is called a set of finite perimeter in  $\mathbb{R}^n$ . In this case, the Radon measure  $\mu_E$  is actually  $-D\chi_E$ , where  $D\chi_E$  is the distributional gradient of the characteristic function of E. A set of density  $\alpha \in [0, 1]$  of E in  $\mathbb{R}^n$  is defined by

$$E^{\alpha} := \{ y \in \mathbb{R}^{n} : \lim_{r \to 0} \frac{|B_{r}(y) \cap E|}{|B_{r}(y)|} = \alpha \},$$
(3)

where |B| as the Lebesgue measure of any Lebesgue measurable set B. Then  $E^0$  is the measure-theoretic exterior of E, while  $E^1$  is the measure-theoretic interior of E.



FIGURE 6. E is an open set where cusp P is a point of density 0



FIGURE 7. E is an open set with several fractures (dotted lines)

The De Giorgi's structure theorem shows that, even though the boundary of E can be very rough, it has nice tangential properties so that there is a notion of measure-theoretic tangent plane. More rigorously, the topological boundary  $\partial E$  of E contains an (n-1)-rectifiable set, known as the reduced boundary of E, denoted as  $\partial^* E$ , which can be covered by a countable union of  $C^1$  surfaces, up to a set of  $\mathscr{H}^{n-1}$ -measure zero. It can be shown that every  $y \in \partial^* E$ has an *inner* unit normal  $\nu_E(y)$  and a tangent plane in the measure-theoretic sense, and (2) reduces to

$$\int_{E} \operatorname{div} \boldsymbol{F} \, \mathrm{d}y = -\int_{\partial^{*}E} \boldsymbol{F}(y) \cdot \boldsymbol{\nu}_{E}(y) \, \mathrm{d}\mathcal{H}^{n-1}(y).$$
(4)

This Gauss-Green formula for Lipschitz vector fields  $\mathbf{F}$  over sets of finite perimeter was proved by De Giorgi (1954–55) and Federer (1945, 1958) in a series of papers. See Federer [12] and the references therein.

## Gauss-Green Formulas and Traces for Sobolev and BV Functions on Lipschitz Domains

It happens in many areas of analysis, such as PDEs and Calculus of Variations, that it is necessary to work with the functions that are not Lipschitz, but only in  $L^p$ ,  $1 \leq p \leq \infty$ . In many of these cases, the functions have distributional derivatives that also belong to  $L^p$ . That is, the corresponding  $\mathbf{F}$  in (4) is a Sobolev vector field. The necessary and sufficient conditions for the existence of traces of Sobolev functions defined on the boundary of the domain have been obtained so that (4) is a valid formula over open sets with Lipschitz boundary.

The development of the theory of Sobolev spaces has been fundamental in analysis. However, for many further applications, this theory is still not sufficient. For example, the characteristic function of a set E of finite perimeter,  $\chi_E$ , belongs to  $L^1$ , but the distributional derivative  $D\chi_E$  does not belong to  $L^1$  which is in fact a Radon measure. Physical solutions in gas dynamics involve shock waves that are discontinuities with jumps. Thus, a larger space of functions, called the space of functions of bounded variation (BV), is necessary, which consists of all functions in  $L^1$  whose distributional derivatives are Radon measures. This space has compactness properties that allow, for instance, to show the existence of minimal surfaces and the well-posedness of BV solutions for hyperbolic conservation laws. Moreover, the Gauss-Green formula (4) is also valid for BV vector fields over Lipschitz domains. See [12, 13, 20] and the references therein.

#### DIVERGENCE-MEASURE FIELDS AND HYPERBOLIC CONSERVATION LAWS

A vector field  $\mathbf{F} \in L^p(\Omega)$ ,  $1 \leq p \leq \infty$ , is called a divergence-measure field if  $div \mathbf{F}$  is a signed Radon measure with finite total variation in  $\Omega$ . Such vector fields form Banach spaces, denoted as  $\mathcal{DM}^p(\Omega)$ , for  $1 \leq p \leq \infty$ .

These spaces arise naturally in the field of hyperbolic conservation laws. Consider a hyperbolic system of conservation laws:

$$\mathbf{u}_t + \nabla_x \cdot \mathbf{f}(\mathbf{u}) = 0$$
 for  $\mathbf{u} = (u_1, ..., u_m) : \mathbb{R}^n \to \mathbb{R}^m$  (5)

where  $(t, x) \in \mathbb{R}^n_+ := \mathbb{R}_+ \times \mathbb{R}^d$ , n := d + 1,  $\mathbf{f} = (\mathbf{f}^1, \mathbf{f}^2, ..., \mathbf{f}^m)$ , and  $\mathbf{f}^i : \mathbb{R}^m \to \mathbb{R}^d$ . A function  $\eta \in C^1(\mathbb{R}^m, \mathbb{R})$  is called an entropy of system (5) if there exists  $\mathbf{q} \in C^1(\mathbb{R}^m, \mathbb{R}^d)$  such that

$$\nabla \mathbf{q}_k(\mathbf{u}) = \nabla \eta(\mathbf{u}) \nabla \mathbf{f}_k(\mathbf{u}) \qquad \text{for } k = 1, 2, ..., d.$$
(6)

Friedrichs-Lax (1971) observed that most systems of conservation laws that result from Continuum Mechanics are endowed with a globally defined, strictly convex entropy. In particular, for the Euler equations for compressible fluids in Lagrangian coordinates,  $\eta = -S$ is such an entropy, where S is the physical thermodynamic entropy of the fluid (*cf.* [11]). The available existence theories show that the solutions of (5) generally fall within the following class of entropy solutions:

An entropy solution **u** of system (5) is characterized by the Lax entropy inequality: For any convex entropy pair  $(\eta, \mathbf{q})$ ,

$$\eta(\mathbf{u})_t + \operatorname{div}_x \mathbf{q}(\mathbf{u}) \leqslant 0 \qquad \text{in the sense of distributions.}$$

$$\tag{7}$$

This implies that there exists a nonnegative measure  $\mu_{\eta} \in \mathcal{M}(\mathbb{R}^{n}_{+})$  such that

$$\operatorname{div}_{(t,x)}(\eta(\mathbf{u}(t,x)), \mathbf{q}(\mathbf{u}(t,x))) = -\mu_{\eta}.$$
(8)

Moreover, for any  $L^{\infty}$  entropy solution **u**, if the system is endowed with a strictly convex entropy, then, for any  $C^2$  entropy pair  $(\eta, \mathbf{q})$  (not necessarily convex for  $\eta$ ), there exists  $\mu_{\eta} \in \mathcal{M}(\mathbb{R}^n_+)$  such that (8) still holds. For these cases,  $(\eta(\mathbf{u}), \mathbf{q}(\mathbf{u}))(t, x)$  is a  $\mathcal{DM}^p(\mathbb{R}^n_+)$ vector field as long as  $(\eta(\mathbf{u}), \mathbf{q}(\mathbf{u})) \in L^p(\mathbb{R}^n_+, \mathbb{R}^m)$  for some  $p \in [1, \infty]$ .

Equation (8) is one of the main motivations to develop a  $\mathcal{DM}$  theory in Chen-Frid [4, 5]. In particular, one of the major issues is whether integration by parts can be performed in (7) to explore to fullest extent possible all the information about the entropy solution **u**. Thus, a concept of normal traces for  $\mathcal{DM}$  fields **F** is necessary to be developed. The existence of weak normal traces is also fundamental for initial-boundary value problems for (5) and for the structure and regularity of entropy solutions **u** (see *e.g.* [4, 5, 8, 11, 19]).

Motivated by hyperbolic conservation laws, the *interior* and *exterior* normal traces need to be constructed as the limit of classical normal traces on one-sided smooth approximations of the domain. Then the surface of a *shock wave* can be approximated with *smooth surfaces* to obtain the *interior* and *exterior* fluxes on the *shock wave*.

### Gauss-Green Formulas and Normal Trace for $\mathcal{DM}^{\infty}$ Fields

We start with the following example:

**Example 1**. Consider the vector field  $\mathbf{F} : \Omega = \mathbb{R}^2 \cap \{y_1 > y_2\} \to \mathbb{R}^2$ :

$$\mathbf{F}(y_1, y_2) = \left(\sin(\frac{1}{y_1 - y_2}), \sin(\frac{1}{y_1 - y_2})\right). \tag{9}$$

Then  $\mathbf{F} \in \mathcal{DM}^{\infty}(\Omega)$  with div $\mathbf{F} = 0$  in  $\Omega$ .

However, for an open square  $E \subset \Omega$  with one face contained in the line,  $\{y_1 = y_2\}$ , the previous Gauss-Green formulas do not apply. Indeed, since  $\mathbf{F}(y_1, y_2)$  is highly oscillatory when  $y_1 - y_2 \to 0$ , it is not clear how the normal trace  $\mathbf{F} \cdot \nu$  on  $\{y_1 = y_2\}$  can be defined in the classical sense, so that the equality between  $\int_E \operatorname{div} \mathbf{F} = 0$  and  $\int_{\partial E} \mathbf{F} \cdot \nu$  holds.

This example shows that a suitable notion of normal traces is required to be developed. See also Chen-Frid [4].

A generalization of (1) to  $\mathcal{DM}^{\infty}(\Omega)$  fields and bounded sets with Lipschitz boundary was derived in Anzellotti [1] and Chen-Frid [4] by different approaches. A further generalization of (1) to  $\mathcal{DM}^{\infty}(\Omega)$  fields and arbitrary bounded sets of finite perimeter,  $E \subseteq \Omega$ , was first obtained in Chen-Torres [7] and Šilhavý [17] independently; see also [9, 10].

**Theorem 1** (Chen-Torres [7] and Šilhavý [17]). Let E be a set of finite perimeter. Then

$$\int_{E^1} \mathrm{d}div \boldsymbol{F} = -\int_{\partial^* E} \mathfrak{F}_{\mathrm{i}} \cdot \nu_E \,\mathrm{d}\mathcal{H}^{n-1}, \quad \int_{E^1 \cup \partial^* E} \mathrm{d}div \boldsymbol{F} = -\int_{\partial^* E} \mathfrak{F}_{\mathrm{e}} \cdot \nu_E \,\mathrm{d}\mathcal{H}^{n-1} \qquad (10)$$

where the interior and exterior normal traces  $\mathfrak{F}_i \cdot \nu_E$  and  $\mathfrak{F}_e \cdot \nu_E$  are bounded functions defined on the reduced boundary of E, both in  $L^{\infty}(\partial^* E; \mathscr{H}^{n-1})$ , and  $E^1$  is the measure-theoretic interior of E as defined in (3).

One approach for the proof of (10) is based on a product rule for  $\mathcal{DM}^{\infty}$  fields (see [7]). Another approach in [9], following [4], is based on a new approximation theorem for sets of finite perimeter, which shows that the level sets of the convolutions  $w_k := \chi_E * \rho_k$  provide smooth approximations essentially from the *interior* (by choosing  $w_k^{-1}(t)$  for  $\frac{1}{2} < t < 1$ ) and the *exterior* (for  $0 < t < \frac{1}{2}$ ). Thus, the function traces are constructed as the limit of classical normal traces over the smooth approximations of E. In this approach, it is also assumed that  $E \subseteq \Omega$ , where  $\Omega$  is the domain of definition of  $\mathbf{F}$ , since the level set  $w_k^{-1}(t)$ (with a suitable fixed  $0 < t < \frac{1}{2}$ ) can intersect the measure-theoretic exterior  $E^0$  of E. Therefore, a critical step in the proof is to show that  $\mathscr{H}^{n-1}(w_k^{-1}(t) \cap E^0)$  converges to zero as  $k \to \infty$ . A basic ingredient of this proof is the fact that, if  $\mathbf{F}$  is a  $\mathcal{DM}^{\infty}$  field, then the Radon measure  $|div\mathbf{F}|$  is absolutely continuous with respect to  $\mathscr{H}^{n-1}$ , as first observed by Chen-Frid [4].

In particular, the formulas in (10) apply to the set of finite perimeter, E, defined as the countable union of open balls with centers on the rational points  $y_k$ , k = 1, 2, ..., of the unit ball in  $\mathbb{R}^n$  and with radius  $2^{-k}$ . We could also apply the formulas to sets E of finite perimeter (*i.e.*  $\mathscr{H}^{n-1}(\partial^* E) < \infty$ ) with a large set of cusps in the topological boundary  $(e.g. \ \mathscr{H}^{n-1}(E^0 \cap \partial E) > 0 \text{ or } \mathscr{H}^{n-1}(E^1 \cap \partial E) > 0)$ . The set, E, could also have points in the boundary that belong  $E^{\alpha}$  for  $\alpha \notin \{0, 1\}$ . For example, the four corners of a square are points of density  $\frac{1}{4}$ . However, Federer's theorem states that  $\mathscr{H}^{n-1}(\partial^* E \setminus \partial^* E) = 0$ , where  $\partial^* E = \mathbb{R}^n \setminus (E^0 \cup E^1)$  is called the essential boundary. Note that  $\partial^* E \subset E^{\frac{1}{2}}$  and  $\partial^* E \subset \partial^* E \subset \partial E$ .

If E is the open disk  $\{y \in \mathbb{R}^2 : |y| < 1\}$  minus one of the radius, the above formulas apply, but the integration is not over the original representative consisting of the disk with a radius removed, since  $E^1 = \{y \in \mathbb{R}^2 : |y| < 1\}$ . In some applications, we may want to integrate on a domain with *fractures* or *cracks*. Since the cracks are part of the topological boundary and belong to the measure-theoretic interior  $E^1$ , the formulas in (10) do not provide such information. In order to prove a Gauss-Green formula that includes this example, we restrict to *open sets* E of finite perimeter with  $\mathscr{H}^{n-1}(\partial E \setminus E^0) < \infty$ . Therefore,  $\partial E$  can still have a large set of *cusps* or points of density 0 (*i.e.* points belonging to  $E^0$ ). **Theorem 2** (Chen-Li-Torres [6]). If E is any bounded set with positive Lebesgue measure (not necessarily open), then there exists a family of sets  $E_k \subseteq E$  such that  $E_k \to E$  in  $L^1$ and  $\sup_k \mathscr{H}^{n-1}(\partial^* E_k) < \infty$  if and only if  $\mathscr{H}^{n-1}(\partial E \setminus E^0) < \infty$ . Furthermore, there exists a function  $g \in L^{\infty}(\partial^* E; \mathscr{H}^{n-1})$  such that, if  $\mathbf{F} \in \mathcal{DM}^{\infty}(E)$ , then

$$\int_{E} \phi \, \mathrm{d}div \mathbf{F} + \int_{E} \mathbf{F} \cdot \nabla \phi \, \mathrm{d}y = \int_{\partial E \setminus E^{0}} g(y) \mathrm{d}\mathscr{H}^{n-1}(y) \qquad \text{for every } \phi \in C^{1}_{c}(\mathbb{R}^{n}).$$
(11)

This approximation result for the bounded set E with positive Lebesgue measure can be accomplished by performing delicate covering arguments, especially by applying the Besicovitch theorem to a covering of the set  $\partial E \cap E^0$ . Moreover, (11) is a formula up to the boundary, since we do not assume that the domain of integration is compactly contained in the domain of  $\mathbf{F}$ . In addition, the Gauss-Green formulas in (10) can be rewritten as integration by parts formulas with  $\phi$  as in (11). More general product rules for  $div(\phi \mathbf{F})$  can be proved to weaken the regularity of  $\phi$ ; see [3, 6] and the references therein.

### Gauss-Green Formulas and Normal Traces for $\mathcal{DM}^p$ Fields

For  $\mathcal{DM}^p$  fields with  $1 \leq p < \infty$ , the situation becomes more delicate.

**Example 2**. Consider the vector field  $\mathbf{F} : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2$  (see Fig. 8):

$$\boldsymbol{F}(y_1, y_2) = \frac{(y_1, y_2)}{y_1^2 + y_2^2}.$$
(12)

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Then  $\mathbf{F} \in \mathcal{DM}_{loc}^p(\mathbb{R}^2)$  for  $1 \leq p < 2$  and  $\operatorname{div} \mathbf{F} = 2\pi \delta_{(0,0)}$ . If  $U = (0,1)^2$ , it is observed in Chen-Frid [5, Example 1.1] that

$$0 = \operatorname{div} \boldsymbol{F}(U) \neq -\int_{\partial U} \boldsymbol{F} \cdot \nu_U \, \mathrm{d} \mathscr{H}^1 = \frac{\pi}{2},$$

where  $\nu_U$  is the inner unit normal to the square. However, if the signed distance function d to  $\partial U$  is used to define  $U^{\varepsilon} := \{y \in \mathbb{R}^n : d(y) > \varepsilon\}$  and  $U_{\varepsilon} := \{y \in \mathbb{R}^n : d(y) > -\varepsilon\}$  for any  $\varepsilon > 0$ , then

$$0 = \operatorname{div} \boldsymbol{F}(U) = -\lim_{\varepsilon \to 0} \int_{\partial U^{\varepsilon}} \boldsymbol{F} \cdot \nu_{U^{\varepsilon}} \, \mathrm{d} \mathscr{H}^{1},$$
  
$$2\pi = \operatorname{div} \boldsymbol{F}(\overline{U}) = -\lim_{\varepsilon \to 0} \int_{\partial U_{\varepsilon}} \boldsymbol{F} \cdot \nu_{U_{\varepsilon}} \, \mathrm{d} \mathscr{H}^{1}.$$

In this sense, the equality is achieved on both sides of the formula.

Indeed, for a  $\mathcal{DM}^p$  field with  $p \neq \infty$ , since  $|div \mathbf{F}|$  is absolutely continuous with respect to  $\mathscr{H}^{n-p'}$  for p' > 1 with  $\frac{1}{p'} + \frac{1}{p} = 1$ . This implies that the approach in [9] does not apply to obtain normal traces for  $\mathcal{DM}^p$  fields for  $p \neq \infty$ .

Then the following questions arise:

- Can the previous formulas be proved in general for any  $\mathbf{F} \in \mathcal{DM}^p(\Omega)$  and for any open set  $U \subset \Omega$ ?
- Since almost all the level sets of the distance function are only sets of finite perimeter, can the formulas with smooth approximations of U be obtained, in place of U<sup>ε</sup> and U<sub>ε</sub>?
- If U has a Lipschitz boundary, do regular Lispchitz deformations of U, as defined in Chen-Frid [4, 5], exist?





FIGURE 9. The normal trace  $\langle \boldsymbol{F} \cdot \boldsymbol{\nu}, \cdot \rangle_{\partial U}$  of the vector field  $\boldsymbol{F}(y_1, y_2) = \frac{(-y_2, y_1)}{y_1^2 + y_2^2}$  with  $U = (-1, 1) \times (-1, 0)$  is not a measure

The answer to all three questions is *affirmative*.

**Theorem 3** (Chen-Comi-Torres [3]). Let  $U \subset \Omega$  be a bounded open set, and let  $\mathbf{F} \in \mathcal{DM}^p(\Omega)$ . Then, for any  $\phi \in C^0 \cap L^\infty(\Omega)$  with  $\nabla \phi \in L^{p'}(\Omega; \mathbb{R}^n)$  for p' > 1 with  $\frac{1}{p'} + \frac{1}{p} = 1$ , there exists a sequence of bounded open sets  $U_k$  with  $C^\infty$  boundary such that  $U_k \Subset U$ ,  $\bigcup_k U_k = U$ , and

$$\int_{U} \phi \, \mathrm{d}div \boldsymbol{F} + \int_{U} \boldsymbol{F} \cdot \nabla \phi \, \mathrm{d}y = -\lim_{k \to \infty} \int_{\partial U_k} \phi \boldsymbol{F} \cdot \nu_{U_k} \, \mathrm{d}\mathscr{H}^{n-1}, \tag{13}$$

where  $\nu_{U_k}$  is the classical inner unit normal to  $U_k$ . For the exterior formula with the same  $\phi \in C^0 \cap L^{\infty}(\Omega)$ , if  $U \subseteq \Omega$  is an open set, there exists a sequence of bounded open sets  $V_k$  with  $C^{\infty}$  boundary such that  $U \subseteq V_k \subset \Omega$ ,  $\bigcap_k V_k = \overline{U}$ , and

$$\int_{\overline{U}} \phi \, \mathrm{d} div \boldsymbol{F} + \int_{\overline{U}} \boldsymbol{F} \cdot \nabla \phi \, \mathrm{d}y = -\lim_{k \to \infty} \int_{\partial V_k} \phi \boldsymbol{F} \cdot \nu_{V_k} \, \mathrm{d}\mathcal{H}^{n-1}, \tag{14}$$

where  $\nu_{V_k}$  is the classical inner unit normal to  $V_k$ .

For the open set U with Lipschitz boundary, it can be proved that the deformations of U obtained with the method of regularized distance are bi-Lipschitz. We can also employ an alternative construction by Nečas (1962) to obtain smooth approximations  $U^{\tau}$  of a bounded open set U with Lipschitz boundary in such a way that the deformation  $\Psi_{\tau}(x)$  mapping  $\partial U$  to  $\partial U^{\tau}$  is bi-Lipschitz and the Jacobians of the deformations  $J^{\partial U}(\Psi_{\tau})$  converge to 1 in  $L^1(\partial U)$  as  $\tau$  approaches zero (see [3, Theorem 8.19]). This shows that any bounded open set with Lipschitz boundary admits a *regular Lipschitz deformation* in the sense of Chen-Frid [4, 5]. Therefore, we can write more explicit Gauss-Green formulas for Lipschitz domains.

**Theorem 4** (Chen-Comi-Torres [3]). If  $U \subseteq \Omega$  is an open set with Lipschitz boundary and  $\mathbf{F} \in \mathcal{DM}^p(\Omega)$  for  $1 \leq p \leq \infty$ , then, for every  $\phi \in C^0(\Omega)$  with  $\nabla \phi \in L^{p'}(\Omega; \mathbb{R}^n)$ , there exists a set  $\mathcal{N} \subset \mathbb{R}$  with  $\mathcal{L}^1(\mathcal{N}) = 0$  such that, for every nonnegative sequence  $\{\varepsilon_k\} \notin \mathcal{N}$  satisfying  $\varepsilon_k \to 0$ ,

$$\int_{U} \phi \, \mathrm{d} div \boldsymbol{F} + \int_{U} \boldsymbol{F} \cdot \nabla \phi \, \mathrm{d} y = -\lim_{k \to \infty} \int_{\partial U} \left( \phi \boldsymbol{F} \cdot \frac{\nabla \rho}{|\nabla \rho|} \right) (\Psi_{\varepsilon_{k}}(y)) J^{\partial U}(\Psi_{\varepsilon_{k}}(y)) \, \mathrm{d} \mathscr{H}^{n-1},$$

and

$$\int_{\overline{U}} \phi \, \mathrm{d}div \boldsymbol{F} + \int_{\overline{U}} \boldsymbol{F} \cdot \nabla \phi \, \mathrm{d}y = -\lim_{k \to \infty} \int_{\partial U} \left( \phi \boldsymbol{F} \cdot \frac{\nabla \rho}{|\nabla \rho|} \right) (\Psi_{-\varepsilon_k}(y)) J^{\partial U}(\Psi_{-\varepsilon_k}(y)) \, \mathrm{d}\mathscr{H}^{n-1}$$

The question you may be wondering now is whether the limit can be realized on the right hand of the previous formulas as an integral on  $\partial U$ . In general, this is not possible (see [18, Example 2.5]). However, in some cases, it is possible to represent the normal trace with a measure supported on  $\partial U$ . In order to see this, for  $\mathbf{F} \in \mathcal{DM}^p(\Omega)$ ,  $1 \leq p \leq \infty$ , and a bounded Borel set  $U \subset \Omega$ , we define the normal trace distribution of  $\mathbf{F}$  on  $\partial U$  as

$$\langle \boldsymbol{F} \cdot \boldsymbol{\nu}, \phi \rangle_{\partial U} := \int_{U} \phi \, \mathrm{d} di \boldsymbol{\nu} \boldsymbol{F} + \int_{U} \boldsymbol{F} \cdot \nabla \phi \, \mathrm{d} y \qquad \text{for any } \phi \in \mathrm{Lip}_{c}(\mathbb{R}^{n}).$$
(15)

The formula presented above shows that the trace distribution in  $\Omega$  can be extended to a functional in  $\{\phi \in C^0 \cap L^{\infty}(\Omega) : \nabla \phi \in L^{p'}(\Omega, \mathbb{R}^n)\}$  so that we can always represent the normal trace distribution as the limit of classical normal traces on smooth approximations of U. Then the question is whether there exists a Radon measure  $\sigma$  concentrated on  $\partial U$  such that  $\langle \mathbf{F} \cdot \nu, \phi \rangle_{\partial U} = \int_{\partial U} \phi \, d\sigma$ . Unfortunately, the answer is not affirmative, as indicated in Fig. 9.

Indeed, it can be shown (see [3, Theorem 4.6]) that  $\langle \boldsymbol{F} \cdot \boldsymbol{\nu}, \phi \rangle_{\partial U}$  can be represented as a measure if and only if  $\chi_U \boldsymbol{F} \in \mathcal{DM}^p(\Omega)$ . Moreover, if  $\langle \boldsymbol{F} \cdot \boldsymbol{\nu}, \cdot \rangle_{\partial U}$  is a measure, then

- (i) For  $p = \infty$ ,  $|\langle \boldsymbol{F} \cdot \boldsymbol{\nu}, \cdot \rangle_{\partial U}| \ll \mathscr{H}^{n-1} \sqcup \partial U$  (*i.e.*  $\mathscr{H}^{n-1}$  restricted to  $\partial U$ );
- (ii) For  $\frac{n}{n-1} \leq p < \infty$ ,

$$|\langle \mathbf{F} \cdot \nu, \cdot \rangle_{\partial U}|(B) = 0$$
 for any Borel set  $B \subset \partial U$  with  $\sigma$ -finite  $\mathscr{H}^{n-p'}$  measure.

This characterization can be used to find classes of vector fields for which the normal trace can be represented by a measure. An important observation is that, for a constant vector field  $\mathbf{F} \equiv \mathbf{v} \in \mathbb{R}^n$ ,

$$\langle \boldsymbol{v} \cdot \boldsymbol{\nu}, \cdot 
angle_{\partial U} = -\operatorname{div}(\chi_U \boldsymbol{v}) = -\sum_{j=1}^n v_j D_{y_j} \chi_U.$$

Thus, in order that  $\sum_{j=1}^{n} v_j D_{y_j} \chi_U$  is a measure, it is not necessary to assume that all the distributional derivatives of  $\chi_U$  are measures (*i.e.*  $\chi_U \in BV(\Omega)$ ), since cancellations could be possible so that the previous sum could still be a measure. Indeed, such an example has been constructed (see [3, Remark 4.14]) for a set  $U \subset \mathbb{R}^2$  without finite perimeter and a vector field  $\mathbf{F} \in \mathcal{DM}^p(\mathbb{R}^2)$  for any  $p \in [1, \infty]$  such that the normal trace of  $\mathbf{F}$  is a measure on  $\partial U$ .

In general, even for an open set with smooth boundary,  $\langle \boldsymbol{F} \cdot \boldsymbol{\nu}, \cdot \rangle_{\partial U} \neq \langle \boldsymbol{F} \cdot \boldsymbol{\nu}, \cdot \rangle_{\partial \overline{U}}$ , since the Radon measure  $div\boldsymbol{F}$  in (15) is *sensitive* to small sets and is not absolutely continuous with respect to the Lebesgue measure in general. Moreover, for  $p = \infty$ , the formulas in (10) show

$$\langle \boldsymbol{F} \cdot \boldsymbol{\nu}, \phi \rangle_{\partial U^1} = -\int_{\partial^* U} \mathfrak{F}_{\mathbf{i}} \cdot \boldsymbol{\nu}_U \phi \, \mathrm{d}\mathscr{H}^{n-1}, \quad \langle \boldsymbol{F} \cdot \boldsymbol{\nu}, \phi \rangle_{\partial (U^1 \cup \partial^* U)} = -\int_{\partial^* U} \mathfrak{F}_{\mathbf{e}} \cdot \boldsymbol{\nu}_U \phi \, \mathrm{d}\mathscr{H}^{n-1}.$$

### Cauchy Flux, Balance Laws, and $\mathcal{DM}$ Fields

In Continuum Physics, the fundamental *principle of balance law* can be stated in the most general terms (*cf.* Dafermos [11]):

A balance law in an open set  $\Omega$  of  $\mathbb{R}^n$  postulates that the production of a vector-valued extensive quantity in any bounded open subset  $U \subseteq \Omega$  is balanced by the Cauchy flux of this quantity through the boundary of U.

In smooth continuum media, the physical principle of balance law can be shown to be equivalent to a corresponding nonlinear system of PDEs – system of balance laws or conservation laws, for smooth solutions (e.g. Fig. 1). Unfortunately, solutions of such PDE systems generically contain discontinuities and singularities such as shock waves and focusing waves. The classical arguments for the equivalence derivation do not apply. To solve this longstanding fundamental problem, it requires the theory of  $\mathcal{DM}$  fields as discussed above. To achieve, we first have to extend the notions of the Cauchy flux and the production to accommodate the discontinuities and singularities in the continuum media.

A side surface in  $\Omega$  is a pair (S, U) so that  $S \subseteq \Omega$  is a Borel set and  $U \subseteq \Omega$  is an open set such that  $S \subset \partial U$ . The side surface (S, U) is often written as S for simplicity, when no confusion arises from the context.

**Definition 1** (Cauchy flux). Let  $\Omega$  be a bounded open set. A Cauchy flux is a functional  $\mathcal{F}$  defined on any side surface S := (S, U) such that the following properties hold:

- (i)  $\mathcal{F}(S_1 \cup S_2) = \mathcal{F}(S_1) + \mathcal{F}(S_2)$  for any pair of disjoint side surfaces  $S_1$  and  $S_2$  in  $\partial U$ , for some  $U \subseteq \Omega$ ;
- (ii) There exists a nonnegative Radon measure  $\sigma$  in  $\Omega$  such that

 $|\mathcal{F}(\partial U)| \leq \sigma(U)$  for every open set  $U \Subset \Omega$ ;

(iii) There exists a nonnegative Borel function  $h \in L^1_{loc}(\Omega)$  such that

$$|\mathcal{F}(S)| \leqslant \int_{S} h \, \mathrm{d}\mathcal{H}^{n-1}$$

for any side surface  $S \subset \partial U$  and any open set  $U \subseteq \Omega$  (the integral could be  $\infty$ , in which case the axiom is also true).

Like the Cauchy flux, we introduce

**Definition 2** (Production). A production is a functional  $\mathcal{P}$ , defined on any bounded open subset  $U \subset \Omega$ , taking value in  $\mathbb{R}^m$  and satisfying the conditions:

$$\mathcal{P}(U_1 \cup U_2) = \mathcal{P}(U_1) + \mathcal{P}(U_2) \qquad if U_1 \cap U_2 = \emptyset, \tag{16}$$

$$|\mathcal{P}(U)| \leqslant \sigma(U). \tag{17}$$

Then the *physical principle of balance law* can be mathematically formulated as

$$\mathcal{F}(\partial U) = \mathcal{P}(U)$$
 for any bounded open subset  $U \subset \Omega$ . (18)

Fuglede's theorem<sup>6</sup> indicates that conditions (16)–(17) imply that there is a production distribution  $P \in \mathcal{M}(\Omega; \mathbb{R}^k)$  such that

$$\mathcal{P}(U) = \int_{U} \mathrm{d}P(y). \tag{19}$$

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Based on the theory of  $\mathcal{DM}$  fields discussed above, the following statement has also been established:

**Theorem 6** (Chen-Comi-Torres [3]). Let  $\mathcal{F}$  be a Cauchy flux in  $\Omega$  with  $h \in L^1_{\text{loc}}(\Omega)$ . Then there exists a unique  $\mathbf{F} \in \mathcal{DM}^1_{\text{loc}}(\Omega)$  such that, for every open set  $U \subseteq \Omega$ , there exists an interior smooth approximation  $U^{\varepsilon}$  of U such that, for a suitable subsequence  $\varepsilon_k \to 0$  as  $k \to \infty$ ,

$$\mathcal{F}(\partial U) = -\lim_{\epsilon_k \to 0} \int_{\partial U^{\varepsilon_k}} \mathbf{F} \cdot \nu_{U^{\varepsilon_k}} \, \mathrm{d}\mathscr{H}^{n-1} = \lim_{\epsilon_k \to 0} \int_{U^{\varepsilon_k}} \mathrm{d}div\mathbf{F} = \int_U \mathrm{d}div\mathbf{F},\tag{20}$$

where  $\boldsymbol{F} \cdot \boldsymbol{\nu}_{U_{\varepsilon_{k}}}$  denotes the classical dot product.

Then (18)-(20) yields the following system of field equations

$$\operatorname{div} \boldsymbol{F}(y) = P(y) \tag{21}$$

in the sense of measures on  $\Omega$ .

We assume that the state of the medium is described by a state vector field  $\mathbf{u}$ , taking value in an open subset U of  $\mathbb{R}^m$ , which determines both the flux density field F and the production density field P at point  $y \in \Omega$  by the *constitutive equations*:

$$\boldsymbol{F}(y) := \boldsymbol{F}(\mathbf{u}(y), y), \quad P(y) := P(\mathbf{u}(y), y), \tag{22}$$

where  $F(\mathbf{u}, y)$  and  $P(\mathbf{u}, y)$  are given smooth functions defined on  $U \times \Omega$ .

Combining (21) with (22) leads to the first-order quasilinear system of PDEs:

$$\operatorname{div} \boldsymbol{F}(\mathbf{u}(y), y) = P(\mathbf{u}(y), y), \tag{23}$$

which is called a system of balance laws.

If  $\mathcal{P} = 0$ , the previous derivation yields

$$\operatorname{div} \boldsymbol{F}(\mathbf{u}(y), y) = 0, \tag{24}$$

which is called a system of conservation laws. When the medium is homogeneous:  $F(\mathbf{u}, y) = F(\mathbf{u})$ , *i.e.* F depends on y only through the state vector, then system (24) becomes

$$\operatorname{div} \boldsymbol{F}(\mathbf{u}(y)) = 0. \tag{25}$$

In particular, when the coordinate system y is described by the time variable t and the space variable  $x = (x_1, \dots, x_d)$ :

$$y = (t, x_1, \cdots, x_d) = (t, x), \quad n = d + 1,$$

and the flux density is written as

$$F(\mathbf{u}) = (\mathbf{u}, \mathbf{f}_1(\mathbf{u}), \cdots, \mathbf{f}_d(\mathbf{u})) = (\mathbf{u}, \mathbf{f}(\mathbf{u})),$$

then we obtain the standard form (5) for systems of conservation laws.

<sup>&</sup>lt;sup>6</sup>Fuglede, B.: On a theorem of F. Riesz. Math. Scand. 3: 283–302, 1955.

#### Entropy Solutions, Hyperbolic Conservation Laws, and $\mathcal{DM}$ Fields

One of the main issues in the theory of hyperbolic conservation laws (5) is to study the behavior of entropy solutions determined by the Lax entropy inequality (7) to explore to the fullest extent possible all questions relating to large-time behavior, uniqueness, stability, structure, and traces of entropy solutions, with neither specific reference to any particular method for constructing the solutions nor additional regularity assumptions.

First, based on the  $\mathcal{DM}$  theory presented earlier, the Cauchy entropy fluxes can be recovered through the Lax entropy inequality for entropy solutions of hyperbolic conservation laws by capturing entropy dissipation. In particular, for any  $L^{\infty}$  entropy solution **u**, we can introduce a functional on any side surface S:

$$\mathcal{F}_{\eta}(S) = \int_{S} (\eta(\mathbf{u}), \mathbf{q}(\mathbf{u})) \cdot \nu \, d\mathcal{H}^{d}, \qquad (26)$$

where  $(\eta(\mathbf{u}), \mathbf{q}(\mathbf{u})) \cdot \nu$  is the normal trace defined earlier, since  $(\eta(\mathbf{u}), \mathbf{q}(\mathbf{u})) \in \mathcal{DM}_{loc}^{\infty}(\mathbb{R}^{n}_{+})$ . It is easy to check that the functional  $\mathcal{F}_{\eta}$  defined by (26) is a Cauchy flux, which is called a Cauchy entropy flux with respect to the entropy  $\eta$ . In particular, when  $\eta$  is convex,

 $\mathcal{F}_{\eta}(S) \ge 0$  on any side surface S.

Furthermore, we can reformulate the balance law of entropy from the recovery of an entropy production by capturing entropy dissipation.

Moreover, it is clear that understanding more properties of  $\mathcal{DM}$  fields can advance our understanding of the behavior of entropy solutions for hyperbolic conservation laws and other related nonlinear equations by selecting appropriate entropy pairs. Successful examples include the stability of Riemann solutions, which may contain rarefaction waves, contact discontinuities, and/or vacuum states, in the class of entropy solutions of the Euler equations for gas dynamics; the decay of periodic entropy solutions; the initial and boundary layer problems; the initial-boundary value problems; and the structure of entropy solutions of nonlinear hyperbolic conservation laws. See [2, 4, 5, 7, 8, 19] and the references therein.

Further connections and applications of  $\mathcal{DM}$  fields include

- The solvability of the vector field  $\mathbf{F}$  for the equation:  $div\mathbf{F} = \mu$  for given  $\mu$ . See [15] and the references therein.
- Image processing via the dual of BV. See [14, 15] and the references therein.

The  $\mathcal{DM}$  theory is useful for the developments of new techniques and tools for entropy methods, measure-theoretic analysis, partial differential equations, free boundary problems, calculus of variations, and related areas, which involve the solutions with discontinuities, singularities, among others.

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