ON THE DUAL OF BV

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ABSTRACT. An important problem in geometric measure theory is the characterization of BV^*, the dual of the space of functions of bounded variation. In this paper we survey recent results that show that the solvability of the equation $\operatorname{div} \boldsymbol{F} = T$ is closely connected to the problem of characterizing BV^*. In particular, the (signed) measures in BV^* can be characterized in terms of the solvability of the equation $\operatorname{div} \boldsymbol{F} = T$.

1. INTRODUCTION

It is an open problem in geometric measure theory to give a full characterization of BV^*, the dual of the space of functions of bounded variation (see Ambrosio-Fusco-Pallara [3, Remark 3.12]). Meyers and Ziemer characterized in [21] the positive measures in $\mathbb{R}^n$ that belong to the dual of $BV(\mathbb{R}^n)$, where $BV(\mathbb{R}^n)$ is the space of all functions in $L^1(\mathbb{R}^n)$ whose distributional gradient is a vector-measure in $\mathbb{R}^n$ with finite total variation. They showed that the positive measure μ belongs to $BV(\mathbb{R}^n)^*$ if and only if μ satisfies the condition

$$\mu(B(x,r)) \leq C r^{n-1}$$

for every open ball $B(x,r) \subset \mathbb{R}^n$ and $C = C(n)$. Besides the classical paper by Meyers and Ziemer, we refer the interested reader to the paper by De Pauw [11], where the author analyzes SBV^*, the dual of the space of special functions of bounded variation.

In Phuc-Torres [22] we showed that there is a connection between the problem of characterizing BV^* and the solvability of the equation $\operatorname{div} \boldsymbol{F} = T$. Indeed, we showed that the (signed) measure μ belongs to $BV(\mathbb{R}^n)^*$ if and only if there exists a bounded vector field $\boldsymbol{F} \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$ such that $\operatorname{div} \boldsymbol{F} = \mu$. Also, we showed that μ belongs to $BV(\mathbb{R}^n)^*$ if and only if

$$(1.1) \qquad\qquad |\mu(U)| \leq C \, \mathcal{H}^{n-1}(\partial U)$$

for any open (or closed) set $U \subset \mathbb{R}^n$ with smooth boundary.

In De Pauw-Torres [13], another BV-type space was considered, the space $BV_{\frac{n}{n-1}}(\mathbb{R}^n)$, defined as the space of all functions $u \in L^{\frac{n}{n-1}}(\mathbb{R}^n)$ such that Du, the distributional gradient of u, is a vector-measure in $\mathbb{R}^n$ with finite total variation. A closed subspace of $BV_{\frac{n}{n-1}}(\mathbb{R}^n)^*$, which is a Banach space denoted as CH_0, was characterized in [13] and it was proven that $T \in CH_0$ if and only if $T = \operatorname{div} \boldsymbol{F}$, for a continuous vector field $\boldsymbol{F}$ vanishing at infinity.

In Phuc-Torres [23], the analysis of $BV(\mathbb{R}^n)^*$ and $BV_{\frac{n}{n-1}}(\mathbb{R}^n)^*$ was continued. It was shown that $BV(\mathbb{R}^n)^*$ and $BV_{\frac{n}{n-1}}(\mathbb{R}^n)^*$ are isometrically isomorphic. It was also shown that the measures in $BV_{\frac{n}{n-1}}(\mathbb{R}^n)^*$ coincide with the measures in $\dot{W}^{1,1}(\mathbb{R}^n)^*$, the dual of the homogeneous Sobolev space $\dot{W}^{1,1}(\mathbb{R}^n)$, in the sense of isometric isomorphism. We remark that the space $\dot{W}^{1,1}(\mathbb{R}^n)^*$ is denoted as the G space in image processing (see Meyer [20]), and that it plays a key role in modeling the noise of an image. Thus, the results in [23] provide a formula to compute the G-norm of a measure (see Remark 3.7) that is more suitable for computations.

It is obvious that if μ is a locally finite signed Radon measure then $\|\mu\| \in BV(\mathbb{R}^n)^*$ implies that $\mu \in BV(\mathbb{R}^n)^*$. The converse was unknown to Meyers and Ziemer as they raised this issue in their

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classical paper [21, page 1356]. In [23], we showed that the converse does not hold true in general by constructing a locally integrable function f such that $f \in BV(\mathbb{R}^n)^*$ but $|f| \notin BV(\mathbb{R}^n)^*$.

Given a bounded open set Ω with Lipschitz boundary, $BV_0(\Omega)$ is the space of functions of bounded variation with zero trace on $\partial\Omega$. All (signed) measures in Ω that belong to $BV_0(\Omega)^*$ were characterized in [23]. It was shown that a locally finite signed measure μ belongs to $BV_0(\Omega)^*$ if and only if (1.1) holds for any smooth open (or closed) set $U \subset\subset \Omega$, and if and only if $\mu = div\,\boldsymbol{F}$ for a vector field $\boldsymbol{F} \in L^\infty(\Omega, \mathbb{R}^n)$. Moreover, the measures in $BV_0(\Omega)^*$ coincide with the measures in $W_0^{1,1}(\Omega)^*$, in the sense of isometric isomorphism.

In the case of $BV(\Omega)$, the space of functions of bounded variation in a bounded open set Ω with Lipschitz boundary (but without the condition of having zero trace on $\partial\Omega$), we shall restrict our attention only to measures in $BV(\Omega)^*$ with bounded total variation in Ω, i.e., finite measures. This is in a sense natural since any *positive* measure that belongs to $BV(\Omega)^*$ must be finite due to the fact that the function $1 \in BV(\Omega)$. We showed in [23] that a finite measure μ belongs to $BV(\Omega)^*$ if and only if (1.1) holds for every smooth open set $U \subset\subset \mathbb{R}^n$, where μ is extended by zero to $\mathbb{R}^n \setminus \Omega$.

The solvability of the equation $div\,\boldsymbol{F} = T$, in various spaces of functions, has been studied in Bourgain-Brezis [5], De Pauw-Pfeffer [12], De Pauw-Torres [13], Phuc-Torres [22] (see also Tadmor [26], Bousquet-Mironescu-Russ [6], Russ [25] and the references therein). A first attempt to solve the equation $div\,\boldsymbol{F} = T$ can be made by solving the Laplace's equation. Thus, we can start by setting $\boldsymbol{F} = \nabla u$ and solving $\Delta u = f \in L^p(U)$, $1 < p < \infty$. In this case, there exists a solution $u \in W^{2,p}(U)$ and hence $\boldsymbol{F} \in W^{1,p}(U)$. However, the limiting cases $p = 1$ and $p = \infty$ can not be solved in general. Indeed, McMullen [19] and Preiss [24] have shown that there exist functions $f \in L^\infty$ (even continuous f) such that the equation $div\,\boldsymbol{F} = f$ has no Lipschitz solution (see also Dacorogna-Fusco-Tartar [8] and Bourgain-Brezis [5, Section 2.2]). Analogously, there exist functions $f \in L^1$ such that the equation $div\,\boldsymbol{F} = f$ has no solution in BV and not even in $L^{n/n-1}$ (see Bourgain-Brezis [5, Section 2.1]).

We recall that if u solves $\Delta u = f \in L^p(U)$ and $n < p < \infty$ then $\nabla u \in C^{0,1-\frac{n}{p}}(U)$. From here we see that this regularity fails for the limiting case $p = \infty$, that would correspond to having a Lipschitz solution, and for the case $p = n$, that would correspond to having a continuous solution. For the case $p = n$, since the equation $\Delta u = f \in L^n$ admits a solution $u \in W^{2,n}$, it follows that $div\,\boldsymbol{F} = f \in L^n$ has a solution $\boldsymbol{F} \in W^{1,n}$. However, $W^{1,n}$ is not contained in L^∞ (since this is a limiting case in the Sobolev embedding theorem) and hence one can not conclude that $\boldsymbol{F} \in L^\infty$. The previous discussion suggests to study the limiting case $f \in L^n$ and to research the existence of vector fields $\boldsymbol{F} \in L^\infty$ or $\boldsymbol{F}$ continuous that solve equation $div\,\boldsymbol{F} = f \in L^n$. Nirenberg provided the example $u(x) = \varphi(x)x_1 |\log\|x\||^\alpha$, $0 < \alpha < \frac{n-1}{n}$, where φ is smooth supported near 0, which shows that solving the equation for the case $p = n$ requires methods other than the Laplace's equation. Indeed, one can check for this particular u that $\Delta u \in L^n(\mathbb{R}^n)$ but $\nabla u \notin L^\infty_{loc}(\mathbb{R}^n)$.

The work by Bourgain-Brezis [5] initiated the analysis of the limiting case $p = n$. The equation $div\,\boldsymbol{F} = f \in L^n_{\#}([0,1]^n)$ was solved in [5] in the class $L^\infty([0,1]^n, \mathbb{R}^n) \cap W^{1,n}([0,1]^n; \mathbb{R}^n)$. The solvability of $div\,\boldsymbol{F} = T$ in the space of continuous vector fields $C(U, \mathbb{R}^n)$ was studied by De Pauw-Pfeffer [12]. They showed that the equation has a continuous solution if and only if T belongs to a class of distributions, the space of strong charges, denoted as $CH_n(U)$. Moreover, $L^n(U) \subset CH_n(U)$, solving in particular the critical case $div\,\boldsymbol{F} = f \in L^n$. The case $p = n$ was also addressed in De Pauw-Torres [13] and it is discussed in Section 5 of this survey paper.

The organization of this paper is as follows. In Section 2 we introduce the space of functions of bounded variation BV. In Section 3 we present the characterization of all (signed) measures in $BV_{\frac{n}{n-1}}(\mathbb{R}^n)^*$. In Section 4 we discuss a counterexample that resolves a question raised by Meyers and Ziemer in [21, page 1356]. In Section 5 we characterize a subspace of $BV_{\frac{n}{n-1}}(\mathbb{R}^n)^*$ in terms of the solvability of $div\,\boldsymbol{F} = T$, in the class of continuous vector fields vanishing at infinity. Finally, in Section 6 we discuss the characterization of all (signed) measures in $BV_0(\Omega)^*$, and all finite measures in the space $BV(\Omega)^*$.

2. Functions of bounded variation

In this section we define all the spaces that will be relevant in this paper.

Definition 2.1. Let Ω be any open set. The space $\mathcal{M}(\Omega)$ consists of all finite (signed) Radon measures μ in Ω; that is, the total variation of μ, denoted as $\|\mu\|$, satisfies $\|\mu\|(\Omega) < \infty$. The space $\mathcal{M}_{loc}(\Omega)$ consists of all locally finite Radon measures μ in Ω; that is, $\|\mu\|(K) < \infty$ for every compact set $K \subset \Omega$.

Note here that $\mathcal{M}_{loc}(\Omega)$ is identified with the dual of the locally convex space $C_c(\Omega)$ (the space of continuous real-valued functions with compact support in Ω) (see [9]), and thus it is a real vector space. For $\mu \in \mathcal{M}_{loc}(\Omega)$, it is not required that either the positive part or the negative part of μ has finite total variation in Ω.

In the next definition by a *vector-valued measure* we mean a Radon measure that takes values in $\mathbb{R}^n$.

Definition 2.2. Let Ω be any open set. The space of functions of bounded variation, denoted as $BV(\Omega)$, is defined as the space of all functions $u \in L^1(\Omega)$ such that the distributional gradient Du is a finite vector-valued measure in Ω. For $\Omega \neq \mathbb{R}^n$, we equip $BV(\Omega)$ with the norm

$$(2.1) \qquad \|u\|_{BV(\Omega)} = \|u\|_{L^1(\Omega)} + \|Du\|(\Omega),$$

where $\|Du\|(\Omega)$ denotes the total variation of the vector-valued measure Du over Ω. For $\Omega = \mathbb{R}^n$, following Meyers-Ziemer [21], we will instead equip $BV(\mathbb{R}^n)$ with the homogeneous norm given by

$$(2.2) \qquad \|u\|_{BV(\mathbb{R}^n)} = \|Du\|(\mathbb{R}^n).$$

Another BV-like space is $BV_{\frac{n}{n-1}}(\mathbb{R}^n)$, defined as the space of all functions in $L^{\frac{n}{n-1}}(\mathbb{R}^n)$ such that Du is a finite vector-valued measure. The space $BV_{\frac{n}{n-1}}(\mathbb{R}^n)$ is a Banach space when equipped with the norm

$$\|u\|_{BV_{\frac{n}{n-1}}(\mathbb{R}^n)} = \|Du\|(\mathbb{R}^n).$$

Remark 2.3. By definition $BV(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$ and thus it is a normed space under the norm (2.2). However, $BV(\mathbb{R}^n)$ is not complete under this norm. Also, we have

$$\|Du\|(\Omega) = \sup\left\{\int_{\Omega} u \, div \, \varphi dx : \varphi \in C_c^1(\Omega) \text{ and } |\varphi(x)| \leq 1 \, \forall x \in \Omega\right\},$$

where $\varphi = (\varphi_1, \varphi_2, ..., \varphi_n)$ and $|\varphi(x)| = (\varphi_1(x)^2 + \varphi_2(x)^2 + \cdots + \varphi_n(x)^2)^{1/2}$. In what follows, we shall also write $\int_{\Omega} |Du|$ instead of $\|Du\|(\Omega)$.

We recall the following Sobolev's inequality for functions in $BV(\mathbb{R}^n)$ whose proof can be found in [3, Theorem 3.47]:

$$(2.3) \qquad \|u\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leq C(n) \|Du\|(\mathbb{R}^n), \quad u \in BV(\mathbb{R}^n).$$

Inequality (2.3) immediately implies the following continuous embedding

$$(2.4) \qquad BV(\mathbb{R}^n) \hookrightarrow BV_{\frac{n}{n-1}}(\mathbb{R}^n).$$

We recall that the standard Sobolev space $W^{1,1}(\Omega)$ is defined as the space of all functions $u \in L^1(\Omega)$ such that $Du \in L^1(\Omega)$. The Sobolev space $W^{1,1}(\Omega)$ is a Banach space with the norm

$$(2.5) \quad \|u\|_{W^{1,1}(\Omega)} = \|u\|_{L^1(\Omega)} + \|Du\|_{L^1(\Omega)} = \int_{\Omega} \left[|u| + (|D_1 u|^2 + |D_2 u|^2 + \cdots + |D_n u|^2)^{\frac{1}{2}}\right] dx.$$

Hereafter, we let $C_c^\infty(\Omega)$ denote the space of smooth functions with compact support in a general open set Ω. We will often refer to the following homogeneous Sobolev space.

Definition 2.4. Let $\dot{W}^{1,1}(\mathbb{R}^n)$ denote the space of all functions $u \in L^{\frac{n}{n-1}}(\mathbb{R}^n)$ such that $Du \in L^1(\mathbb{R}^n)$. Equivalently, the space $\dot{W}^{1,1}(\mathbb{R}^n)$ can also be defined as the closure of $C_c^\infty(\mathbb{R}^n)$ in $BV_{\frac{n}{n-1}}(\mathbb{R}^n)$ (i.e., in the norm $\|Du\|_{L^1(\mathbb{R}^n)}$). Thus, $u \in \dot{W}^{1,1}(\mathbb{R}^n)$ if and only if there exists a sequence $u_k \in C_c^\infty(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} |D(u_k - u)| dx = 0$, and moreover,

$$\dot{W}^{1,1}(\mathbb{R}^n) \hookrightarrow BV_{\frac{n}{n-1}}(\mathbb{R}^n).$$

We recall that the open bounded set Ω has Lipschitz boundary if for each $x \in \partial\Omega$, there exist $r > 0$ and a Lipschitz mapping $h : \mathbb{R}^{n-1} \to \mathbb{R}$ such that, upon rotating and relabeling the coordinate axes if necessary, we have

$$\Omega \cap B(x, r) = \{y = (y_1, \ldots, y_{n-1}, y_n) : h(y_1, \ldots, y_{n-1}) < y_n\} \cap B(x, r).$$

Remark 2.5. Let Ω be a bounded open set with Lipschitz boundary. We denote by $W_0^{1,1}(\Omega)$ the Sobolev space consisting of all functions in $W^{1,1}(\Omega)$ with zero trace on $\partial\Omega$. Then it is well-known that $C_c^\infty(\Omega)$ is dense in $W_0^{1,1}(\Omega)$. We will explain in Section 6 the precise definition of $BV_0(\Omega)$, the space of all functions in $BV(\Omega)$ with zero trace on $\partial\Omega$ (see (6.1)). In this paper we equip the two spaces, $BV_0(\Omega)$ and $W_0^{1,1}(\Omega)$, with the equivalent norms (see (6.2)) to (2.1) and (2.5), respectively, given by

$$\|u\|_{BV_0(\Omega)} = \|Du\|(\Omega), \quad \text{and} \quad \|u\|_{W_0^{1,1}(\Omega)} = \int_\Omega |Du| dx.$$

Definition 2.6. For any open set Ω, we let $BV_c(\Omega)$ denote the space of functions in $BV(\Omega)$ with compact support in Ω. Also, $BV^\infty(\Omega)$ and $BV_0^\infty(\Omega)$ denote the space of bounded functions in $BV(\Omega)$ and $BV_0(\Omega)$, respectively. Finally, $BV_c^\infty(\Omega)$ is the space of all bounded functions in $BV(\Omega)$ with compact support in Ω.

3. CHARACTERIZATION OF (SIGNED) MEASURES IN $BV_{\frac{n}{n-1}}(\mathbb{R}^n)^*$

The fundamental step in the characterization of the (signed) measures in $BV_{\frac{n}{n-1}}(\mathbb{R}^n)^*$ is the fact that $BV_c^\infty(\mathbb{R}^n)$ is dense in $BV_{\frac{n}{n-1}}(\mathbb{R}^n)$. Indeed, we have the following ([23, Theorem 3.1]:

Theorem 3.1. *Let $u \in BV_{\frac{n}{n-1}}(\mathbb{R}^n)$, $u \geq 0$, and $\phi_k \in C_c^\infty(\mathbb{R}^n)$ be a nondecreasing sequence of smooth functions satisfying:*

(3.1) $$0 \leq \phi_k \leq 1, \ \phi_k \equiv 1 \text{ on } B_k(0), \ \phi_k \equiv 0 \text{ on } \mathbb{R}^n \setminus B_{2k}(0) \text{ and } |D\phi_k| \leq c/k.$$

Then

(3.2) $$\lim_{k\to\infty} \|(\phi_k u) - u\|_{BV_{\frac{n}{n-1}}(\mathbb{R}^n)} = 0,$$

and for each fixed $k > 0$ we have

(3.3) $$\lim_{j\to\infty} \|(\phi_k u) \wedge j - \phi_k u\|_{BV_{\frac{n}{n-1}}(\mathbb{R}^n)} = 0.$$

In particular, $BV_c^\infty(\mathbb{R}^n)$ is dense in $BV_{\frac{n}{n-1}}(\mathbb{R}^n)$.

With this density result we can show that:

(3.4) $$BV(\mathbb{R}^n)^* \text{ and } BV_{\frac{n}{n-1}}(\mathbb{R}^n)^* \text{ are isometrically isomorphic.}$$

Indeed, we define the map

$$S : BV_{\frac{n}{n-1}}(\mathbb{R}^n)^* \to BV(\mathbb{R}^n)^*$$

as

$$S(T) = T \llcorner BV(\mathbb{R}^n).$$

Clearly (see [23, Corollary 3.3]), Theorem 3.1 implies that S is an isometry. We now proceed to make precise our definitions of measures in $\dot{W}^{1,1}(\mathbb{R}^n)^*$ and $BV_{\frac{n}{n-1}}(\mathbb{R}^n)^*$.

Definition 3.2. We let

$$\mathcal{M}_{loc}(\mathbb{R}^n) \cap \dot{W}^{1,1}(\mathbb{R}^n)^* := \{T \in \dot{W}^{1,1}(\mathbb{R}^n)^* : T(\varphi) = \int_{\mathbb{R}^n} \varphi d\mu \text{ for some } \mu \in \mathcal{M}_{loc}(\mathbb{R}^n), \forall \varphi \in C_c^\infty(\mathbb{R}^n)\}.$$

Therefore, if $\mu \in \mathcal{M}_{loc}(\mathbb{R}^n) \cap \dot{W}^{1,1}(\mathbb{R}^n)^*$, then the action $\langle \mu, u \rangle$ can be uniquely defined for all $u \in \dot{W}^{1,1}(\mathbb{R}^n)$.

Definition 3.3. We let

$$\mathcal{M}_{loc} \cap BV_{\frac{n}{n-1}}(\mathbb{R}^n)^* := \{T \in BV_{\frac{n}{n-1}}(\mathbb{R}^n)^* : T(\varphi) = \int_{\mathbb{R}^n} \varphi^* d\mu \text{ for some } \mu \in \mathcal{M}_{loc}, \forall \varphi \in BV_c^\infty(\mathbb{R}^n)\},$$

where φ^* is the precise representative of φ in $BV_c^\infty(\mathbb{R}^n)$ (see [3, Corollary 3.8] for the definition of precise representative). Thus, if $\mu \in \mathcal{M}_{loc} \cap BV_{\frac{n}{n-1}}(\mathbb{R}^n)^*$, then the action $\langle \mu, u \rangle$ can be uniquely defined for all $u \in BV_{\frac{n}{n-1}}(\mathbb{R}^n)$.

In Definition 3.3, if we use $C_c^\infty(\mathbb{R}^n)$ instead of $BV_c^\infty(\mathbb{R}^n)$, then by the Hahn-Banach Theorem there exists a non-zero $T \in BV_{\frac{n}{n-1}}(\mathbb{R}^n)^*$ that is represented by the zero measure, which would cause a problem of injectivity in Theorem 3.6.

The following lemma (see [23, Lemma 4.1]) characterizes all the distributions in $\dot{W}^{1,1}(\mathbb{R}^n)^*$ and is a key ingredient in the characterization of the measures in $BV_{\frac{n}{n-1}}(\mathbb{R}^n)^*$. We recall that $\dot{W}^{1,1}(\mathbb{R}^n)$ is the homogeneous Sobolev space introduced in Definition 2.4 and that it is also known as the G space in image processing.

Lemma 3.4. *The distribution T belongs to $\dot{W}^{1,1}(\mathbb{R}^n)^*$ if and only if $T = \operatorname{div} \boldsymbol{F}$ for some vector field $\boldsymbol{F} \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$. Moreover,*

$$\|T\|_{\dot{W}^{1,1}(\mathbb{R}^n)^*} = \min\{\|\boldsymbol{F}\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^n)}\},$$

where the minimum is taken over all $\boldsymbol{F} \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$ such that $\operatorname{div} \boldsymbol{F} = T$. Here we use the norm

$$\|\boldsymbol{F}\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^n)} := \left\|(F_1^2 + F_2^2 + \cdots + F_n^2)^{1/2}\right\|_{L^\infty(\mathbb{R}^n)} \text{ for } \boldsymbol{F} = (F_1, \ldots, F_n).$$

The following theorem characterizes all the signed measures in $BV_{\frac{n}{n-1}}(\mathbb{R}^n)^*$ (see [23, Theorem 4.4]. This result was first proven in Phuc-Torres [22] for the space $BV(\mathbb{R}^n)^*$ with no sharp control on the involving constants. In Phuc-Torres [23] we offered a new and direct proof of (**i**) $\Rightarrow$ (**ii**). We also clarified the first part of (**iii**). The second part of (**iii**) relies on a delicate argument using the coarea formula. Moreover, our proof of (**ii**) $\Rightarrow$ (**iii**) yields a sharp constant that will be needed for the proof of Theorem 3.6 below. The density obtained in Theorem 3.1 allows to show (**iii**) $\Rightarrow$ (**iv**). Indeed, we note that the main idea in the proof of Theorem 3.5 is to show first that, if $div\,\boldsymbol{F} = \mu$, then the action of the measure μ can be defined for any function $u \in BV_c^\infty(\mathbb{R}^n)$ and thus, since $BV_c^\infty(\mathbb{R}^n)$ is dense in $BV_{\frac{n}{n-1}}(\mathbb{R}^n)$, the action of μ can be extended to all $BV_{\frac{n}{n-1}}(\mathbb{R}^n)$, which shows that $\mu \in BV_{\frac{n}{n-1}}(\mathbb{R}^n)^*$. Finally, Lemma 3.4 yields (**iv**) $\Rightarrow$ (**i**).

Theorem 3.5. *Let $\mu \in \mathcal{M}_{loc}(\mathbb{R}^n)$ be a locally finite signed measure. The following are equivalent:*

(**i**) *There exists a vector field $\boldsymbol{F} \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$ such that $\operatorname{div} \boldsymbol{F} = \mu$ in the sense of distributions.*

(**ii**) *There is a constant C such that*

$$|\mu(U)| \leq C \mathcal{H}^{n-1}(\partial U)$$

for any smooth bounded open (or closed) set U with $\mathcal{H}^{n-1}(\partial U) < +\infty$.

(**iii**) *$\mathcal{H}^{n-1}(A) = 0$ implies $\|\mu\|(A) = 0$ for all Borel sets A and there is a constant C such that, for all $u \in BV_c^\infty(\mathbb{R}^n)$,*

$$|\langle \mu, u \rangle| := \left| \int_{\mathbb{R}^n} u^* d\mu \right| \leq C \int_{\mathbb{R}^n} |Du|,$$

where u^* is the precise representative in the class of u that is defined $\mathcal{H}^{n-1}$-almost everywhere.

(iv) $\mu \in BV_{\frac{n}{n-1}}(\mathbb{R}^n)^*$. *The action of μ on any $u \in BV_{\frac{n}{n-1}}(\mathbb{R}^n)$ is defined (uniquely) as*

$$\langle \mu, u \rangle := \lim_{k \to \infty} \langle \mu, u_k \rangle = \lim_{k \to \infty} \int_{\mathbb{R}^n} u_k^* d\mu,$$

where $u_k \in BV_c^\infty(\mathbb{R}^n)$ converges to u in $BV_{\frac{n}{n-1}}(\mathbb{R}^n)$. In particular, if $u \in BV_c^\infty(\mathbb{R}^n)$ then

$$\langle \mu, u \rangle = \int_{\mathbb{R}^n} u^* d\mu,$$

and moreover, if μ is a non-negative measure then, for all $u \in BV_{\frac{n}{n-1}}(\mathbb{R}^n)$,

$$\langle \mu, u \rangle = \int_{\mathbb{R}^n} u^* d\mu.$$

We recall the spaces introduced in Definitions 3.2 and 3.3. With the previous theorem, we can prove the following new result (see [23, Theorem 4.7]).

Theorem 3.6. *Let $\mathcal{E} := \mathcal{M}_{loc}(\mathbb{R}^n) \cap BV_{\frac{n}{n-1}}(\mathbb{R}^n)^*$ and $\mathcal{F} := \mathcal{M}_{loc}(\mathbb{R}^n) \cap \dot{W}^{1,1}(\mathbb{R}^n)^*$. Then $\mathcal{E}$ and $\mathcal{F}$ are isometrically isomorphic.*

In the following remark we discuss the connection between Theorem 3.5, Theorem 3.6 and image processing.

Remark 3.7. The space $\dot{W}^{1,1}(\mathbb{R}^n)^*$ is denoted as the G space in image processing (see Meyer [20]), and it plays a key role in modeling the noise of an image. It is mentioned in [20] that it is more convenient to work with G instead of $BV_{\frac{n}{n-1}}(\mathbb{R}^n)^*$. Indeed, except for the characterization of the (signed) measures treated in this paper and the results in De Pauw-Torres [11], the full characterization of $BV_{\frac{n}{n-1}}(\mathbb{R}^n)^*$ is unknown. However, G can be easily characterized; see Lemma 3.4. Our previous results Theorem 3.5 and Theorem 3.6 show that, when restricted to measures, both spaces coincide. Moreover, the norm of any (signed) measure $\mu \in G$ can be computed as

$$(3.5) \qquad \qquad \|\mu\|_G = \sup \frac{|\mu(U)|}{\mathcal{H}^{n-1}(\partial U)},$$

where the sup is taken over all open sets $U \subset \mathbb{R}^n$ with smooth boundary and $\mathcal{H}^{n-1}(\partial U) < +\infty$. Hence, our results give an alternative to the more abstract computation of $\|\mu\|_G$ given, by Lemma 3.4, as

$$\|\mu\|_G = \min\{\|\boldsymbol{F}\|_{L^\infty(\mathbb{R}^n, \mathbb{R}^n)}\},$$

where the minimum is taken over all $\boldsymbol{F} \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$ such that $\operatorname{div} \boldsymbol{F} = T$. We refer the reader to Kindermann-Osher-Xu [17] for an algorithm based on the level set method to compute (3.5) for the case when μ is a function $f \in L^2(\mathbb{R}^2)$ with zero mean. Also, in the two-dimensional case, when μ is a function $f \in L^2(\mathbb{R}^2)$, the isometry of measures in Theorem 3.6 could be deduced from [16, Lemma 3.1].

4. On an issue raised by Meyers and Ziemer

Using Theorem 3.5, we constructed in [23, Section 5] a locally integrable function f such that $f \in BV(\mathbb{R}^n)^*$ but $|f| \notin BV(\mathbb{R}^n)^*$. We mention that this kind of highly oscillatory function appeared in [18] in a different context. It is clear that if μ is a locally finite signed Radon measure then $\|\mu\| \in BV(\mathbb{R}^n)^*$ implies that $\mu \in BV(\mathbb{R}^n)^*$. The converse was unknown to Meyers and Ziemer as they raised this issue in their classical paper [21, page 1356]. The following Proposition shows that the converse does not hold true in general.

Proposition 4.1. *Let $f(x) = \epsilon|x|^{-1-\epsilon} \sin(|x|^{-\epsilon}) + (n-1)|x|^{-1} \cos(|x|^{-\epsilon})$, where $0 < \epsilon < n-1$ is fixed. Then*

$$(4.1) \qquad \qquad f(x) = \operatorname{div}[x|x|^{-1} \cos(|x|^{-\epsilon})].$$

Moreover, there exists a sequence $\{r_k\}$ decreasing to zero such that

$$(4.2) \qquad \int_{B(0,r_k)} f^+(x)dx \geq c\, r_k^{n-1-\epsilon}$$

for a constant $c = c(n,\epsilon) > 0$ independent of k. Here f^+ is the positive part of f. Thus by Theorem 3.5 we see that f belongs to $BV(\mathbb{R}^n)^*$, whereas $|f|$ does not.

5. Characterization of a subspace of $BV_{\frac{n}{n-1}}(\mathbb{R}^n)^*$ whose elements are the divergence of continuous vector fields vanishing at infinity

In this section we characterize a subspace of $BV_{\frac{n}{n-1}}(\mathbb{R}^n)^*$, denoted as $CH_0(\mathbb{R}^n)$, and consisting of all *charges vanishing at infinity*. We start by introducing a notion of convergence in $BV_{\frac{n}{n-1}}(\mathbb{R}^n)$ and the definition of *charge vanishing at infinity*.

Definition 5.1. Given a sequence $\{u_j\}$ in $BV_{\frac{n}{n-1}}(\mathbb{R}^n)$, we write $u_j \twoheadrightarrow 0$ whenever

(1) $\sup_j \|Du_j\|(\mathbb{R}^n) < \infty$;
(2) $u_j \rightharpoonup 0$ weakly in $L^{\frac{n}{n-1}}(\mathbb{R}^n)$.

A *charge vanishing at infinity* is a linear functional $T : BV_{\frac{n}{n-1}}(\mathbb{R}^n) \to \mathbb{R}$ such that $\langle u_j, T \rangle \to 0$ whenever $u_j \twoheadrightarrow 0$. The set of charges vanishing at infinity is denoted as $CH_0(\mathbb{R}^n)$.

We showed in [13] that the equation $div\, \boldsymbol{F} = T$ has a solution $\boldsymbol{F} \in C_0(\mathbb{R}^n, \mathbb{R}^n)$ if and only if $T \in CH_0(\mathbb{R}^n)$, where $C_0(\mathbb{R}^n, \mathbb{R}^n)$ is the space of continuous vector fields vanishing at infinity. We recall that $\boldsymbol{F} \in C_0(\mathbb{R}^n, \mathbb{R}^n)$ if and only if, for every $\varepsilon > 0$, there exists a compact set $K \subset \mathbb{R}^n$ such that $|\boldsymbol{F}(x)| \leq \varepsilon$ whenever $x \in \mathbb{R}^n \setminus K$. The terminology *charge vanishing at infinity* was motivated by the fact that, given $T \in CH_0(\mathbb{R}^n)$ and $\varepsilon > 0$, there exists a compact set $K \subset \mathbb{R}^n$ such that

$$|\langle u, T \rangle| \leq \varepsilon \|Du\|(\mathbb{R}^n),$$

whenever $u \in BV_{\frac{n}{n-1}}(\mathbb{R}^n)$ and $K \cap \operatorname{supp} u = \emptyset$.

A compactness property can be proven in $BV_{\frac{n}{n-1}}(\mathbb{R}^n)$ (see [13, Proposition 2.6]) which states that if $\{u_j\}$ is a bounded sequence in $BV_{\frac{n}{n-1}}(\mathbb{R}^n)$, i.e. $\sup_j \|Du_j\|(\mathbb{R}^n) < \infty$, then there exists a subsequence $\{u_{j_k}\}$ of $\{u_j\}$ and $u \in BV_{\frac{n}{n-1}}(\mathbb{R}^n)$ such that $u_{j_k} \twoheadrightarrow u$. The compactness in $BV_{\frac{n}{n-1}}(\mathbb{R}^n)$ implies that

$$(5.1) \qquad \|T\|_{CH_0} := \sup\{\langle u, T \rangle : u \in BV_{\frac{n}{n-1}}(\mathbb{R}^n) \text{ and } \|Du\|(\mathbb{R}^n) \leq 1\} < \infty$$

and hence $\|\cdot\|_{CH_0}$ defines a norm in $CH_0(\mathbb{R}^n)$. Moreover, $CH_0(\mathbb{R}^n)$ with the norm $\|\cdot\|_{CH_0}$ is a Banach space. From (5.1) we see that

$$CH_0(\mathbb{R}^n) \subset BV_{\frac{n}{n-1}}(\mathbb{R}^n)^*.$$

We now consider the divergence operator

$$div : C_0(\mathbb{R}^n, \mathbb{R}^n) \to CH_0(\mathbb{R}^n)$$

given as

$$div\,(\boldsymbol{F}) : BV_{\frac{n}{n-1}}(\mathbb{R}^n) \to \mathbb{R}, \quad \boldsymbol{F} \in C_0(\mathbb{R}^n, \mathbb{R}^n),$$

$$div\,(\boldsymbol{F})(u) = -\int_{\mathbb{R}^n} \langle \boldsymbol{F}, d(Du) \rangle, \ u \in BV_{\frac{n}{n-1}}(\mathbb{R}^n).$$

It can be proven that this operator is well defined. Moreover, we have that

$$div : C_0(\mathbb{R}^n, \mathbb{R}^n) \to CH_0(\mathbb{R}^n)$$

is a bounded linear operator satisfying

$$\|div\,(\boldsymbol{F})\|_{CH_0} \leq \|\boldsymbol{F}\|_\infty.$$

The question now is whether the divergence operator defined above is surjective. Indeed, this is true so we have:

Theorem 5.2. *There exists $\boldsymbol{F} \in C_0(\mathbb{R}^n, \mathbb{R}^n)$ such that*

$$\operatorname{div} \boldsymbol{F} = T$$

if and only if $T \in CH_0(\mathbb{R}^n)$.

Since

$$CH_0(\mathbb{R}^n) \subset BV_{\frac{n}{n-1}}(\mathbb{R}^n)^*,$$

we have then characterized a closed subspace of $BV_{\frac{n}{n-1}}(\mathbb{R}^n)^*$. Moreover, since (see [13, Proposition 3.4]):

$$L^n(\mathbb{R}^n) \subset CH_0(\mathbb{R}^n),$$

the theorem implies that to each $f \in L^n(\mathbb{R}^n)$ there corresponds a continuous vector field vanishing at infinity, $\boldsymbol{F} \in C_0(\mathbb{R}^n, \mathbb{R}^n)$, such that

$$\operatorname{div} \boldsymbol{F} = f \tag{5.2}$$

in the sense that

$$-\int_{\mathbb{R}^n} \langle \boldsymbol{F}, d(Du) \rangle = \int_{\mathbb{R}^n} fu, \quad u \in BV_{\frac{n}{n-1}}(\mathbb{R}^n), \tag{5.3}$$

and in particular in the sense of distributions. Therefore, (5.2) addresses the solvability of the critical case $p = n$ discussed in the introduction. We now present the main idea of the proof of Theorem 5.2 (see [13, Theorem 6.1] for the details), which uses the adjoint operator:

$$
\begin{array}{ccc}
CH_0(\mathbb{R}^n)^* & \xrightarrow{\ \operatorname{div}^*\ } & C_0(\mathbb{R}^n; \mathbb{R}^n)^* \\
ev \big\| & & \big\| \\
BV_{\frac{n}{n-1}}(\mathbb{R}^n) & \xrightarrow[\ -D\ ]{} & \mathcal{M}(\mathbb{R}^n, \mathbb{R}^n)
\end{array}
$$

and the fact that the evaluation map

$$ev : BV_{\frac{n}{n-1}}(\mathbb{R}^n) \to CH_0(\mathbb{R}^n)^*$$

is a bijection, where

$$\langle T, ev(u) \rangle = \langle u, T \rangle, \quad u \in BV_{\frac{n}{n-1}}(\mathbb{R}^n), \quad T \in CH_0(\mathbb{R}^n).$$

Moreover, the evaluation map is in fact an isomorphism of the Banach spaces $BV_{\frac{n}{n-1}}(\mathbb{R}^n)$ and $CH_0(\mathbb{R}^n)^*$, according to the Open Mapping Theorem. Since the Range of div is dense in $CH_0(\mathbb{R}^n)$, in order to show that $\operatorname{div}(C_0(\mathbb{R}^n, \mathbb{R}^n))$ is closed in $CH_0(\mathbb{R}^n)$, and hence div is a surjective operator, it suffices to show that the Range of the adjoint operator div^* is closed in $C_0(\mathbb{R}^n, \mathbb{R}^n)^*$, according to the Closed Range Theorem. This can be seen as follows. Let $\{\alpha_j\}$ be a sequence in $CH_0(\mathbb{R}^n)^*$ such that $\operatorname{div}^*(\alpha_j) \to \mu$. Let $\{u_j\}$ in $BV_{\frac{n}{n-1}}(\mathbb{R}^n)$ such that $\alpha_j = ev(u_j)$. Hence

$$\|\operatorname{div}^*(\alpha_j)\|_{\mathcal{M}} = \|(\operatorname{div}^* \circ ev)(u_j)\|_{\mathcal{M}} = \|Du_j\|_{\mathcal{M}}.$$

Since $\{\operatorname{div}^*(\alpha_j)\}$ is bounded then $\sup_j \|Du_j\|_{\mathcal{M}} < \infty$. Therefore, the compactness property described at the beginning of this section implies that there exists a subsequence $\{u_{j_k}\}$ and $u \in BV_{\frac{n}{n-1}}(\mathbb{R}^n)$ such that $u - u_{j_k} \twoheadrightarrow 0$. Since

$$\int_{\mathbb{R}^n} \langle \boldsymbol{F}, d(Du) \rangle = -\int_{\mathbb{R}^n} u \operatorname{div} \boldsymbol{F} = -\lim_k \int_{\mathbb{R}^n} u_{j_k} \operatorname{div} \boldsymbol{F} = \lim_k \int_{\mathbb{R}^n} \langle \boldsymbol{F}, d(Du_{j_k}) \rangle,$$

then $\int_{\mathbb{R}^n} \langle \boldsymbol{F}, d(Du) \rangle = \int_{\mathbb{R}^n} \langle \boldsymbol{F}, d\mu \rangle$ since $Du_j \to \mu$, from which it follows that $Du = \mu$.

6. Characterization of (signed) measures in $BV_0(\Omega)^*$

In this section we let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. We recall that functions in $BV(\Omega)$ (see for example [15, Theorem 2.10] and [4, Theorem 10.2.1]) have traces on $\partial\Omega$. More precisely, if $u \in BV(\Omega)$, there exists a function $\varphi \in L^1(\partial\Omega)$ such that, for $\mathcal{H}^{n-1}$-almost every $x \in \partial\Omega$,

$$\lim_{r \to 0} r^{-n} \int_{B(x,r) \cap \Omega} |u(y) - \varphi(x)| dy = 0.$$

From the construction of the trace φ (see [15, Lemma 2.4]), we see that φ is uniquely determined. Therefore, we have a well defined operator

$$\gamma_0 : BV(\Omega) \to L^1(\partial\Omega).$$

The trace operator γ_0 is continuous from $BV(\Omega)$ equipped with the intermediate convergence onto $L^1(\Omega)$ equipped with the strong convergence. By the intermediate convergence of $BV(\Omega)$ we mean that the sequence $\{u_k\} \in BV(\Omega)$ converges to $u \in BV(\Omega)$ in the sense of intermediate (or strict) convergence if

$$u_k \to u \text{ strongly in } L^1(\Omega) \text{ and } \int_\Omega |Du_k| \to \int_\Omega |Du|.$$

It is natural to define $BV_0(\Omega)$ as follows:

Definition 6.1. Let

$$BV_0(\Omega) = \ker(\gamma_0).$$

We also define another BV function space with a zero boundary condition.

Definition 6.2. Let

$$\mathbb{BV}_0(\Omega) := \overline{C_c^\infty(\Omega)},$$

where the closure is taken with respect to the intermediate convergence of $BV(\Omega)$.

Both spaces $\mathbb{BV}_0(\Omega)$ and $BV_0(\Omega)$ are the same. Clearly, $\mathbb{BV}_0(\Omega) \subset BV_0(\Omega)$ holds by the continuity of the trace operator. In order to prove $BV_0(\Omega) \subset \mathbb{BV}_0(\Omega)$, it is necessary to show that the space $BV_c(\Omega)$ is dense in $BV_0(\Omega)$ in the strong topology of $BV(\Omega)$. This can be proven by using Giusti [15, Inequality (2.10)] (see [23, Theorem 6.3] for details). With this result at hand, given $u \in BV_0(\Omega)$, we can find by convolution a sequence in $C_c^\infty(\Omega)$ that converges to u in the intermediate convergence. Hence, we have

$$(6.1) \qquad\qquad \mathbb{BV}_0(\Omega) = BV_0(\Omega).$$

We note that (6.1) implies the following Sobolev's inequality for functions in $BV_0(\Omega)$

$$(6.2) \qquad\qquad \|u\|_{L^{\frac{n}{n-1}}(\Omega)} \leq C \|Du\|(\Omega), \ u \in BV_0(\Omega),$$

for a constant $C = C(n)$.

Also, from (6.2), we see that the full norm $\|u\|_{BV(\Omega)} = \|u\|_{L^1(\Omega)} + \|Du\|(\Omega)$ is equivalent to $\|Du\|(\Omega)$ whenever $u \in BV_0(\Omega)$ (or $\mathbb{BV}_0(\Omega)$) and Ω is a bounded open set with Lipschitz boundary. Thus, for the rest of the paper we will equip $BV_0(\Omega)$ with the homogeneous norm:

$$(6.3) \qquad\qquad \|u\|_{BV_0(\Omega)} = \|Du\|(\Omega).$$

The fact that $BV_c(\Omega)$ is dense in $BV_0(\Omega)$ in the strong topology of $BV(\Omega)$ implies the following Corollary (see [23, Lemma 6.5]) which is the analogous (for bounded domains) to Theorem 3.1, and it is a key element in the characterization of measures in $BV_0(\Omega)^*$.

Corollary 6.3. *Let Ω be any bounded open set with Lipschitz boundary. Then $BV_c^\infty(\Omega)$ is dense in $BV_0(\Omega)$.*

We now proceed to make precise the definitions of measures in the spaces $W_0^{1,1}(\Omega)^*$ and $BV_0(\Omega)^*$.

Definition 6.4. For a bounded open set Ω with Lipschitz boundary, we let

$$\mathcal{M}_{loc}(\Omega) \cap W_0^{1,1}(\Omega)^* := \{T \in W_0^{1,1}(\Omega)^* : T(\varphi) = \int_\Omega \varphi d\mu \text{ for some } \mu \in \mathcal{M}_{loc}(\Omega), \forall \varphi \in C_c^\infty(\Omega)\}.$$

Therefore, if $\mu \in \mathcal{M}_{loc}(\Omega) \cap W_0^{1,1}(\Omega)^*$, then the action $\langle \mu, u \rangle$ can be uniquely defined for all $u \in W_0^{1,1}(\Omega)$.

Definition 6.5. For a bounded open set Ω with Lipschitz boundary, we let

$$\mathcal{M}_{loc}(\Omega) \cap BV_0(\Omega)^* := \{T \in BV_0(\Omega)^* : T(\varphi) = \int_\Omega \varphi^* d\mu \text{ for some } \mu \in \mathcal{M}_{loc}(\Omega), \forall \varphi \in BV_c^\infty(\Omega)\},$$

where φ^* is the precise representative of φ. Thus, if $\mu \in \mathcal{M}_{loc}(\Omega) \cap BV_0(\Omega)^*$, then the action $\langle \mu, u \rangle$ can be uniquely defined for all $u \in BV_0(\Omega)$.

We will use the following characterization of $W_0^{1,1}(\Omega)^*$ whose proof is analogous to that of Lemma 3.4.

Lemma 6.6. *Let Ω be any bounded open set with Lipschitz boundary. The distribution T belongs to $W_0^{1,1}(\Omega)^*$ if and only if $T = \operatorname{div} \boldsymbol{F}$ for some vector field $\boldsymbol{F} \in L^\infty(\Omega, \mathbb{R}^n)$. Moreover,*

$$\|T\|_{W_0^{1,1}(\Omega)^*} = \min\{\|\boldsymbol{F}\|_{L^\infty(\Omega, \mathbb{R}^n)}\},$$

where the minimum is taken over all $\boldsymbol{F} \in L^\infty(\Omega, \mathbb{R}^n)$ such that $\operatorname{div} \boldsymbol{F} = T$. Here we use the norm

$$\|\boldsymbol{F}\|_{L^\infty(\Omega, \mathbb{R}^n)} := \left\|(F_1^2 + F_2^2 + \cdots + F_n^2)^{1/2}\right\|_{L^\infty(\Omega)} \text{ for } \boldsymbol{F} = (F_1, \ldots, F_n).$$

We are now ready to state the main result of this section (see [23, Theorem 7.4]), whose proof uses Corollary 6.3 (for the proof of (**iii**) $\Rightarrow$ (**iv**)) and Lemma 6.6 (for the proof of (**iv**) $\Rightarrow$ (**i**)).

Theorem 6.7. *Let Ω be any bounded open set with Lipschitz boundary and $\mu \in \mathcal{M}_{loc}(\Omega)$. Then, the following are equivalent:*

(**i**) *There exists a vector field $\boldsymbol{F} \in L^\infty(\Omega, \mathbb{R}^n)$ such that $\operatorname{div} \boldsymbol{F} = \mu$.*

(**ii**) *$|\mu(U)| \leq C \mathcal{H}^{n-1}(\partial U)$ for any smooth open (or closed) set $U \subset\subset \Omega$ with $\mathcal{H}^{n-1}(\partial U) < +\infty$.*

(**iii**) *$\mathcal{H}^{n-1}(A) = 0$ implies $\|\mu\|(A) = 0$ for all Borel sets $A \subset \Omega$ and there is a constant C such that, for all $u \in BV_c^\infty(\Omega)$,*

$$|\langle \mu, u \rangle| := \left|\int_\Omega u^* d\mu\right| \leq C \int_\Omega |Du|,$$

where u^* is the precise representative in the class of u that is defined $\mathcal{H}^{n-1}$-almost everywhere.

(**iv**) *$\mu \in BV_0(\Omega)^*$. The action of μ on any $u \in BV_0(\Omega)$ is defined (uniquely) as*

$$\langle \mu, u \rangle := \lim_{k \to \infty} \langle \mu, u_k \rangle = \lim_{k \to \infty} \int_\Omega u_k^* d\mu,$$

where $u_k \in BV_c^\infty(\Omega)$ converges to u in $BV_0(\Omega)$. In particular, if $u \in BV_c^\infty(\Omega)$ then

$$\langle \mu, u \rangle = \int_\Omega u^* d\mu,$$

and moreover, if μ is a non-negative measure then, for all $u \in BV_0(\Omega)$,

$$\langle \mu, u \rangle = \int_\Omega u^* d\mu.$$

Remark 6.8. If Ω is a bounded domain containing the origin then the function f given in Proposition 4.1 belongs to $BV_0(\Omega)^*$ but $|f|$ does not.

Theorem 6.7 immediately imply the following new result which states that the set of measures in $BV_0(\Omega)^*$ coincides with that of $W_0^{1,1}(\Omega)^*$.

ON THE DUAL OF BV 11

Theorem 6.9. *The normed spaces $\mathcal{M}_{loc}(\Omega) \cap BV_0(\Omega)^*$ and $\mathcal{M}_{loc}(\Omega) \cap W_0^{1,1}(\Omega)^*$ are isometrically isomorphic.*

The proof of Theorem 6.9 is similar to that of Theorem 3.6 but this time one uses Theorem 6.7 and Corollary 6.3 in place of Theorem 3.5 and Theorem 3.1, respectively (see [23, Theorem 7.6] for the details of the proof).

7. Finite measures in $BV(\Omega)^*$

The characterization of measures in the space $BV(\Omega)^*$, where $BV(\Omega)$ is the space of functions of bounded variation in a bounded open set Ω with Lipschitz boundary (but without the condition of having zero trace on $\partial\Omega$), offers more difficulties since we have to work with the full norm $\|u\|_{BV(\Omega)} = \|u\|_{L^1(\Omega)} + \|Du\|(\Omega)$. However, we can still characterize all *finite* signed measures that belong to $BV(\Omega)^*$. Note that the finiteness condition here is necessary at least for *positive* measures in $BV(\Omega)^*$. By a measure $\mu \in BV(\Omega)^*$ we mean that the inequality

$$\left| \int_\Omega u^* d\mu \right| \leq C \|u\|_{BV(\Omega)}$$

holds for all $u \in BV^\infty(\Omega)$. Since $BV^\infty(\Omega)$ is dense in $BV(\Omega)$ in the strong topology of $BV(\Omega)$, we see that such a μ can be uniquely extended to be a continuous linear functional in $BV(\Omega)$. We have the following (see [23, Theorem 8.2]):

Theorem 7.1. *Let Ω be an open set with Lipschitz boundary and let μ be a finite signed measure in Ω. Extend μ by zero to $\mathbb{R}^n \setminus \Omega$ by setting $\|\mu\|(\mathbb{R}^n \setminus \Omega) = 0$. Then, $\mu \in BV(\Omega)^*$ if and only if*

$$(7.1) \qquad\qquad |\mu(U)| \leq C\, \mathcal{H}^{n-1}(\partial U)$$

for every smooth open set $U \subset \mathbb{R}^n$ and a constant $C = C(\Omega, \mu)$.

The main ingredient in the proof of Theorem 7.1 is the following result, concerning the extension of BV functions, and whose proof can be found in [27, Lemma 5.10.14]:

Lemma 7.2. *Let Ω be an open set with Lipschitz boundary and $u \in BV(\Omega)$. Then, the extension of u to $\mathbb{R}^n$ defined by*

$$u_0(x) = \begin{cases} u(x), & x \in \Omega \\ 0, & x \in \mathbb{R}^n \backslash \Omega \end{cases}$$

satisfies that $u_0 \in BV(\mathbb{R}^n)$ and

$$\|u_0\|_{BV(\mathbb{R}^n)} \leq C \|u\|_{BV(\Omega)},$$

where $C = C(\Omega)$.

It is easy to see that if μ is a positive measure in $BV(\Omega)^*$ then its action on $BV(\Omega)$ is given by

$$\langle \mu, u \rangle = \int_\Omega u^* d\mu, \text{ for all } u \in BV(\Omega).$$

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