SIAM J. MATH. ANAL.
Vol. 36, No. 2, pp. 523–551

PLANE-LIKE MINIMAL SURFACES IN PERIODIC MEDIA WITH EXCLUSIONS*

MONICA TORRES†

Abstract. We consider minimal surfaces in a medium with exclusions (voids). This extends the results given in [*Comm. Pure Appl. Math.*, 54 (2001), pp. 1403–1441] to the case of a degenerate metric such that the area of a surface of codimension 1 is measured by neglecting the parts inside the exclusions. We prove that, given any plane in the medium, there is at least one minimal surface that always stays at a bounded distance from the plane. We also explore the connections of this problem with the theory of homogenization of Hamilton–Jacobi equations.

Key words. minimal surfaces, sets of finite perimeter, homogenization, periodic media

AMS subject classification. 35R99

DOI. 10.1137/S0036141001399970

1. Introduction.

The recent results in [14] consider a generalization of the problem of minimal surfaces in periodic media and show that, given a metric with periodic coefficients, there exists a number M so that one can find a minimizer in any strip of width M. The width M is independent of the orientation of the strip. Moreover, the minimizers constructed in [14] have the property that, when folded to the fundamental domain, they are laminations. For a discussion on the history of the problem of constructing minimizers that are asymptotic to a plane we refer the reader to [14] and the references therein.

The goal of this paper is to extend the results of [14] to a situation where the medium has exclusions, i.e., regions for which the metric vanishes. We also discuss the behavior of the minimizers near the exclusions, which is an issue not considered in [14]. Since similar situations of media with exclusions appear naturally in the theory of homogenization, we consider in this paper the relation of the minimizers with the theory of homogenization, and we develop several explicit calculations.

We recall that minimal surfaces can be studied using geometric measure theory (see, e.g., [26, 34]) in which the surfaces are interpreted as currents, i.e., dual to forms. Then the laminations can be interpreted as homologically minimizing currents (see, for instance, [6, 5, 4]). One can also study minimal surfaces by considering the surfaces as boundaries of sets in which the perimeter is defined in a weak sense (see, e.g., [27]).

In this paper we will follow the approach of locally finite perimeter sets, which is the one followed in [14]. For the problem considered in this paper, this approach is more advantageous because the fundamental domain is a manifold with boundary, and the theory of homologically minimizing currents in manifolds with boundary is not readily available to our knowledge. We refer the reader to [27, 25, 2] for a comprehensive survey on the theory of sets of finite perimeter.

The setting of the problem is as follows: the space $\mathbb{R}^n$ is considered as the lattice of cubes $[0,1]^n + \mathbb{Z}^n$ where each cube has an internal exclusion. If I denotes the

*Received by the editors December 20, 2001; accepted for publication (in revised form) November 14, 2003; published electronically July 29, 2004. This work was supported by a Research Assistanship from Luis A. Caffarelli.

http://www.siam.org/journals/sima/36-2/39997.html

†Department of Mathematics, Northwestern University, 2033 Sheridan Rd., Evanston, IL 60208-2730 (torres@math.northwestern.edu).

exclusion contained in $Y = [0,1]^n$, we assume the following:

1. I is compact, connected, and has Lipschitz boundary.
2. The distance between I and the boundary of Y, which we shall denote by α, is strictly positive.
3. Any other exclusion is of the form $I + z$ for some $z \in \mathbb{Z}^n$; i.e., the exclusions are periodic.

Once we have set up the domain for our problem, we proceed to explain our definition of *minimal surface*, which is made precise in section 2. If Σ is a surface in $\mathbb{R}^n$ of codimension one, we consider the following procedure for measuring the area of Σ: the portions that are inside the exclusions do not contribute to the area, and outside the exclusions the area is measured in the standard way. We say that Σ is a *minimal surface* if Σ minimizes area outside the exclusions. This means, loosely speaking, that any compact perturbation to Σ increases its area outside the exclusions.

We can now introduce the main result of this paper, which reads as follows: Under the assumptions 1, 2, and 3 given above, there exists a universal constant C (that depends only on n an α) such that, for every $(n-1)$-dimensional hyperplane Π, we can find a minimal surface Σ satisfying $d(\Pi, \Sigma) \leq C$.

The minimizers constructed in this paper are regular away from the boundaries of the exclusions. This follows directly from standard interior regularity theory for minimal surfaces (see Remark A.2). For the case when the exclusions have C^2 boundaries, the regularity of the minimizers near the boundary of the exclusions is a consequence of [29], where techniques of geometric measure theory are used to prove optimal regularity for codimension one minimal surfaces with a free boundary.

An important property of the surfaces constructed in this paper is that they meet the exclusions orthogonally. This means, loosely speaking, that the intersection of the minimizers with the exclusions looks like two perpendicular hyperplanes (in a small neighborhood). This orthogonality result can be deduced (once we have the regularity of the minimizers up to the boundary of the exclusions) by studying the first variation of the area. An analysis of the Euler–Lagrange equation is done in [31], where numerical and theoretical analysis for minimal surfaces involving two media is performed. We discuss the orthogonality property in section 6, and we explain how it can be obtained from [31]. For a proof of this orthogonality property, in the context of geometric measure theory, we refer the reader to [29].

The existence of plane-like minimizers implies that, in spite of having a heterogeneous media, the minimizer looks like a plane (homogeneous media) when seen from a far distance. This suggests connections with the theory of homogenization of PDEs, which studies the asymptotic behavior of a family of PDEs that oscillate with small period of size $\epsilon > 0$. The last section of this paper begins to explore the connection with the theory of homogenization of Hamilton–Jacobi equations. Hamilton–Jacobi equations arise in optimal control, differential games, geometric optics, calculus of variations, etc., and their solutions are understood in the *viscosity sense*. We refer the reader to [8, 23, 7] and the references therein for the definitions and basic properties of viscosity solutions that we will use in this paper.

The study of asymptotics of solutions of Hamilton–Jacobi equations is a fundamental question, as well as their applications to mathematical sciences. The homogenization of Hamilton–Jacobi equations has been extensively studied (see, for instance, [32, 21, 22, 15, 9]). The homogenized equation is also a Hamilton–Jacobi equation, and the corresponding Hamiltonian, usually denoted by $\overline{H}$, is called the *effective Hamiltonian*. It is a difficult but interesting task to find explicit formulas for $\overline{H}$. The references [22, 19, 20, 17, 16, 24] contain results in this direction. In this paper, we introduce a particular example, and we perform several explicit computations in search of its corresponding effective Hamiltonian. The homogenization of Hamilton–Jacobi equations in perforated domains was treated in [30], where both the Neumann-type and the Dirichlet boundary value problems were considered. A generalization of [30] has been studied in [1].

The organization of the paper is as follows.

Section 2 contains the proof of the existence of minimizers.

Section 3 uses some subadditivity properties of sets of finite perimeter to define an *infimal minimizer* which is contained in all the other minimizers and satisfies several monotonicity properties. The results presented in section 3 are contained in [14], but for clarity of the exposition we present again the proofs with more detail.

Section 4 deals with the proof of a geometric property that is specific to the infimal minimizer. This property is analogous to the so-called Birkhoff property in Aubry–Mather theory.

Section 5 contains the proof of the main theorem, which relies on the fact that minimizers must satisfy some *density estimates*. The geometric property proven in section 4, together with the density estimates, allows us to prove that the infimal minimizer is contained in a band whose width is independent of the direction of the plane.

Section 6 discusses the behavior of the minimizers near the boundaries of the exclusions.

Section 7 explores the connection with the theory of homogenization of Hamilton–Jacobi equations and contains several explicit computations.

We present at the end an appendix that includes the main definitions concerning sets of finite perimeter, as well as several remarks regarding some conventions and notation that we are using throughout the paper.

2. Existence of minimizers. We proceed now to prove the existence of minimizers. We refer the reader to the appendix for the definition and main properties of sets of finite perimeter. As explained before, our setting in this paper is $\mathbb{R}^n$ with exclusions (voids) that satisfy the three properties stated in the introduction.

We denote I as the exclusion contained in $[0,1]^n$. We let $\mathcal{I}$ denote the union of all exclusions and O its complement; i.e.,

$$\mathcal{I} = \bigcup_{k\in\mathbb{Z}^n} (I+k), \tag{1}$$

$$O = \mathbb{R}^n \backslash \mathcal{I}. \tag{2}$$

We let $\omega \in \mathbb{R}^n$, and we consider first the case when $\omega \in \mathbb{Q}^n$. Given $\tilde{M} \in \mathbb{R}$, we define

$$\Gamma_{\omega,\tilde{M}} = \left\{ x \in \mathbb{R}^n : x \cdot \frac{\omega}{|\omega|} \leq \tilde{M} \right\}, \tag{3}$$

where $\frac{\omega}{|\omega|}$ is the outward unit normal to $\partial\Gamma_{\omega,\tilde{M}}$. We denote T_k as the translation operator by $k \in \mathbb{Z}^n$; that is, $T_k(x) = x+k$, $x \in \mathbb{R}^n$. Given $N \in \mathbb{N}^+$ and $M > 0$, we define

$$\begin{aligned} A_{S_1,S_2} = \{ &E : E \text{ is a set of finite perimeter}, \\ &S_1 \subset E \subset S_2, T_{Nk}E = E \ \forall k \in \mathbb{Z}^n, \omega \cdot k = 0\}, \end{aligned} \tag{4}$$

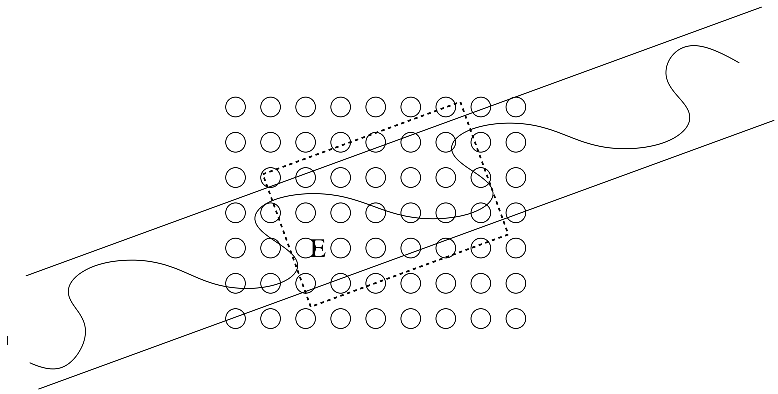

FIG. 1. *Diagram showing parallel plane restrictions and the period for minimization.*

where $S_1 = \Gamma_{\omega,0}$ and $S_2 = \Gamma_{\omega,M}$. We will refer to the sets $\Pi_1 \equiv \{x \in \mathbb{R}^n : x \cdot \omega = 0\}$ and $\Pi_2 \equiv \{x \in \mathbb{R}^n : x \cdot \frac{\omega}{|\omega|} = M\}$ as the parallel plane restrictions. Throughout this paper, we consider (without loss of generality) sets of finite perimeter that satisfy Remark A.1.

Since ω is rational, the sets in A_{S_1,S_2} can be identified with sets in the manifold

$$\Gamma_{\omega,M}/\approx, \tag{5}$$

where $\approx$ is the equivalence relation defined by

$$x \approx y \Longleftrightarrow x = y + Nk \quad \text{for some} \quad k \in \mathbb{Z}^n, \omega \cdot k = 0. \tag{6}$$

The space defined in (5) is $[-\infty, M] \times \mathbb{T}^{n-1}$. Moreover, we can identify the period of the class A_{S_1,S_2} as $[-\epsilon, M+\epsilon] \times \mathbb{T}^{n-1}$ for a fixed $\epsilon > 0$ (see Figure 1). We define

$$\Omega = ([-\epsilon, M+\epsilon] \times \mathbb{T}^{n-1}) \setminus \mathcal{I}. \tag{7}$$

For each set $E \in A_{S_1,S_2}$, we consider

$$J(E) = \int_\Omega |D\varphi_E|, \tag{8}$$

where the measure $|D\varphi_E|$ is introduced in Definition A.4. We let $\beta = \inf_{E \in A_{S_1,S_2}} J(E)$ and $\{E_j\}$ be a sequence such that $J(E_j) \to \beta$. This implies that the sequence $\{\int_\Omega |D\varphi_{E_j}|\}$ is uniformly bounded. Since the exclusions have at least Lipschitz boundary, it follows from Theorem A.2 that $BV(\Omega)$ is relatively compact in $L^1(\Omega)$. Therefore, there exists a convergent subsequence, which we denote again by $\{E_j\}$, in $L^1(\Omega)$. We let $E_0 \in L^1(\Omega)$ be the limit. Using Proposition A.1 we obtain

$$\int_\Omega |D\varphi_{E_0}| \leq \liminf \int_\Omega |D\varphi_{E_j}|.$$

Thus,

$$J(E_0) = \inf_{E \in A_{S_1,S_2}} J(E).$$

We make the following definitions.

DEFINITION 2.1. *Any* $E \in A_{S_1,S_2}$ *that satisfies* $J(E) = J(E_0)$ *shall be called a minimizer corresponding to the class* A_{S_1,S_2}*, or simply a minimizer, when it is not necessary to specify the class.*

DEFINITION 2.2. *We say that the minimizer* E *is an unconstrained minimizer if there exists a universal constant* $\tilde{M} > 0$ *such that, for all* $M \geq \tilde{M}$ *and all* $\epsilon \geq 0$, E *is a minimizer corresponding to the class* $A_{\Gamma_{\omega,-\epsilon},\Gamma_{\omega,M}}$.

DEFINITION 2.3. *We say that the minimizer* E *is a class* A *minimizer if, for any open ball* B_R,

$$\int_{B_R \cap O} |D\varphi_E|$$
$$= \inf \left\{ \int_{B_R \cap O} |D\varphi_F| : F \text{ is a set of finite perimeter, } \operatorname{spt}(\varphi_F - \varphi_E) \subset B_R \right\}.$$

DEFINITION 2.4. *We say that* $\Sigma \subset \mathbb{R}^n$ *is* a minimal surface *if* $\Sigma = \partial E$, *where* E *is a class* A *minimizer.*

Remark 2.1. We shall prove later (Proposition 5.2) that if the distance between the two restrictions Π_1 and Π_2 is large enough (independently of the slope of the restrictions), then there exists at least one unconstrained class A minimizer. That is, if the distance between Π_1 and Π_2 is large enough, then the restrictions do not interfere in the minimization, which means that they do not prevent the minimizers from doing "better."

The following lemma tells us that, without loss of generality, we can assume that minimizers are closed sets.

LEMMA 2.1. *If* E *is a minimizer corresponding to the class* A_{S_1,S_2}*, then there exists a closed set* $\tilde{E}$*, which is also a minimizer for the class* A_{S_1,S_2}.

Proof. Define $\tilde{E} = E \cup \partial E$ (see Definition A.8). We have that $\tilde{E}$ is closed. We need to prove that $\tilde{E}$ and E differ (outside the exclusions) on a set of $\mathcal{L}^n$-measure zero. Since the restrictions Π_1 and Π_2 have $\mathcal{L}^n$-measure zero, we need only to consider the set $\mathcal{K} \equiv \partial E \cap O \cap B_{\Pi_1,\Pi_2}$, where B_{Π_1,Π_2} is the open slab enclosed by Π_1 and Π_2. Since E minimizes area outside the exclusions, it follows from Lemma A.5 that if $x \in \mathcal{K}$ has density γ_x, then $0 < \gamma_x < 1$ (see Definition A.6 for the definition of density of a point), which implies that such x is not a Lebesgue point for φ_E. Therefore, from Definition A.6 we obtain that $\mathcal{L}^n(\mathcal{K}) = 0$. We can now prove that $\tilde{E}$ is a minimizer, which is a consequence of the fact that the sets E and $\tilde{E}$ differ (outside the exclusions) on a set of $\mathcal{L}^n$-measure zero. In fact, if $V \subset O$ is any open set, we have

$$\begin{aligned} \int_V |D\varphi_E| &= \sup \left\{ \int_V \varphi_E \operatorname{div} g : g \in C_0^1(V;\mathbb{R}^n), \quad |g(x)| \leq 1, \quad \text{for} \quad x \in V \right\} \\ &= \sup \left\{ \int_V \varphi_{\tilde{E}} \operatorname{div} g : g \in C_0^1(V;\mathbb{R}^n), \quad |g(x)| \leq 1, \quad \text{for} \quad x \in V \right\} \\ &= \int_V |D\varphi_{\tilde{E}}|, \end{aligned}$$

which proves that both measures coincide outside the exclusions. □

Remark 2.2. From now on, we shall assume that minimizers are closed sets.

We now proceed to prove that a minimizer (minus the exclusions) is connected for the case when the exclusions are simply connected sets and have at least C^1

boundaries. We remark that we do not need the connectivity of the minimizers in any of the proofs in this paper, but we present the result since it is interesting by itself.

LEMMA 2.2. *Let E be a minimizer corresponding to the class A_{S_1,S_2}. Assume that the exclusions are simply connected and have at least C^1 boundaries; then $E \cap O$ is connected.*

Proof. We let $\tilde{E} = E \cap O$. We prove that $\tilde{E}_{int}$ is connected. We proceed by contradiction and assume that

$$\tilde{E}_{int} = A \cup B, \tag{9}$$

where A,B are two disjoints open sets. Since $\Gamma_{\omega,0} \cap O$ is connected, it must be contained in either A or B. We assume that $\Gamma_{\omega,0} \cap O \subset A$, and we let $F = \mathbb{R}^n \setminus \tilde{E}$. Since E minimizes area outside the exclusions it follows that the points in ∂F have uniform density; i.e., there exists a universal constant C such that

$$|F \cap B(x,r)| \geq Cr^n, \quad x \in \partial F, \quad r \leq r_0, \tag{10}$$

for some small enough universal constant r_0. We prove this claim in Lemma A.6. We now proceed to prove that (10) implies that we can approximate $\tilde{E}_{int}$ from inside with smooth sets. We recall (see [2]) that sets of finite perimeter in $\mathbb{R}^n$ can be approximated in measure by open sets with smooth boundaries in such a way that we also have convergence of perimeters to perimeters. It is not, in general, possible to approximate a set of finite perimeter E by C^∞ sets contained inside E, nor it is possible from outside (see [27, p. 24] for a counterexample). However, in our case, we prove in Lemma A.7 that we can find sequences of sets $\{A_t\}$, $\{B_t\}$ with smooth boundaries satisfying

$$A_t \subset\subset A, \quad B_t \subset\subset B \tag{11}$$

and

$$\operatorname{Per}(A \cup B) = \lim_{t \to 0} \operatorname{Per}(A_t \cup B_t), \quad A_t \to A \quad B_t \to B \quad \text{in measure.} \tag{12}$$

From (11), (12), and the lower semicontinuity property given in Proposition A.1 we obtain

$$\begin{aligned} \operatorname{Per}(A \cup B) &= \lim_{t \to 0} \operatorname{Per}(A_t \cup B_t) \\ &= \lim_{t \to 0} \operatorname{Per}(A_t) + \lim_{t \to 0} \operatorname{Per}(B_t) \\ &\geq \operatorname{Per}(A) + \operatorname{Per}(B). \end{aligned}$$

This is a contradiction since we can eliminate B and obtain a set with less perimeter. ∎

3. Infimal minimizer. The minimizer we have just constructed may not be unique. However, we can prove the existence of an infimal minimizer, that is, a minimizer that is contained in any other minimizer. The results presented in this section are contained in [14], but, for clarity of the exposition, we present here the proofs with more detail.

In this section, Ω denotes the set defined in (7).

THEOREM 3.1. *There exists $E_* \in A_{S_1,S_2}$ such that, if E is any other minimizer, then $E_* \subset E$. We refer to E_* as the infimal minimizer.*

Proof. We denote $\mathcal{B}$ as the set of all minimizers. We have that $\mathcal{B} \subset L^1(\Omega)$. If $E_1, E_2 \in \mathcal{B}$, by Theorem A.4 we have

$$\operatorname{Per}(E_1 \cap E_2, \Omega) + \operatorname{Per}(E_1 \cup E_2, \Omega) \leq \operatorname{Per}(E_1, \Omega) + \operatorname{Per}(E_2, \Omega). \tag{13}$$

Since $E_1 \cup E_2$ is an admissible set we have $\operatorname{Per}(E_1 \cup E_2, \Omega) \geq \operatorname{Per}(E_1, \Omega)$. Since $\operatorname{Per}(E_1, \Omega) = \operatorname{Per}(E_2, \Omega)$ and using inequality (13), it follows that

$$\operatorname{Per}(E_1 \cap E_2, \Omega) \leq \operatorname{Per}(E_1, \Omega),$$

which implies that $E_1 \cap E_2$ is also a minimizer. Since we can uniformly bound the perimeters of minimizers in Ω, it follows from Proposition A.1 and Theorem A.2 that $\mathcal{B}$ is a compact subset of $L^1(\Omega)$. Since $L^1(\Omega)$ is separable, $\mathcal{B}$ is also separable. We let $\{E_j\}$ denote a dense subset of $\mathcal{B}$, and we define

$$\tilde{E_n} = \bigcap_{j=1}^{n} E_j.$$

Since $\tilde{E_n}$ is a minimizer and $\tilde{E}_{n+1} \subset \tilde{E_n}$ with $|\tilde{E_1} \cap \Omega| < \infty$, it follows that $|\tilde{E_n} \cap \Omega| \to |\bigcap_{n=1}^{\infty} \tilde{E_n} \cap \Omega|$, and therefore $\tilde{E_n} \to \bigcap_{n=1}^{\infty} \tilde{E_n}$ in $L^1(\Omega)$. We define

$$E_* = \bigcap_{n=1}^{\infty} \tilde{E_n}.$$

By Proposition A.1

$$\operatorname{Per}(E_*, \Omega) \leq \liminf \operatorname{Per}(\tilde{E_n}, \Omega),$$

which implies that E_* is a minimizer.

If E denotes any other minimizer we claim that $|(E_* \backslash E) \cap \Omega| = 0$. We proceed by contradiction and assume this is not true; i.e., $|(E_* \backslash E) \cap \Omega| > \delta > 0$. Since $\{E_j\}$ is a dense subset of $\mathcal{B}$, we can find E_k such that $|(E_k \backslash E) \cap \Omega| < \frac{\epsilon}{2}, \epsilon < \delta$. We choose N large enough such that $\tilde{E}_N \subset E_k$ and $|(E_* \backslash \tilde{E}_N) \cap \Omega| < \frac{\epsilon}{2}$. We have

$$\begin{aligned} |(E_* \backslash E) \cap \Omega| &\leq |(E_* \backslash \tilde{E}_N) \cap \Omega| + |(\tilde{E}_N \backslash E) \cap \Omega| \\ &\leq |(E_* \backslash \tilde{E}_N) \cap \Omega| + |(E_k \backslash E) \cap \Omega| \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon < \delta, \end{aligned}$$

which is a contradiction. Since E and E_* are both minimizers and are closed, if follows from Remark A.1 that $E_* \subset E$. ◻

COROLLARY 3.1. *The infimal minimizer is unique.*

We let $M_1 < 0$ and $M_2 > 0$ be such that $T_2 := \Gamma_{\omega, M_2} \subset S_2$. We have $T_1 := \Gamma_{\omega, M_1} \subset S_1$ and $T_1 \subset T_2$. The following proposition shall be used later to establish properties of the infimal minimizer.

PROPOSITION 3.1. *If E is a minimizer corresponding to the class A_{S_1,S_2} and L a minimizer corresponding to the class A_{T_1,T_2}, then*

(a) *$E \cap L$ is a minimizer corresponding to the class A_{T_1,T_2};*

(b) *$E \cup L$ is a minimizer corresponding to the class A_{S_1,S_2};*

(c) $E_{*,T_1,T_2} \subset E_{*,S_1,S_2}$.

Proof. We note that $E \cup L \in A_{S1,S2}$ and $E \cap L \in A_{T_1,T_2}$. Using Theorem A.4 and since $\mathrm{Per}(E,\Omega) \leq \mathrm{Per}(E \cup L,\Omega)$, it follows that

$$\begin{aligned}\mathrm{Per}(E\cap L,\Omega)+\mathrm{Per}(E,\Omega) &\leq \mathrm{Per}(E\cap L,\Omega)+\mathrm{Per}(E\cup L,\Omega)\\ &\leq \mathrm{Per}(E,\Omega)+\mathrm{Per}(L,\Omega),\end{aligned}$$

which implies that $\mathrm{Per}(E\cap L,\Omega) \leq \mathrm{Per}(L,\Omega)$; i.e., $E \cap L$ is a minimizer in the class A_{T_1,T_2}. In the same way we prove (b). In order to prove (c) we note that, by (a), $E_{*,T_1,T_2} \cap E_{*,S_1,S_2}$ is a minimizer corresponding to the class A_{T_1,T_2}, and hence

$$\begin{aligned}&E_{*,T_1,T_2} \subset (E_{*,T_1,T_2} \cap E_{*,S_1,S_2})\\ \Rightarrow\ &E_{*,T_1,T_2} \subset E_{*,S_1,S_2}. \qquad \square\end{aligned}$$

4. Birkhoff property. We denote E as the infimal minimizer corresponding to the class A_{S_1,S_2}. We recall that T_k denotes the translation operator by $k \in \mathbb{Z}^n$; that is, $T_k(x) = x + k$, $x \in \mathbb{R}^n$. The infimal minimizer satisfies an important geometric property (quite analogous to the property called Birkhoff in Aubry–Mather theory), which is proven in [14].

LEMMA 4.1. *If $k \in \mathbb{Z}^n$, we have the following:*

(a) *If $k \cdot \omega \leq 0$, then $T_k E \subset E$.*

(b) *If $k \cdot \omega \geq 0$, then $E \subset T_k E$.*

Proof. (a) We let $T_1 = T_k(S_1)$ and $T_2 = T_k(S_2)$, where as before $S_1 = \{x \in \mathbb{R}^n : x\cdot\omega \leq 0\}$ and $S_2 = \{x \in \mathbb{R}^n : x \cdot \omega \leq M\}$. If $k \cdot \omega \leq 0$ we have that $T_1 \subset S_1$, $T_2 \subset S_2$, and $T_1 \subset T_2$. We note that $T_k E$ is the infimal minimizer in A_{T_1,T_2}. By Proposition 3.1(c) we have $T_k E \subset E$.

(b) If $k \cdot \omega \geq 0$, we have that $S_1 \subset T_1$, $S_2 \subset T_2$, and $T_1 \subset T_2$. Since $T_k E$ is the infimal minimizer in A_{T_1,T_2}, by Proposition 3.1(c) it follows that $E \subset T_k E$. $\square$

We make the following important observation.

Remark 4.1. From (a) and (b) above, we have that if $k \cdot \omega = 0$, then $T_k E = E$. This implies that even though in the minimization of (8) the size of the period of the candidate sets is given by the number N (recall the definition (4)), the infimal mininimizer E has indeed a periodicity that depends only on the slope of the restrictions.

DEFINITION 4.1. *Given any two hyperplanes Π and $\tilde{\Pi}$ parallel to the restrictions, we denote $B_{\Pi,\tilde{\Pi}}$ as the open* slab *enclosed by Π and $\tilde{\Pi}$.*

The following two results are needed in order to handle the exclusions. They play the analogous role that the lower estimates in [14] play for the case without exclusions.

LEMMA 4.2. *If $\mathcal{C} \subset B_{\Pi_1,\Pi_2}$ is a cube of edge length $l \geq 5$ with sides parallel to the coordinate axis and integer vertices, we have the following:*

(a) *If $\mathcal{C} \subset (\mathbb{R}^n \backslash E)$, then there exists $0 < M_a < M$ such that $E \subset \Gamma_{w,M_a}$.*

(b) *If $\mathcal{C} \subset E$, then there exist $0 < M_b < M$ such that $\Gamma_{w,M_b} \subset E$.*

Proof. (a) We denote $\tilde{\Pi}$ as the hyperplane parallel to the restrictions Π_1 and Π_2 in such a way that the intersection $\tilde{\Pi} \cap \mathcal{C}$ consists only of the edge of $\mathcal{C}$ that is closer to the lower restriction Π_1. The equation of $\tilde{\Pi}$ is $x \cdot \frac{\omega}{|\omega|} = \tilde{M}$ for some $0 < \tilde{M} < M$. We define

$$\mathcal{D} = \bigcup_{\omega\cdot k \geq 0} T_k C. \tag{14}$$

If $\omega \cdot k \geq 0$, we claim that $T_k C \subset \mathbb{R}^n \setminus E$. In fact, if this is not true, there exist $x \in C$, $y \in E$ such that $T_k(x) = y$. Then $T_{-k}(y) = x$, which is a contradiction since

Lemma 4.1 implies $T_{-k}E \subset E$. We conclude that the set $\mathcal{D} \subset \mathbb{R}^n \setminus E$. We note that $\mathcal{D}$ contains the set $\{x \cdot \frac{\omega}{|\omega|} \geq \tilde{M} + \sqrt{n}\}$. If we define $M_a = \tilde{M} + \sqrt{n}$, we obtain that $E \subset \Gamma_{w,M_a}$.

(b) We denote $\tilde{\Pi}$ as the hyperplane parallel to the restrictions Π_1 and Π_2 in such a way that the intersection $\tilde{\Pi} \cap \mathcal{C}$ consists only of the edge of $\mathcal{C}$ that is closer to the upper restriction Π_2. The equation of $\tilde{\Pi}$ is $x \cdot \frac{\omega}{|\omega|} = \tilde{M}$ for some $0 < \tilde{M} < M$. We define

$$\mathcal{G} = \bigcup_{\omega \cdot k \leq 0} T_k C. \tag{15}$$

If $\omega \cdot k \leq 0$, it follows from Lemma 4.1 that $T_k E \subset E$, and therefore $T_k C \subset E$. We conclude that $\mathcal{G} \subset E$. We note that $\mathcal{G}$ contains the set $\{x \cdot \frac{\omega}{|\omega|} \leq \tilde{M} - \sqrt{n}\}$. If we define $M_b = \tilde{M} - \sqrt{n}$, we obtain $\Gamma_{w,M_b} \subset E$. ∎

We use the previous lemma to prove the following proposition.

Proposition 4.1. *If $\mathcal{C} \subset B_{\Pi_1,\Pi_2}$ is a cube of edge length $l \geq 5$, with sides parallel to the coordinate axis and integer vertices, then we cannot have $\mathcal{C} \subset E$.*

Proof. We proceed by contradiction. We let M_b be the number given by Lemma 4.2(b), and we define $\Pi_b = \{x \in \mathbb{R}^n : x \cdot \frac{\omega}{|\omega|} = M_b\}$. By subtracting a small number $\epsilon > 0$ to M_b, if necessary, we can assume that $|\omega| M_b \in \mathbb{Q}$. Since $l \geq 5$, there exists $p \in \mathbb{Z}^n$ such that $p \in C \cap \Gamma_{w,M_b}$. We define $M_c = p \cdot \frac{\omega}{|\omega|}$, and we take $k \in \{x \cdot \frac{\omega}{|\omega|} = M_b - M_c\} \cap \mathbb{Z}^n$ (which can be chosen because $|\omega|(M_b - M_c) \in \mathbb{Q}$). Since $M_b - M_c = k \cdot \frac{\omega}{|\omega|}$ we have

$$\begin{aligned} T_{-k}(\Pi_b) &= \left\{ x - k : x \cdot \frac{\omega}{|\omega|} = M_b \right\} \\ &= \left\{ y : y \cdot \frac{\omega}{|\omega|} = M_b - k \cdot \frac{\omega}{|\omega|} \right\} \\ &= \left\{ y : y \cdot \frac{\omega}{|\omega|} = M_c \right\} := \Pi_c. \end{aligned}$$

The plane Π_c divides E in two parts, say E_1 and E_2, where $\Pi_1 \subset E_1$ and $\Pi_b \subset E_2$. We consider now the set $E_1 \cup T_{-k}(E_2 \backslash B_{\Pi_b,\Pi_c})$. Clearly, this set is also a minimizer contained (and not equal) in E. This contradicts the fact that E is the infimal minimizer, that is, a minimizer that is contained in any other minimizer. ∎

5. Proof of the main theorem. We proceed in this section to prove the main theorem. We recall that we are considering $\mathbb{R}^n$ as the lattice $[0,1]^n + \mathbb{Z}^n$ with periodic exclusions; i.e., each cube $[0,1]^n + k$ with $k \in \mathbb{Z}^n$ has an internal exclusion. If I denotes the exclusion contained $Y = [0,1]^n$, we assume the following:

1. I is compact, connected, and has Lipschitz boundary.
2. The distance between I and the boundary of Y, which we denote by α, is strictly positive.
3. Any other exclusion is of the form $I + z$ for some $z \in \mathbb{Z}^n$; i.e., the exclusions are periodic.

Remark 5.1. From now on, given the restrictions S_1 and S_2, *we work with the unique infimal minimizer E corresponding to the class A_{S_1,S_2}.*

Remark 5.2. In order to clarify exposition we use the same C to denote different universal constants.

We now state the main theorem.

THEOREM 5.1. *Assume that the exclusions satisfy* 1, 2, *and* 3 *above. Then there exists a universal constant C (that depends only on n and α) such that, for every $(n-1)$-dimensional hyperplane Π, we can find a minimal surface Σ satisfying $d(\Pi, \Sigma) \leq C$.*

We recall from Definition 2.4 that a surface Σ is a minimal surface if it is the boundary of a class A minimizer (recall Definition 2.3), which means that any compact perturbation to Σ will increase its area outside the exclusions. The tool used to prove Theorem 5.1 is essentially a covering argument. This argument is similar to the one used in [14] to obtain the theorem for the case without exclusions. However, in our case we need to make several adjustments in order to extend the theorem to the case with exclusions. Lemmas 5.1, 5.3, and 5.4 are needed to handle the presence of exclusions. Using these lemmas we prove Propositions 5.1 and 5.2. Then Theorem 5.1 follows, for the case ω rational, from Proposition 5.3. Finally, we consider the case ω irrational at the end of this section.

LEMMA 5.1. *We let E denote the infimal minimizer corresponding to the class A_{S_1,S_2}, and we let $x \in \partial E$. If Q_q is a closed cube of edge length q (or a closed ball of radius q) containing x and such that $Q_q \cap \Pi_1 = \emptyset$ and $Q_q \cap \Pi_2 = \emptyset$, then*

$$\operatorname{Per}(E, Q_q^0 \cap O) \leq C q^{n-1},$$

where Q_q^0 denotes the interior of the set Q_q.

Proof. We can consider the set E as a candidate in the class with a period large enough (choosing N large enough in the definition (4)) in such a way that Q_q is completely contained inside the period $[0, M] \times \mathbb{T}^{n-1}$. Using Remark 4.1, it follows that the set E is a minimizer for the new class. Proceeding as in Lemmas A.1 and A.2 we can prove that, for almost every $0 < s < q$,

$$\operatorname{Per}(E \backslash Q_s, Q_q^0 \cap O) = \operatorname{Per}(E, (Q_q^0 \backslash Q_s) \cap O) + \mathcal{H}_{n-1}(\partial Q_s \cap E \cap O). \tag{16}$$

(In fact, we can use Lemma A.1 with $f(x) = \varphi_E$, $A = (Q_q^0 \backslash Q_s) \cap O$, and $\Omega = Q_q^0 \cap O$.) Since E is a minimizer we have

$$\operatorname{Per}(E, Q_q^0 \cap O) \leq \operatorname{Per}(E \backslash Q_s, Q_q^0 \cap O). \tag{17}$$

From (16) and (17) we obtain that, for almost every $0 < s < q$,

$$\begin{aligned} \operatorname{Per}(E, Q_q^0 \cap O) &\leq \operatorname{Per}(E, (Q_q^0 \backslash Q_s) \cap O) + \mathcal{H}_{n-1}(\partial Q_s \cap E \cap O) \\ &\leq \operatorname{Per}(E, (Q_q^0 \backslash Q_s) \cap O) + C s^{n-1}. \end{aligned} \tag{18}$$

We now choose a sequence $\{s_j\} \to q$ such that (18) holds for each s_j. If we let $j \to \infty$, we conclude that $\int_{Q_q^0 \cap O} |D\varphi_E| \leq C q^{n-1}$. We note that we can use $C = 2n$ if Q_q is a cube and $C = n w_n$ (where w_n is the volume of the n-dimensional unit ball) if Q_q is a ball. $\square$

LEMMA 5.2. *We let E denote the infimal minimizer corresponding to the class A_{S_1,S_2}, and we let $y \in \partial E$. We assume that there exists $\tilde{r} > 0$ that satisfies $B(y, \tilde{r}) \cap \Pi_1 = \emptyset$, $B(y, \tilde{r}) \cap \Pi_2 = \emptyset$, and $B(y, \tilde{r}) \subset O$. Then there exists a universal constant $C > 0$ such that, for all $r \leq \tilde{r}$,*

$$\int_{B(y,r)} |D\varphi_E| \geq C r^{n-1}. \tag{19}$$

Proof. Since E minimizes area outside the exclusions we have, for all $r \leq \tilde{r}$,

$$\int_{B(y,r)} |D\varphi_E| \leq \mathcal{H}_{n-1}(E \cap \partial B(y,r)). \tag{20}$$

We define $V(r) = |E \cap B(y,r)|$, $r \leq \tilde{r}$. Using the isoperimetric inequality given in Lemma A.3 we have that

$$|E \cap B(y,r)| \leq C[\mathrm{Per}(E \cap B(y,r))]^{\frac{n}{n-1}}. \tag{21}$$

From Lemma A.2 and using (20) and (21) it follows that, for almost every $r \leq \tilde{r}$,

$$\begin{aligned} |E \cap B(y,r)| &\leq C[\mathrm{Per}(E \cap B(y,r), \mathbb{R}^n)]^{\frac{n}{n-1}} \\ &= C[\mathrm{Per}(E, B(y,r)) + \mathcal{H}_{n-1}(E \cap \partial B(y,r))]^{\frac{n}{n-1}} \\ &\leq C[\mathcal{H}_{n-1}(E \cap \partial B(y,r))]^{\frac{n}{n-1}}. \end{aligned}$$

Due to Remark A.1 it follows that $V(r) > 0$ for all $r \leq \tilde{r}$. Since $V'(r) = \mathcal{H}_{n-1}(E \cap \partial B(y,r))$ we have, for almost every $r \leq \tilde{r}$,

$$V(r) \leq CV'(r)^{\frac{n}{n-1}}. \tag{22}$$

If we divide (22) by $V(r)$, we obtain $C \leq V(r)^{\frac{1-n}{n}} V'(r) = (V(r)^{\frac{1}{n}})'$. If we integrate, we obtain $V(r)^{\frac{1}{n}} \geq Cr$; i.e.,$V(r) \geq Cr^n$ for all $r \leq \tilde{r}$. In the same way we can prove that $|(\mathbb{R}^n \backslash E) \cap B(y,r)| \geq Cr^n$,$r \leq \tilde{r}$. The isoperimetric inequality stated in Lemma A.4 gives us

$$\begin{aligned} \min\{|(\mathbb{R}^n \backslash E) \cap B(y,r)|, |E \cap B(y,r)|\} &\leq C \left(\int_{B(y,r)} |D\varphi_E| \right)^{\frac{n}{n-1}} \\ &\Rightarrow \\ Cr^n &\leq \left(\int_{B(y,r)} |D\varphi_E| \right)^{\frac{n}{n-1}}. \end{aligned}$$

We conclude that

$$\int_{B(y,r)} |D\varphi_E| \geq Cr^{n-1}.$$

This completes the proof of the lemma. □

LEMMA 5.3. *We let E denote the infimal minimizer for the class A_{S_1,S_2}, and we take $x \in \partial E \cap O$. We assume that $x \in Y$, where $Y = [0,1]^n + k$ for some $k \in \mathbb{Z}^n$, and we denote I as the exclusion contained in Y. We assume also that Y does not intersect the parallel plane restrictions Π_1 and Π_2. Then $\partial E \cap \partial Y_\alpha \neq \emptyset$, where $Y_\alpha = \{x \in Y : d(x,I) \geq \frac{\alpha}{2}\}$.*

Proof. We proceed by contradiction and assume that $\partial E \cap Y_\alpha = \emptyset$. This implies that $Y_\alpha \subset E_{int}$ or $Y_\alpha \subset \mathbb{R}^n \backslash E$. Assume that $Y_\alpha \subset E_{int}$. We define $\tilde{E} = E \cup Y$. From Lemma 5.2 it follows that $\tilde{E}$ has strictly less area than E, which is a contradiction. If we assume now that $Y_\alpha \subset \mathbb{R}^n \backslash E$, then we can define $\tilde{E} = E \setminus Y$. Again, Lemma 5.2 implies that the set $\tilde{E}$ has strictly less area than E, which is a contradiction. We conclude that $\partial E \cap \partial Y_\alpha \neq \emptyset$. □

LEMMA 5.4. *We let E denote the infimal minimizer corresponding to the class A_{S_1,S_2}, and we let $x \in \partial E \cap O$. We assume that $x \in Y$, where $Y = [0,1]^n + k$ for* *some $k \in \mathbb{Z}^n$. We assume also that Y is far away from the parallel plane restrictions Π_1 and Π_2. Then there exists a cube $\mathcal{C}_x$ of edge length 2 and a universal constant $\beta > 0$, such that $x \in \mathcal{C}_x$ and $\mathcal{C}_x$ contains at least $\beta > 0$ amount of area, where β is a universal constant.*

Proof. From Lemma 5.3 there exists $y \in \partial E \cap Y$ such that $d(y, I) \geq \frac{\alpha}{2}$, where I is the exclusion contained in Y. If we make a dyadic decomposition of Y, we get 2^n cubes of side $\frac{1}{2}$ contained in Y. The point y must be contained in one of these dyadic cubes, say $\tilde{Y}$. Both Y and $\tilde{Y}$ have a common vertex, say v. We denote $\mathcal{C}_x$ as the cube of edge length 2 with its center in v. We note that $B(y, \frac{\alpha}{4})$ satisfies the hypothesis of Lemma 5.3, and thus we obtain the existence of the required constant β (in fact, $\beta = C(\frac{\alpha}{4})^n$). This completes the proof of the lemma. □

We shall use Vitali's covering lemma (see [25, Chapter 1]).

LEMMA 5.5. *Let $\mathcal{F}$ be any collection of nondegenerate closed cubes in $\mathbb{R}^n$ with edges parallel to the coordinate axis and satisfying*

$$\sup\{\text{diagonal} \quad \mathcal{C} : \mathcal{C} \in \mathcal{F}\} < \infty.$$

Then there exists a countable family $\mathcal{G}$ of disjoints cubes in $\mathcal{F}$ such that

$$\bigcup_{\mathcal{C}\in\mathcal{F}} \mathcal{C} \subset \bigcup_{\mathcal{C}\in\mathcal{G}} \hat{\mathcal{C}},$$

where $\hat{\mathcal{C}}$ is concentric with $\mathcal{C}$, and with edge length five times the edge length of $\mathcal{C}$.

Proof. The proof is the same as with balls, using the fact that the cubes are oriented in the same way as the coordinate axis. □

We have the following.

Remark 5.1. If we have a cube $\mathcal{C}$ in $\mathbb{R}^n$ of edge length l, then we can have at most $3^n - 1$ cubes of edge length l that intersect $\mathcal{C}$ without intersecting among themselves in a set of positive measure.

We now prove the following.

PROPOSITION 5.1. *There exists a universal constant $\tilde{M}$ such that for all $M \geq \tilde{M}$, if E denotes the infimal minimizer corresponding to A_{S_1,S_2}, where $S_1 = \Gamma_{\omega,0}$ and $S_2 = \Gamma_{\omega,M}$, then $d(\Pi_1, \partial E) < \tilde{M}$.*

Proof. We define $\tau = 5$, and we fix λ to be a multiple of 2τ and satisfying

$$\lambda > \frac{2^{2n} n \tau^n (3^n - 1)}{\beta}. \tag{23}$$

We let $\tilde{M} = 2\lambda\sqrt{n}$, and we note that $2\lambda\sqrt{n}$ is the length of the diagonal of the cube of edge length 2λ. We fix $M \geq \tilde{M}$ and denote E as the infimal minimizer corresponding to the class A_{S_1,S_2}, where $S_1 = \Gamma_{\omega,0}$ and $S_2 = \Gamma_{\omega,M}$. Our choice of λ allows us to fit a cube $\tilde{\mathcal{C}}$ of edge length 2λ in between $\Pi_1 = \{x \in \mathbb{R}^n : x \cdot \omega = 0\}$ and $\Pi_2 = \{x \in \mathbb{R}^n : x \cdot \frac{\omega}{|\omega|} = M\}$, with $\tilde{\mathcal{C}}$ having integer vertices, and edges parallel to the coordinate axis and intersecting Π_1 in a line. We claim that

$$d(\partial E, \Pi_1) < \tilde{M}. \tag{24}$$

We let $\mathcal{C}$ be the cube of edge length λ that is concentric with the cube $\tilde{\mathcal{C}}$. One of the following must happen:

1. $\mathcal{C} \subset \mathbb{R}^n \backslash E$. In this case, Lemma 4.2(a) implies the inequality (24).

2. $\mathcal{C} \cap E \neq \emptyset$. In this case, due to Proposition 4.1, we cannot have $\mathcal{C} \subset E$. Therefore, $\mathcal{C}$ must intersect ∂E.

For each $x \in \partial E \cap O \cap \mathcal{C}$ we denote $\mathcal{C}_x$ as the cube of edge length 2 constructed in Lemma 5.4. Therefore, we have a cover $\{\cup \mathcal{C}_x\}$ for $\partial E \cap \mathcal{C} \cap O$. By Lemma 5.5 we can extract a countable disjoint family $\{\mathcal{C}_i\}$ such that

$$\bigcup \mathcal{C}_x \subset \bigcup \hat{\mathcal{C}}_i, \tag{25}$$

where $\hat{\mathcal{C}}_i$ is concentric with $\mathcal{C}_i$ and has edge length 2τ. From Lemma 5.1 we have

$$\int_{(\cup \mathcal{C}_i)\cap O} |D\varphi_E| \leq 2n(2\lambda)^{n-1}. \tag{26}$$

From (26) and Lemma 5.4 it follows that the disjoint family has a finite number of cubes, say K, given by

$$K \leq \frac{2^n n \lambda^{n-1}}{\beta}. \tag{27}$$

Since λ is a multiple of 2τ, we can divide $\mathcal{C}$ in $\frac{\lambda^n}{(2\tau)^n}$ cubes of edge length 2τ, each cube having integer vertices and edges parallel to the coordinate axis. We note that the cubes do not intersect in sets of positive measure. Let us refer to this collection of cubes as $\mathcal{B}$. By Remark 5.1, out of the collection $\mathcal{B}$, at most

$$\frac{(3^n-1)2^n n\lambda^{n-1}}{\beta} \tag{28}$$

intersect ∂E. Due to our choice of λ, we have

$$\frac{(3^n-1)2^n n\lambda^{n-1}}{\beta} < \frac{\lambda^n}{(2\tau)^n}.$$

This implies that there exists $\mathcal{C}^{'} \in \mathcal{B}$ such that $\mathcal{C}^{'} \cap \partial E = \emptyset$. Due to Proposition 4.1 we must have $\mathcal{C}^{'} \subset \mathbb{R}^n \backslash E$, and the inequality (24) follows from Lemma 4.2(a). ◻

We have the following.

Proposition 5.2. *If E denotes the infimal minimizer corresponding to A_{S_1,S_2}, where $S_1 = \Gamma_{\omega,0}$ and $S_2 = \Gamma_{\omega,2\lambda\sqrt{n}}$, then E is an unconstrained minimizer.*

Proof. From inequality (24) we have that, for all $M > \tilde{M} = 2\lambda\sqrt{n}$, E is a minimizer for the class $A_{\Gamma_{\omega,0},\Gamma_{\omega,M}}$. We fix $\gamma > 0$. We claim that E is a minimizer for the class $A_{\Gamma_{\omega,-\gamma},\Gamma_{\omega,\tilde{M}}}$. We proceed by contradiction and assume this is not true. Therefore, the infimal minimizer, say $\tilde{E}$, corresponding to the class $A_{\Gamma_{\omega,-\gamma},\Gamma_{\omega,\tilde{M}}}$ has less perimeter than E. We choose $k \in \mathbb{Z}^n$ in such a way that $\Gamma_{\omega,0} \subset T_k\tilde{E}$. We obtain a contradiction since $T_k\tilde{E}$ is contained in the class $A_{\Gamma_{\omega,0},\Gamma_{\omega,\tilde{M}+k\cdot\frac{w}{|w|}}}$ and has less perimeter than E, which is a minimizer for this class. ◻

Proposition 5.3. *If E denotes the infimal minimizer corresponding to A_{S_1,S_2}, where $S_1 = \Gamma_{\omega,0}$ and $S_2 = \Gamma_{\omega,2\lambda\sqrt{n}}$, then E is a class A minimizer.*

Proof. We let L denote any set that coincides with E outside the ball B_{R-1}. We consider E as competing in a class with a period and distance between the plane restrictions large enough so that B_{R-1} is completely contained in one period. In order to do this, we choose $M > 0$ and N in (4) large enough in such a way that B_{R-1}

is contained in the period $[-M, M] \times \mathbb{T}^{n-1}$ corresponding to the class $A_{\Gamma_{\omega,-M},\Gamma_{\omega,M}}$. Using Proposition 5.2 and Remark 4.1 it follows that E is a minimizer in this new class, and therefore $\mathrm{Per}(E, B_R \cap O) \leq \mathrm{Per}(L, B_R \cap O)$. Since R is arbitrary, the proposition follows. □

This completes the proof of Theorem 5.1 for the case ω rational.

5.1. The case ω irrational. We now proceed to consider the case when the slope ω of the plane is irrational. Given $\omega \in \mathbb{R}^n \backslash \mathbb{Q}^n$, there exists a sequence $\{\omega_j\} \in \mathbb{Q}^n$ with $\omega_j \to \omega$. For each ω_j, we let $\{E_{\omega_j}\}$ denote the corresponding class A minimizers given by Theorem 5.1. From Lemma 5.1 we have

$$\mathrm{Per}(E_{\omega_j}, B_R \cap O) \leq CR^{n-1}.$$

Thus, $\{E_{\omega_j}\}$ has a subsequence that is convergent in $L^1(B_R \cap O)$. By applying the diagonal procedure, we obtain a subsequence of $\{E_{\omega_j}\}$ (which we will denote again as $\{E_{\omega_j}\}$) and a set E_ω such that $E_{\omega_j} \to E_\omega$ in $L^1_{loc}(\mathbb{R}^n \cap O)$. We need to check that E_ω is a class A minimizer. We let L denote any set that coincides with E_w outside the ball B_{R-1}. We define, for each j and $R \leq r \leq R+1$,

$$F_j^r = \begin{cases} L & \text{in} \quad B_r, \\ E_j & \text{in} \quad B_{R+1} \setminus \overline{B}_r. \end{cases}$$

Since each E_j is a class A minimizer we have

$$\begin{aligned} \int_{B_{R+1}\cap O} |D\varphi_{E_j}| &\leq \int_{B_{R+1}\cap O} |D\varphi_{F_j^r}| \\ &= \int_{B_r\cap O} |D\varphi_L| + \int_{\partial B_r\cap O} |D\varphi_{F_j^r}| + \int_{(B_{R+1}\setminus\overline{B}_r)\cap O} |D\varphi_{E_j}| \\ &= \int_{B_r\cap O} |D\varphi_L| + \int_{\partial B_r\cap O} |(\varphi_L)^r_{tr} - (\varphi_{E_j})^r_{tr}| d\mathcal{H}_{n-1} \\ &\quad + \int_{(B_{R+1}\setminus\overline{B}_r)\cap O} |D\varphi_{E_j}|, \end{aligned}$$

where $(\varphi_L)^r_{tr}$ and $(\varphi_{E_j})^r_{tr}$ are the traces (see Theorem A.3) of φ_L and φ_{E_j} on ∂B_r, respectively. We recall that, for almost every $R \leq r \leq R+1$, the traces $(\varphi_L)^r_{tr}$ and $(\varphi_{E_j})^r_{tr}$ coincide with the corresponding characteristic functions (see [27]). Using this fact and passing the last term in the right-hand side of the previous inequality to the left we obtain, for almost every $R \leq r \leq R+1$,

$$\int_{B_r\cap O} |D\varphi_{E_j}| \leq \int_{B_r\cap O} |D\varphi_L| + \int_{\partial B_r\cap O} |\varphi_L - \varphi_{E_j}| d\mathcal{H}_{n-1}. \tag{29}$$

We have the identity

$$\int_{(B_{R+1}\setminus\overline{B}_R)\cap O} |\varphi_{E_j} - \varphi_L| = \int_R^{R+1} \int_{\partial B_r\cap O} |\varphi_{E_j} - \varphi_L| d\mathcal{H}_{n-1} dr. \tag{30}$$

Since $E_w = L$ in $B_{R+1} \setminus \overline{B_R}$ it follows that $E_j \to L$ in $L^1((B_{R+1} \setminus \overline{B}_R) \cap O)$. This implies that (30) converges to 0 as $j \to \infty$, and therefore there exists a subsequence of $\{E_j\}$ (that we shall denote again as E_j) such that, for almost every $R \leq r \leq R+1$,

$$\int_{\partial B_r\cap O} |\varphi_{E_j} - \varphi_L| d\mathcal{H}_{n-1} \to 0. \tag{31}$$

From (29) and (31), it follows that, for almost every $R \le r \le R+1$,

$$\limsup_{r\to o} \int_{B_r\cap O} |D\varphi_{E_j}| \le \int_{B_r\cap O} |D\varphi_L|, \tag{32}$$

and hence

$$\begin{aligned}\int_{B_r\cap O} |D\varphi_{E_w}| &\le \liminf \int_{B_r\cap O} |D\varphi_{E_j}| \\ &\le \int_{B_r\cap O} |D\varphi_L|.\end{aligned}$$

Since $L = E_w$ in $B_{R+1} \setminus \overline{B}_R$ we conclude that

$$\int_{B_R\cap O} |D\varphi_{E_w}| \le \int_{B_R\cap O} |D\varphi_L|, \tag{33}$$

which proves that E_w is a class A minimizer. Clearly, we also have $d(E_w, \Pi_1) \le 2\lambda\sqrt{n}$. □

6. Behavior of the minimizers near the boundaries of the exclusions. It is an easy exercise to check that, for $n = 2$, minimizers must enter the exclusions *orthogonally*. In higher dimensions, the analogous result can be deduced (once we have the regularity of the minimizers up to the boundary of the exclusions) by studying the first variation of the area. An analysis of the Euler–Lagrange equation is done in [31], and we explain in this section how to use the results in [31] to obtain the fact that the minimizers must enter the exclusions *orthogonally*. For a proof of this orthogonality property, using techniques of geometric measure theory, we refer the reader to [29]. When the exclusions have C^2 boundary, the regularity of the minimizers near the boundaries of the exclusions is proven in [29].

In order to show how the orthogonality result follows from the work in [31], we must first recall that the minimal surface problem can also be studied by considering the surfaces as graphs of functions (nonparametric approach; cf. [27]). We can think of the nonparametric minimal surface problem as the problem of minimizing the energy among a class of functions with fixed boundary data and where the density at each point is one. In [31], the nonparametric minimal surface problem involving two different media is considered (the density at each point is given by a positive, piecewise smooth function), and the Euler–Lagrange equation is derived from the variational form. The solution has a jump across the interface that separates the two media, and a jump condition is derived that generalizes Snell's law to higher dimensions.

For the case $n = 2$, following [31] we consider a two-dimensional domain $D = [a,b] \times [c,d]$, and we seek a function $u(x,y)$ which minimizes the functional

$$\begin{aligned}E(u) &= \int_D c(x,y,u(x,y))\sqrt{1+|Du(x,y)|^2}\,dxdy, \\ u(x,y)|_{\partial D} &= u_0(x,y),\end{aligned} \tag{34}$$

where $u_0(x,y)$ is a given boundary condition and $c(x,y,z)$ is a positive piecewise smooth function which has a finite jump across a surface $S = \{(x,y,z) : g(x,y,z) = 0\}$. We assume that the graph of the minimizer of (34) intersects the surface S at a curve Γ. We denote γ as the projection of Γ on the (x,y)-plane. The curve γ divides the set D in two regions, D_1 and D_2. It is proven in [31] that if the surface S can be expressed locally as the graph of the function $z = \phi(x, y)$, then the jump of the derivatives of $u(x, y)$ across the surface S must satisfy the following generalized Snell's law in three dimensions (which can be extended to higher dimensions):

$$c^- \mathbf{n_1} \cdot \mathbf{m}|_\Gamma = c^+ \mathbf{n_2} \cdot \mathbf{m}|_\Gamma, \tag{35}$$

where c^- and c^+ are the weights of the two different media, $\mathbf{n_i} = \frac{(-u_x, -u_y, 1)}{\sqrt{1+u_x^2+u_y^2}}$ are the normal directions of the surface $u(x, y)$ in D_i, $i = 1, 2$, and $\mathbf{m} = \frac{(-\phi_x, -\phi_y, 1)}{\sqrt{1+\phi_x^2+\phi_y^2}}$ is the unit normal direction of S.

We note that if we consider the case $c^- = \epsilon$, $c^+ = 1$ and then compute the limit in (35) as $\epsilon \to 0$ we obtain

$$\mathbf{n_2} \cdot \mathbf{m}|_\Gamma = 0, \tag{36}$$

which implies that $\mathbf{n_2}$ and $\mathbf{m}$ are orthogonal vectors. We conclude from this that the minimizer E meets (on its regular points) the boundary of the exclusions orthogonally.

7. Connection with homogenization of Hamilton–Jacobi equations. In this section we explore some connections with the theory of homogenization of Hamilton–Jacobi equations. We first recall some of the main issues concerning the homogenization of Hamilton–Jacobi equations, and then we present the connection with the degenerate metric considered earlier.

We consider, for each $0 < \epsilon \leq 1$, the viscosity solution u^ϵ of the following problem:

$$H\left(Du^\epsilon, \frac{x}{\epsilon}\right) = 0 \text{ in } \mathbb{R}^n, \tag{37}$$

where $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a periodic function in the second variable. Under a suitable hypothesis (see, for instance, [22]) we can homogenize (37); i.e., the sequence of viscosity solutions $\{u^\epsilon\}$ converges as $\epsilon \to 0$ to the viscosity solution u of the averaged problem

$$\overline{H}(Du) = 0 \text{ in } \mathbb{R}^n,$$

where $\overline{H} : \mathbb{R}^n \to \mathbb{R}$ is defined as follows: for each $p \in \mathbb{R}^n$, $\overline{H}(p)$ is the unique number for which the PDE

$$\begin{gathered} H(p + D_y v, y) = \overline{H}(p) \text{ in } \mathbb{R}^n, \\ v \text{ is } [0,1]^n\text{-periodic} \end{gathered} \tag{38}$$

has a viscosity solution.

As explained in the introduction, the function $\overline{H}$ is called the effective Hamiltonian, and an interesting endeavor is to study the structure of $\overline{H}$ in order to find explicit formulas for it. This is still largely an open problem, and [22, 19, 20, 24, 17, 16] contain results in this direction. The goal of this section is to provide a particular example of (37) for which we can explicitly compute the limiting function u.

We recall our earlier consideration of $\mathbb{R}^n$ as the lattice of cubes $[0, 1]^n + \mathbb{Z}^n$ where each cube of side 1 has an internal exclusion. The exclusions satisfy properties 1, 2, and 3 stated in the introduction. In this section we work with surfaces of codimension $(n - 1)$, i.e., curves, instead of surfaces of codimension 1.

We fix $x \in \mathbb{R}^n$, and for each $0 < \epsilon \leq 1$ we consider the sequence of lattices $\epsilon([0, 1]^n + \mathbb{Z}^n)$. We let $\mathcal{J}$ denote the set of all curves joining the origin with x. We use the degenerate metric introduced in this paper to measure the length of each curve $l \in \mathcal{J}$. The length of l at the scale ϵ, that is, when we consider l as residing in the configuration $\epsilon([0,1]^n + \mathbb{Z}^n)$, is obtained by neglecting the portions inside the exclusions. This length depends on ϵ since the configuration of the lattice changes as we let $\epsilon \to 0$. We let l_ϵ denote the curve of minimal length and denote this optimal length by $d_\epsilon(0,x)$. We shall refer to the number $d_\epsilon(0,x)$ as the *smallest distance between* 0 *and* x *at the scale* ϵ.

We define, for each $0 < \epsilon \le 1$, the sequence of functions

$$u^\epsilon(x) = d_\epsilon(0,x), \quad x \in \mathbb{R}^n. \tag{39}$$

We have the following.

THEOREM 7.1. *If $\mathcal{I}$ denotes the union of all exclusions and $O = \mathbb{R}^n \backslash \mathcal{I}$, then*

$$\begin{cases} |Du_\epsilon| = 1 & \text{in} \quad \epsilon O, \\ u_\epsilon & \text{is constant on each connected component of} \quad \epsilon\mathcal{I}. \end{cases} \tag{40}$$

Proof. Without loss of generality we can assume $\epsilon = 1$. We define

$$v(x) = d_1(x,0), \quad x \in \mathbb{R}^n. \tag{41}$$

$v(x)$ is the smallest distance from x to the origin, and since we compute the length of a path $l \in \mathcal{J}$ by neglecting the portions inside the exclusions, we have that v is constant on each exclusion, which is connected. We prove now that v solves the eikonal equation $|Dv| = 1$ in the viscosity sense outside the exclusions. We prove first that v is a viscosity subsolution of $|Du| = 1$. If φ is a C^1 function such that $v - \varphi$ has a local maximum at the point $x_0 \in O$, we need to prove that $|D\varphi(x_0)| \le 1$. Since $v - \varphi$ has a local maximum at x_0 it follows that $v(x) - v(x_0) \le \varphi(x) - \varphi(x_0)$ for all x in a neighborhood of x_0. Therefore, for all z satisfying $|z| = 1$ and for all h small enough, we have

$$\begin{aligned} v(x_0 + hz) - v(x_0) \le \varphi(x_0 + hz) - \varphi(x_0) &= \int_0^h \frac{d}{ds}\varphi(x_0 + sz) \\ &= \int_0^h D\varphi(x_0 + sz) \cdot z ds \le \int_0^h D\varphi(x_0) \cdot z ds + Ch^2. \end{aligned}$$

If we define $z_0 = -\frac{D\varphi(x_0)}{|D\varphi(x_0)|}$, then

$$v(x_0 + hz_0) - v(x_0) \le -\int_0^h |D\varphi(x_0)| ds + Ch^2 = -h|D\varphi(x_0)| + Ch^2. \tag{42}$$

We now use the fact that v is a Lipschitz function, and from (42) we obtain

$$h|D\varphi(x_0)| \le v(x_0) - v(x_0 + hz_0) + Ch^2 \le |hz_0| + Ch^2,$$

and hence

$$|D\varphi(x_0)| \le 1 + Ch.$$

By letting $h \to 0$, we conclude that $|D\varphi(x_0)| \le 1$. We now prove that v is a supersolution. If φ is a C^1 function such that $v - \varphi$ has a local minimum at the point $x_0 \in O$, we need to prove that $|D\varphi(x_0)| \geq 1$. Since $v - \varphi$ has a local minimum at x_0, it follows that $v(x) - v(x_0) \geq \varphi(x) - \varphi(x_0)$ for all x in a neighborhood of x_0. Therefore, if h is small enough, we have

$$v(x_0 + hz) - v(x_0) \geq \varphi(x_0 + hz) - \varphi(x_0) = \int_0^h \frac{d}{ds}\varphi(x_0 + sz) = \int_0^h D\varphi(x_0 + sz) \cdot z ds$$

$$\geq \int_0^h D\varphi(x_0) \cdot z ds - Ch^2 \geq -h|D\varphi(x_0)| - Ch^2 \tag{43}$$

for all $|z| = 1$. We fix h small enough. We note that $v(x_0) = \inf_{|z|=1}\{h + v(x_0 + hz)\}$, and hence there exists a point z_0 such that $v(x_0 + hz_0) + h \leq v(x_0) + h^2$. From (43) we obtain $h|D\varphi(x_0)| \geq v(x_0) - v(x_0 + hz_0) - Ch^2 \geq h - h^2 - Ch^2$, and hence $|D\varphi(x_0)| \geq 1 - h - Ch$. Letting $h \to 0$ we obtain $|D\varphi(x_0)| \geq 1$. □

From standard theory of viscosity solutions we have that $\{u_\epsilon\}$ contains a subsequence that converges uniformly to a function u_0. Constructing the PDE that u_0 solves (i.e., the homogenization of (40)) is difficult. We present in this section some partial results toward this homogenization.

We proceed to compute u_0 *for the particular case* $n = 2$ and we assume, in addition to properties 1, 2, and 3 given in the introduction, that the exclusions are *balls of radius* ρ. Given two fixed points P and Q in the plane, we let $l_\epsilon^\rho(P, Q)$ denote the optimal path joining P and Q at the scale ϵ. We denote $d_\epsilon^\rho(P, Q)$ as the length of $l_\epsilon^\rho(P, Q)$. The behavior of $d_\epsilon^\rho(P, Q)$ depends on the value of ρ, where $0 < \rho \leq \frac{1}{2}$ (we note that the radius of the exclusions at the scale ϵ is $\epsilon\rho$). We have, for any $0 < \epsilon \leq 1$,

$$0 \leq d_\epsilon^\rho(P, Q) \leq |P - Q|_2. \tag{44}$$

Thus, fixing ρ and letting $\epsilon \to 0$ it follows that $\{d_\epsilon^\rho(P, Q)\}$ contains a subsequence that converges to a number, say $d_0^\rho(P, Q)$.

If we assume that X and Y are centers of exclusions at the scale ϵ, we can replace $l_\epsilon^\rho(X, Y)$ inside the exclusions with lines so that this optimal path is composed of a sequence of segments. We can classify (after a suitable translation and/or rotation) these segments in the following four categories:

1. a segment joining the points $(0, 0)$ and $(\frac{i}{\epsilon}, \frac{j}{\epsilon})$, where $i, j \in \mathbb{Z}^+$ are relatively prime and $j < i$;
2. the segment joining the points $(0, 0)$ and $(\frac{1}{\epsilon}, 0)$;
3. the segment joining the points $(0, 0)$ and $(0, \frac{1}{\epsilon})$;
4. the segment joining $(0, 0)$ and $(\frac{1}{\epsilon}, \frac{1}{\epsilon})$.

We identify a segment of type 1 with the pair $[i, j]$, a segment of types 2 or 3 with $[1, 0]$, and a segment of type 4 with $[1, 1]$. Therefore, any optimal path joining two points that are centers of exclusions is composed of a sequence of segments belonging to the set

$$\mathcal{P} = \{[i, j] : i, j \in \mathbb{Z}^+, i, j \quad \text{are relatively prime, } j < i\} \cup \{[1, 1]\} \cup \{[1, 0]\}.$$

We prove in the next theorem that if ρ is large enough, then the optimal path joining two centers of exclusions is composed only of segments of the type $[1, 0]$.

Theorem 7.2. *If* $\rho > \frac{2-\sqrt{2}}{2}$ *then, for any* $0 < \epsilon \leq 1$, *the optimal path connecting two points that are centers of exclusions is composed only of segments of the type* $[1, 0]$. *Moreover, if* P *and* Q *are any two points, we have that*

$$\lim_{\epsilon \to 0} d_\epsilon^\rho(P, Q) = (1 - 2\rho)|P - Q|_1.$$

Proof. We fix $\epsilon > 0$, and thus in this proof we work in the domain $\epsilon([0,1]^n + \mathbb{Z}^n)$. We denote $X = (x_1, x_2)$ and $Y = (y_1, y_2)$ as two points that are centers of exclusions. We can assume, without loss of generality, that $y_1 \geq x_1$ and $y_2 \geq x_2$. We proceed by contradiction and assume that $l_\epsilon^\rho(X,Y)$ has a segment of the type 1 or 4. Therefore, the path $l_\epsilon^\rho(X,Y)$ contains a segment joining the points $\epsilon(i+\frac{1}{2}, j+\frac{1}{2})$ and $\epsilon(i+\frac{1}{2}+m, j+\frac{1}{2}+n)$, where $n \leq m$, $n \geq 2$, and m and n are prime relative to each other. Since $l_\epsilon^\rho(X,Y)$ is the optimal path we have that

$$\begin{aligned} m(\epsilon - 2\epsilon\rho) + n(\epsilon - 2\epsilon\rho) &\geq \sqrt{\epsilon^2 m^2 + \epsilon^2 n^2} - 2\epsilon\rho, \\ m(1-2\rho) + n(1-2\rho) &\geq \sqrt{m^2+n^2} - 2\rho, \\ m + n - \sqrt{m^2+n^2} &\geq 2(m+n-1)\rho, \\ &\Rightarrow \rho \leq \frac{m+n-\sqrt{m^2+n^2}}{2(m+n-1)}. \end{aligned}$$

We claim that $\frac{m+n-\sqrt{m^2+n^2}}{2(m+n-1)} \leq \frac{2-\sqrt{2}}{2}$. To prove this, we consider the function $f(x) = \frac{x+n-\sqrt{x^2+n^2}}{2(x+n-1)}$ and its derivative $f'(x) = \frac{1}{2}\frac{n^2-\sqrt{x^2+n^2}+x-xn}{\sqrt{n^2+x^2}(x+n+1)^2}$. We note that $f'(x) \leq 0$ if $x \geq 0$. This implies that f is decreasing, and thus $f(m) \leq f(n)$. By a simple substitution it follows that $f(n) = \frac{2n-\sqrt{2}n}{2(2n-1)} = \frac{2-\sqrt{2}}{2}(\frac{n}{2n-1}) \leq \frac{2-\sqrt{2}}{2}$ (since $\frac{n}{2n-1} \leq 1$). Hence, $\rho \leq f(m) \leq \frac{2-\sqrt{2}}{2}$, which contradicts the fact that $\rho > \frac{2-\sqrt{2}}{2}$. This proves the first part of the theorem. Because of the above result, we can explicitly compute $d_\epsilon^\rho(X,Y)$:

$$\begin{aligned} d_\epsilon^\rho(X,Y) &= \frac{y_2-x_2}{\epsilon}(\epsilon - 2\epsilon\rho) + \frac{y_1-x_1}{\epsilon}(\epsilon-2\epsilon\rho) \\ &= (1-2\rho)[(y_2-x_2)+(y_1-x_1)] \\ &= (1-2\rho)|Y-X|_1. \end{aligned} \tag{45}$$

We now denote P and Q as any two points in the plane. If P' and Q' are the closest centers of exclusions to P and Q, respectively, we have

$$d_\epsilon^\rho(P',Q') - \sqrt{2}\epsilon \leq d_\epsilon^\rho(P,Q) \leq d_\epsilon^\rho(P',Q') + \sqrt{2}\epsilon.$$

From (45) we have

$$(1-2\rho)|P'-Q'|_1 - \sqrt{2}\epsilon \leq d_\epsilon^\rho(P,Q) \leq (1-2\rho)|P'-Q'|_1 + \sqrt{2}\epsilon.$$

Using the triangle inequality, again

$$\begin{aligned} (1-2\rho)(|P-Q|_1 - 2\sqrt{2}\epsilon) - \sqrt{2}\epsilon \leq d_\epsilon^\rho(P,Q)& \\ \leq (1-2\rho)(|P-Q|_1 + 2\sqrt{2}\epsilon) + \sqrt{2}\epsilon.& \end{aligned}$$

Letting $\epsilon \to 0$ yields

$$\begin{gathered} 1-2\rho \leq \frac{d_0^\rho(P,Q)}{|P-Q|_1} \leq 1-2\rho \\ \Rightarrow \\ d_0^\rho(P,Q) = (1-2\rho)|P-Q|_1. \qquad \square \end{gathered}$$

We now wish to study the behavior of the optimal path as $\rho \to 0$. As ρ decreases, new paths (new segments of the collection $\mathcal{P}$) become available. For each segment $[i,j]$ there exists a critical radius $\rho_{[i,j]}$, which is the largest radius for which $d_1^{\rho_{[i,j]}}((0,0),(i,j)) = \sqrt{i^2+j^2}$.

Since $\mathcal{P}$ is countable, we can enumerate the sequence $\{\rho_{[i,j]}\}$ in such a way that the coordinate i is always increasing. We have the following lemma.

LEMMA 7.1. $\lim_{i\to\infty} \rho_{[i,j]} = 0$.

Proof. We recall that $[i,j]$ represents the segment joining $(0,0)$ with (i,j). We denote by $P=(p_1,p_2)$ the closest point (other than the extremes) with integer coordinates to the segment, and we denote this distance as d. The point P satisfies the equation $|\frac{i}{j} - \frac{p_1}{p_2}| = \frac{1}{jp_2}$, that is, $|ip_2 - jp_1| = 1 = A(p)$, where $A(p)$ is the area of the parallelogram spanned by (i,j) and (p_1,p_2). Hence we have that $\frac{1}{2} = d\frac{\sqrt{i^2+j^2}}{2}$, which implies that $d = \frac{1}{\sqrt{i^2+j^2}}$. We define $l = \sqrt{i^2+j^2}$, $l_1 = \sqrt{p_1^2+p_2^2}$, and $l_2 = \sqrt{(i-p_1)^2+(j-p_2)^2}$. Solving the equation $l-2\rho = l_1+l_2-4\rho$ for ρ, we obtain the critical radius for which the segment joining $(0,0)$ with (i,j) is a better path than the one joining the points $(0,0),(p_1,p_2)$, and (i,j). We have that $\rho = \frac{l_1+l_2-l}{2}$. Since $l_1+l_2 \le 2d+l$ it follows that $\rho \le \frac{2d+l-l}{2} = d = \frac{1}{\sqrt{i^2+j^2}}$. Since $\rho_{[i,j]} \le \rho$ the lemma holds. ◻

An easy computation gives us $\rho_{[1,1]} = \frac{2-\sqrt{2}}{2}$, $\rho_{[2,1]} = \frac{1+\sqrt{2}-\sqrt{5}}{2}$, and $\rho_{[3,1]} = \frac{1+\sqrt{5}-\sqrt{10}}{2}$. We have the following theorem.

THEOREM 7.3. *Let P, Q be any two points in the plane. Then we have the following:*

(a) *If* $\frac{1+\sqrt{2}-\sqrt{5}}{2} < \rho \le \frac{2-\sqrt{2}}{2}$, *we have*

$$\lim_{\epsilon\to 0} d_\epsilon^\rho(P,Q) = |P-Q|_{1,1},$$

where $|\cdot|_{1,1} : \mathbb{R}^2 \to \mathbb{R}^+$ *is given by*

$|(x,y)|_{1,1} = (\sqrt{2}-1)|x| + (1-2\rho)|y|$ *if* $|x| \le |y|$ *and*

$|(x,y)|_{1,1} = (\sqrt{2}-1)|y| + (1-2\rho)|x|$ *if* $|y| \le |x|$.

(b) *If* $\frac{1+\sqrt{5}-\sqrt{10}}{2} < \rho \le \frac{1+\sqrt{2}-\sqrt{5}}{2}$, *we have*

$$\lim_{\epsilon\to 0} d_\epsilon^\rho(P,Q) = |P-Q|_{2,1},$$

where $|\cdot|_{2,1} : \mathbb{R}^2 \to \mathbb{R}^+$ *is given by*

$|(x,y)|_{2,1} = (1-2\rho)|x| + (\sqrt{5}-2+2\rho)|y|$ *if* $|y| \le \frac{|x|}{2}$,

$|(x,y)|_{2,1} = (2\sqrt{2}-\sqrt{5}-2\rho)|y| + (\sqrt{5}-\sqrt{2})|x|$ *if* $\frac{|x|}{2} < |y| \le |x|$,

$|(x,y)|_{2,1} = (1-2\rho)|y| + (\sqrt{5}-2-2\rho)|x|$ *if* $|y| \ge 2|x|$, *and*

$|(x,y)|_{2,1} = (2\sqrt{2}-\sqrt{5}+2\rho)|x| + (\sqrt{5}-\sqrt{2})|y|$ *if* $|x| \le |y| < 2|x|$.

Moreover, $|\cdot|_{1,1}$ *and* $|\cdot|_{2,1}$ *define norms in* $\mathbb{R}^2$.

Proof. We denote $X=(x_1,x_2)$ and $Y=(y_1,y_2)$ as centers of exclusions (at the scale ϵ). We can assume that $x_1 \le y_1$ and $x_2 \le y_2$. In order to prove (a), we consider first the case when $y_2 - x_2 \le y_1 - x_1$. Solving the equation $\sqrt{5}-2\rho = \sqrt{2}+1-4\rho$, we obtain $\rho = \frac{\sqrt{2}+1-\sqrt{5}}{2}$, the critical radius for which the next segment $[2,1]$ becomes available. Therefore, if ρ belongs to the interval given in (a), the only paths available are $[1,0]$ and $[1,1]$. Thus, the optimal path $l_\epsilon^\rho(X,Y)$ has as many segments $[1,1]$ as possible, since for this interval $[1,1]$ is better than two segments of type $[1,0]$. Hence, we can compute $d_\epsilon^\rho(X,Y)$ explicitly, and we obtain

$$\begin{aligned} d_\epsilon^\rho(X,Y) &= \frac{y_2-x_2}{\epsilon}(\sqrt{2}\epsilon - 2\epsilon\rho) + \frac{(y_1-x_1)-(y_2-x_2)}{\epsilon}(\epsilon - 2\epsilon\rho) \\ &= (\sqrt{2}-1)(y_2-x_2) + (1-2\rho)(y_1-x_1) \\ &= |Y-X|_{1,1}. \end{aligned}$$

The case $y_2 - x_2 \geq y_1 - x_1$ is computed in the same way, except that we interchange the roles of the coordinates. To prove (a) we can proceed now in exactly the same way (provided that $|\cdot|_{1,1}$ is a norm) as in Theorem 7.2. We need to check that $|\cdot|_{1,1}$ defines a norm in $\mathbb{R}^2$. We need only to show that the triangle inequality holds, and there are several cases to verify.

We let $(x,y),(w,z)$ be any two points in the plane, and we consider the case $|y| \leq |x|$, $|w| \leq |z|$ and $|x+w| \leq |y+z|$. We need to prove that $(\sqrt{2}-1)|x+w| + (1-2\rho)|y+z| \leq (\sqrt{2}-1)(|y|+|w|) + (1-2\rho)(|x|+|z|)$; that is, $(\sqrt{2}-1)(|x+w|-|y|-|w|) + (1-2\rho)(|y+z|-|x|-|z|) \leq 0$. Using the triangle inequality for real numbers we can see that the last inequality is true since $|x|-|y| \geq 0$ and $1-2\rho > \sqrt{2}-1$ for ρ in the interval given in (a).

Considering now the case $|y| \leq |x|$, $|w| \geq |z|$, and $|x+w| \leq |y+z|$, we need to prove that $(\sqrt{2}-1)|x+w| + (1-2\rho)|y+z| \leq (\sqrt{2}-1)(|y|+|z|) + (1-2\rho)(|x|+|w|)$; that is, $(\sqrt{2}-1)(|x+w|-|y|-|z|) + (1-2\rho)(|y+z|-|x|-|w|) \leq 0$. Using the triangle inequality for real numbers we can see that the last inequality is true.

Proceeding to the case $|y| \leq |x|$, $|w| \leq |z|$, and $|x+w| \geq |y+z|$, we need to prove that $(\sqrt{2}-1)|y+z| + (1-2\rho)|x+w| \leq (\sqrt{2}-1)(|y|+|w|) + (1-2\rho)(|x|+|z|)$; that is, $(\sqrt{2}-1)(|y+z|-|y|-|w|) + (1-2\rho)(|x+w|-|x|-|z|) \leq 0$. Using the triangle inequality for real numbers we can see that the last inequality is true since $|z|-|w| \geq 0$ and $1-2\rho > \sqrt{2}-1$ for ρ in the interval given in (a).

Finally, we check that $|y| \leq |x|$, $|w| \geq |z|$, and $|x+w| \geq |y+z|$. We need to prove that $(\sqrt{2}-1)|y+z| + (1-2\rho)|x+w| \leq (\sqrt{2}-1)(|y|+|z|) + (1-2\rho)(|x|+|w|)$; that is, $(\sqrt{2}-1)(|y+z|-|y|-|z|) + (1-2\rho)(|x+w|-|x|-|w|) \leq 0$, which is true due to the triangle inequality. There are four more cases corresponding to $|y| \geq |x|$, but they are proven in the same way. The unit ball for this norm is a polygon with eight edges as shown in Figure 2.

To prove (b) we note that by solving the equation $\sqrt{10} - 2\rho = \sqrt{5} + 1 - 4\rho$ we obtain $\rho = \frac{\sqrt{5}+1-\sqrt{10}}{2}$, the critical radius for which the next segment $[3,1]$ becomes available. If $p > 0$ and $q \geq 0$ are two integers satisfying $-p+2q \leq 0$, then, for ρ in the interval given in (b), the best path joining $(0,0)$ with (p,q) consists only of segments of the type $[2,1]$ and $[1,0]$. Furthermore, this path takes as many $[2,1]$ segments as possible and then completes the trajectory with segments $[1,0]$. If $-p+2q > 0$ and $q < p$, the best path consists only of segments of the type $[2,1]$ and $[1,1]$, and this path takes as many $[2,1]$ segments as possible and then completes the trajectory with segments $[1,1]$. Thus, we can compute $d_\epsilon^\rho(X,Y)$ exactly as before and proceed as in Theorem 7.2. The unit ball for $|\cdot|_{2,1}$ is a polygon with 16 edges as shown in Figure 2. ∎

The following theorem gives an asymptotic behavior of d_0^ρ.

THEOREM 7.4. *Let P, Q be any two points in the plane. Then*

$$\lim_{\rho\to 0} d_0^\rho(P,Q) = |P-Q|_2.$$

Proof. We denote X and Y as two points that are centers of exclusions (at the scale ϵ). The optimal path $l_\epsilon^\rho(X,Y)$ intersects a finite numbers of balls, say N. We

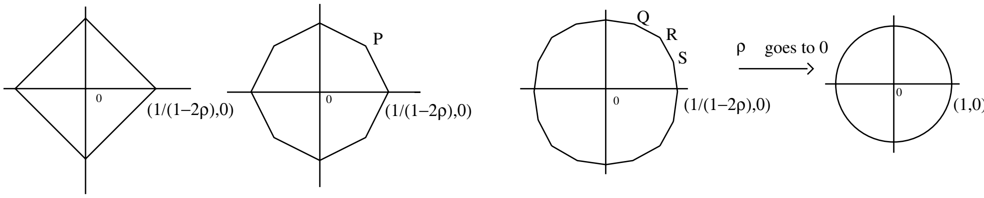

FIG. 2. *Unit balls for limiting norms.* $P = R = (\frac{1}{\sqrt{2}-2\rho}, \frac{1}{\sqrt{2}-2\rho})$, $Q = (\frac{1}{\sqrt{5}-2\rho}, \frac{2}{\sqrt{5}-2\rho})$, $S = (\frac{2}{\sqrt{5}-2\rho}, \frac{1}{\sqrt{5}-2\rho})$.

define

$$\tilde{d}^{\rho}_{\epsilon}(X,Y) = d^{\rho}_{\epsilon}(X,Y) + (N-1)(2\epsilon\rho). \tag{46}$$

Since the distance between two centers of exclusions is at least ϵ it follows that

$$\begin{aligned} N &\le \frac{\tilde{d}^{\rho}_{\epsilon}(X,Y)}{\epsilon} + 1 \\ \Rightarrow N - 1 &\le \frac{\tilde{d}^{\rho}_{\epsilon}(X,Y)}{\epsilon}. \end{aligned} \tag{47}$$

Hence, using (46) and (47) we obtain

$$\begin{aligned} d^{\rho}_{\epsilon}(X,Y) &\ge \tilde{d}^{\rho}_{\epsilon}(X,Y) - \frac{\tilde{d}^{\rho}_{\epsilon}(X,Y)}{\epsilon}(2\epsilon\rho) \\ &= \tilde{d}^{\rho}_{\epsilon}(X,Y)(1-2\rho) \\ &\ge (1-2\rho)|X-Y|_2. \end{aligned} \tag{48}$$

We denote P' and Q' as the closest points to P and Q, respectively (at the scale ϵ), in such a way that both P' and Q' are centers of exclusions. We have

$$d^{\rho}_{\epsilon}(P',Q') - \sqrt{2}\epsilon \le d^{\rho}_{\epsilon}(P,Q) \le |P-Q|_2.$$

Using (48), we obtain

$$(1-2\rho)|P'-Q'|_2 - \sqrt{2}\epsilon \le d^{\rho}_{\epsilon}(P,Q) \le |P-Q|_2.$$

Using the triangle inequality we have

$$(1-2\rho)(|P-Q|_2 - \sqrt{2}\epsilon) - \sqrt{2}\epsilon \le d^{\rho}_{\epsilon}(P,Q) \le |P-Q|_2.$$

Letting $\epsilon \to 0$ we have

$$1 - 2\rho \le \frac{d^{\rho}_0(P,Q)}{|P-Q|_2} \le 1.$$

This implies

$$\lim_{\rho\to 0} d^{\rho}_0(P,Q) = |P-Q|_2. \qquad \square$$

Figure 2 shows the unit balls of norms d^{ρ}_0 for the cases $\frac{2-\sqrt{2}}{2} < \rho < 0.5$, $\frac{1+\sqrt{2}-\sqrt{5}}{2} < \rho \le \frac{2-\sqrt{2}}{2}$, and $\frac{1+\sqrt{5}-\sqrt{10}}{2} < \rho \le \frac{1+\sqrt{2}-\sqrt{5}}{2}$. Our results suggest that as ρ gets smaller the behavior of the unit ball changes, though it is always polygonal with more and mores edges until it becomes a circle in the limit. That is, *as* $\rho \to 0$, *the sequence of norms converges to the Euclidean norm.*

Remark 7.1. The norms d_0^ρ can be thought of as an example of the so-called stable norms (see, for instance, [33, 28, 11, 6, 10, 5, 4] and the references therein). However, in this paper we are interested in looking at these norms in the context of Hamilton–Jacobi equations in order to provide an explicit example of homogenization of Hamilton–Jacobi equations. As mentioned earlier, finding explicit formulas for the effective Hamiltonian $\overline{H}$ is essentially still an open problem.

Remark 7.2. Theorems 7.2 and 7.3 provide, for $n = 2$, an explicit formula for u_0, which is the uniform limit of the solutions of (40). The homogenization of (40) is difficult to achieve. The construction of the corresponding effective Hamiltonian $\overline{H}$ does not follow from [30] due to the behavior of the functions u^ϵ on the boundaries of the exclusions.

Appendix A.

We refer to the standard references [27, 25, 2] for the details related to the theory of sets of finite perimeter.

DEFINITION A.1. *Throughout this paper, we denote $B(x,r)$ as the open ball centered at x and radius r (we shall also use the notation B_r when $x = 0$). We denote $\mathcal{H}_k$ as the k-dimensional Hausdorff measure in $\mathbb{R}^n$, and $\mathcal{L}^n$ denotes the Lebesgue measure in $\mathbb{R}^n$. We recall that $\mathcal{H}_n = \mathcal{L}^n$. At times we shall denote $|E|$ as the $\mathcal{L}^n$-Lebesgue measure of E.*

DEFINITION A.2 (see [27, p. 4]). *We let $\Omega \subset \mathbb{R}^n$ denote an open set. If $f \in L^1(\Omega)$, we define*

$$\int_\Omega |Df| = \sup\left\{\int_\Omega f \operatorname{div} g : g \in C_0^1(\Omega;\mathbb{R}^n), \quad |g(x)| \le 1, \quad \text{for} \quad x \in \Omega\right\}.$$

DEFINITION A.3. *A function $f \in L^1(\Omega)$ is said to have bounded variation in Ω if $\int_\Omega |Df| < \infty$. We define $BV(\Omega)$ as the space of all functions in $L^1(\Omega)$ with bounded variation. With the norm $|f|_{BV} = |f|_{L^1(\Omega)} + \int_\Omega |Df|$, $BV(\Omega)$ is a Banach space. If $f \in BV(\Omega)$, then Df, the gradient of f in the sense of distributions, is a vector valued Radon measure in Ω with total variation $|Df|$. Thus we may extend the definition of $\int_A |Df|$ to include cases where $A \subset \Omega$ is not necessarily open.*

DEFINITION A.4. *If E denotes a Borel set, we define the perimeter of E in Ω as*

$$\mathrm{Per}(E,\Omega) = \int_\Omega |D\varphi_E|,$$

where φ_E is the characteristic function of the set E. If $\mathrm{Per}(E,\Omega) < \infty$ *for every bounded open set Ω, then E is called a set of locally finite perimeter in $\mathbb{R}^n$. For simplicity, we will denote a set of locally finite perimeter in $\mathbb{R}^n$ simply as a* set of finite perimeter. *Also, at times we shall denote* $\mathrm{Per}(E,\mathbb{R}^n)$ *simply as* $\mathrm{Per}(E)$.

DEFINITION A.5 (see [27, p. 43]). *Let E be a set of finite perimeter. We call the* reduced boundary *of E, denoted as $\partial^* E$, the set of all points $x \in \operatorname{supp}|D\varphi_E|$ such that*

- $\int_{B(x,r)} |D\varphi_E| > 0$ *for all $r > 0$;*
- *the limit* $\nu(x) = \lim_{r\to 0} \frac{\int_{B(x,r)} D\varphi_E}{\int_{B(x,r)} |D\varphi_E|}$ *exists;*
- $|\nu(x)| = 1$.

Definition A.6. *For every* $\gamma \in [0,1]$ *and every* $\mathcal{L}^n$*-measurable set* $E \subset \mathbb{R}^n$*, we define*

$$E_\gamma = \left\{ x \in \mathbb{R}^n : \lim_{r \to 0} \frac{|B(x,r) \cap E|}{|B(x,r)|} = \gamma \right\}, \tag{49}$$

the set of all points with density γ*. If* E *is a set of finite perimeter, then (cf.* [2]*) the limit in* (49) *exists for* $\mathcal{H}_{n-1}$*-almost every* x*. The sets* E_1 *and* E_0 *are the measure theoretic interior and exterior of* E*, respectively.*

Definition A.7. *We say that the set of finite perimeter* E *has* least area *in the open set* Ω *if*

$$\int_\Omega |D\varphi_E| = \inf \left\{ \int_\Omega |D\varphi_F| : F \text{ is a set of finite perimeter, } \operatorname{support}(\varphi_F - \varphi_E) \subset \Omega \right\}.$$

Definition A.8. *If* E *is a set of finite perimeter, we denote* ∂E *as the topological boundary of* E*. We note that* $E_{int} \subset E_1$ *and* $E_{out} \subset E_0$*, where* E_{int} *denotes the topological interior of the set* E*, and* $E_{out} = (\mathbb{R}^n \setminus E)_{int}$*. We define*

$$\partial^s E = \mathbb{R}^n \setminus (E_0 \cup E_1).$$

The set $\partial^s E$ *is called the* essential boundary *of* E*. We have*

$$\partial^* E \subset E_{\frac{1}{2}} \subset \partial^s E$$

and

$$\mathcal{H}_{n-1}(\partial^s E \setminus \partial^* E) = 0.$$

We have that

$$|D\varphi_E| = \mathcal{H}_{n-1}|_{\partial^* E}. \tag{50}$$

Remark A.1. When considering functions in BV we are really considering equivalence classes of functions, and changing a function on a set of measure zero gives the same function. The same is true for sets of finite perimeter, and, therefore, since we are concerned only with equivalence classes of sets, *we assume throughout this paper that a set of finite perimter* E *is the representative given by Theorem* A.1. With this convention, there is no ambiguity when speaking of the topological boundary of a set of finite perimeter.

Remark A.2. Standard interior regularity theory [12, 13, 26, 27, 18] implies that, if $n \leq 7$ and E is a set of finite perimeter that has least area in the open set Ω, then $\partial E \cap \Omega$ is a smooth surface. If $n > 7$, $\partial E \cap \Omega$ can have singularities, but they have zero $\mathcal{H}_k$-measure for any $k > n - 8$. At times we will use the word “surface” to denote the boundary of a set of finite perimeter, although this boundary could have singularities.

Proposition A.1 (see [27, p. 7]). *If* $\{f_j\}$ *denote a sequence of functions in* $BV(\Omega)$ *that converge in* $L^1_{loc}(\Omega)$ *to a function* f*, then the following semicontinuity property holds:*

$$\int_\Omega |Df| \leq \liminf_{j \to \infty} \int_\Omega |Df_j|.$$

THEOREM A.1 (see [27, p. 42]). *If E is a Borel set, then there exists a Borel set $\tilde{E}$ equivalent to E (that is, differs only by a set of $\mathcal{L}^n$-measure zero) and such that*

$$0 < |\tilde{E} \cap B(x,r)| < \omega_n r^n$$

for all $x \in \partial\tilde{E}$ and all $r > 0$, where ω_n is the measure of the unit ball in $\mathbb{R}^n$.

THEOREM A.2 (see [27, p. 17]). *If Ω is a bounded open set in $\mathbb{R}^n$ with Lipschitz continuous boundary, then sets of functions uniformly bounded in a BV norm are relatively compact in $L^1(\Omega)$.*

Since we are regarding $BV(\Omega)$ as a subset of $L^1(\Omega)$, it makes no sense to talk about the value of a BV function on sets of measure zero. However, it is important to be able to talk about the value of a BV function on the boundary of a set even though such a boundary may have measure zero; that is, we need a notion of *trace of a BV function on the boundary of the set*. The following theorem provides such a trace, which depends on the value of the function on the surroundings of the set.

THEOREM A.3 (see [27, p. 37]). *If Ω is a bounded open set with Lipschitz continuous boundary $\partial\Omega$ and $f \in BV(\Omega)$, then there exists a function $f_{tr} \in L^1(\partial\Omega)$ such that, for $\mathcal{H}_{n-1}$-almost all $x \in \partial\Omega$,*

$$\lim_{r\to 0} \int_{B(x,r)\cap\Omega} |f(z) - f_{tr}(x)| dz = 0,$$

and f_{tr} is called the trace function.

THEOREM A.4 (see [27, p. 172]). *We let A and B denote two sets of finite perimeter. If Ω is any open set, then*

$$\mathrm{Per}(A\cap B,\Omega) + \mathrm{Per}(A\cup B,\Omega) \le \mathrm{Per}(A,\Omega) + \mathrm{Per}(B,\Omega).$$

Proof. We let f, g be two smooth functions with $0 \le f \le 1$, $0 \le g \le 1$. We define $\Psi = f + g - fg$ and $\Phi = fg$. We note that

$$\int_\Omega |D\Psi| \le \int_\Omega (1-f)|Dg| + \int_\Omega (1-g)|Df|,$$
$$\int_\Omega |D\Phi| \le \int_\Omega f|Dg| + \int_\Omega g|Df|.$$

This implies

$$\int_\Omega |D\Phi| + \int_\Omega |D\Psi| \le \int_\Omega |Df| + \int_\Omega |Dg|. \tag{51}$$

We can find [27] sequences of smooth functions f_j and g_j such that $f_j \to \varphi_A, g_j \to \varphi_B$ in $L^1(\Omega)$ and $\int_\Omega |Df_j| \to \int_\Omega |D\varphi_A|$, $\int_\Omega |Dg_j| \to \int_\Omega |D\varphi_B|$. Since $\Psi_j = f_j + g_j - f_j g_j \to \varphi_{A\cup B}$, $\Phi_j = f_j g_j \to \varphi_{A\cap B}$, the theorem follows from (51) and Proposition A.1. ◻

THEOREM A.5 (see [27, p. 173]). *Let $E = E_1 \cup E_2$, and let $\mathcal{H}_{n-1}(\overline{E}_1 \cap \overline{E}_2) = 0$. Then for any open set A we have*

$$\int_A |D\varphi_E| = \int_A |D\varphi_{E_1}| + \int_A |D\varphi_{E_2}|. \tag{52}$$

Moreover, if E has least area in A, the same is true for E_1 and E_2.

LEMMA A.1 (see [27, p. 28]). *Let $f \in BV(\Omega)$. If $A \subset\subset \Omega$ is an open set with Lipschitz continuous boundary ∂A, then $f|_A$ and $f|_{\Omega\setminus\overline{A}}$ belong to $BV(A)$ and $BV(\Omega\setminus\overline{A})$, respectively, and*

$$\int_{\partial A} |Df| = \int_{\partial A} |f_A^- - f_A^+| d\mathcal{H}_{n-1},$$

where $f_A^- = (f_A)_{tr}$ and $f_A^+ = (f|_{\Omega\setminus\overline{A}})_{tr}$, the traces on ∂A of $f|_A$ and $f|_{\Omega\setminus\overline{A}}$, respectively.

LEMMA A.2. *If E is a set of finite perimeter and $x \in \mathbb{R}^n$, then, for almost every r,*

$$\mathrm{Per}(E \cap B(x,r), \mathbb{R}^n) = \mathrm{Per}(E, B(x,r)) + \mathcal{H}_{n-1}(E \cap \partial B(x,r)).$$

Proof. We denote

$$F(x) = \begin{cases} \varphi_E(x), & x \in B(x,r), \\ 0, & x \in \mathbb{R}^n \setminus B(x,r). \end{cases} \tag{53}$$

From Lemma A.1 and using (53) we have

$$\int_{\mathbb{R}^n} |DF| = \int_{B(x,r)} |D\varphi_E| + \int_{\partial B(x,r)} |(\varphi_E)_{tr}| d\mathcal{H}_{n-1}. \tag{54}$$

The lemma follows from (54) since $\int_{\mathbb{R}^n} |DF| = \mathrm{Per}(E \cap B(x,r), \mathbb{R}^n)$ and $\varphi_E = (\varphi_E)_{tr}$ for almost every r. ◻

LEMMA A.3 (see [27, p. 25]). *If E is a set of finite perimeter and $x \in \mathbb{R}^n$, then, for every r,*

$$|E|^{\frac{n-1}{n}} \leq C(n)\mathrm{Per}(E, \mathbb{R}^n).$$

LEMMA A.4 (see [27, p. 25]). *If E is a set of finite perimeter and $x \in \mathbb{R}^n$, then, for every r,*

$$\min\{|E \cap B(x,r)|, |(\mathbb{R}^n \setminus E) \cap B(x,r)|\}^{\frac{n-1}{n}} \leq C(n) \int_{B(x,r)} |D\varphi_E|.$$

LEMMA A.5. *Let E be a set of finite perimeter that minimizes area in the open set Ω. If $x \in \partial E \cap \Omega$ has density γ_x (see Definition A.6), then $0 < \gamma_x < 1$.*

Proof. We take $x \in \partial E \cap \Omega$. Let $r_0 > 0$ such that $B(x, r_0) \subset \Omega$. We now prove that there exist universal constants C_1, C_2 such that, for all $r \leq r_0$,

$$|B(x,r) \cap E| \geq C_1 r^n, \quad |B(x,r) \cap (\mathbb{R}^n \setminus E)| \geq C_2 r^n. \tag{55}$$

The computation that gives the first part of (55) is contained in the proof of Lemma 5.2. The second part (i.e., for the complement of E) is proven in the same way, and we present here again the argument since (55) is a fundamental property of minimal surfaces. We let $F = \mathbb{R}^n \setminus E$. For all $r \leq r_0$ we have

$$\int_{B(x,r)} |D\varphi_F| \leq \mathcal{H}_{n-1}(F \cap \partial B(x,r)). \tag{56}$$

We define $V(r) = |F \cap B(x,r)|$, $r \le r_0$. Using the isoperimetric inequality given in Lemma A.3 we have that

$$|F \cap B(x,r)| \le C[\mathrm{Per}(F \cap B(x,r), \mathbb{R}^n)]^{\frac{n}{n-1}}.$$

Proceeding as in Lemma A.2 we can prove that $\mathrm{Per}(F \cap B(x,r), \mathbb{R}^n) = \mathrm{Per}(F, B(x,r)) + \mathcal{H}_{n-1}(F \cap \partial B(x,r))$ for almost every $r \le r_0$, and hence

$$\begin{aligned} |F \cap B(x,r)| &\le C[\mathrm{Per}(F \cap B(x,r), \mathbb{R}^n)]^{\frac{n}{n-1}} \\ &= C[\mathrm{Per}(F, B(x,r)) + \mathcal{H}_{n-1}(F \cap \partial B(x,r))]^{\frac{n}{n-1}} \\ &\le C[\mathcal{H}_{n-1}(F \cap \partial B(x,r))]^{\frac{n}{n-1}}. \end{aligned}$$

Due to Remark A.1 it follows that $V(r) > 0$ for all $r \le r_0$. Since $V'(r) = \mathcal{H}_{n-1}(F \cap \partial B(x,r))$ we have , for almost every $r \le r_0$,

$$V(r) \le C V'(r)^{\frac{n}{n-1}}. \tag{57}$$

If we divide (57) by $V(r)$ and integrate we obtain $V(r)^{\frac{1}{n}} \ge Cr$; i.e.,$V(r) \ge Cr^n$. Now, from (55) we have

$$C_1 r^n \le |B(x,r) \cap E| = |B(x,r)| - |B(x,r) \cap (\mathbb{R}^n \setminus E)| \le |B(x,r)| - C_2 r^n.$$

Therefore

$$0 < \tilde{C}_1 \le \frac{|B(x,r) \cap E|}{|B(x,r)|} \le \tilde{C}_2 < 1,$$

where $\tilde{C}_1$ and $\tilde{C}_2$ are two universal constants. Taking limit as $r \to 0$ and from Definition A.6 we obtain that $0 < \gamma_x < 1$. □

LEMMA A.6. *If E is a minimizer corresponding to the class A_{S_1,S_2}, and if the exclusions have at least C^1 boundaries, then there exists a universal constant C such that the set $F \equiv \mathbb{R}^n \setminus (E \cap O)$ satisfies*

$$|F \cap B(x,r)| \ge Cr^n \tag{58}$$

for all $x \in \partial F$, $r \le r_0$, where r_0 is a universal constant.

Proof. We take $x \in \partial F$ and $r < \frac{\alpha}{2}$. We have different situations according to the location of $B(x,r)$. In each case, however, the density estimate (58) can be obtained as in Lemma A.5 from the isoperimetric inequality given in Lemma A.3. In fact, if $B(x,r)$ does not intersect any exclusion or the parallel plane restrictions, then we proceed exactly as in Lemma A.5. We consider now the cases

1. $B(x,r)$ intersects Π_1 (the lower parallel plane restriction) and/or an exclusion.
2. $B(x,r)$ intersects Π_2 (the upper parallel plane restriction) and/or an exclusion.

In case 1, we proceed as in Lemma A.5 with $V(r) = |F \cap B(x,r) \cap O|$, applying the isoperimetric inequality given in Lemma A.3 to the domain $|F \cap B(x,r) \cap O|$. In order to estimate $\mathrm{Per}(F \cap B(x,r) \cap O)$ we use the fact that ∂E is a free boundary (in the sense that we do not impose any restriction as to how the minimizer E meets the exclusions), and hence $\mathrm{Per}(F, B(x,r) \cap O) \le \mathcal{H}_{n-1}(\partial B(x,r) \cap F \cap O)$. If $B(x,r)$ intersects the exclusion I, then (while computing $\mathrm{Per}(F \cap B(x,r) \cap O)$) we can estimate $H_{n-1}(\partial I \cap B(x,r) \cap F)$ by performing a change of variables to flatten the boundary of the exclusion. In case 2, if $B(x,r)$ intersects Π_2 and more than half the ball $B(x,r)$ is outside the restrictions, then (58) is clear, but, if not, then we consider $B(x, \frac{r}{2})$, and we proceed as in case 1. ∎

LEMMA A.7. *Let E be a set of finite perimeter in $\mathbb{R}^n$, and let $F = \mathbb{R}^n \setminus E$. If there exists a universal constant C such that*

$$|F \cap B(x,r)| \geq Cr^n \tag{59}$$

for all $x \in \partial F$ and all $r \leq \tilde{r}$, then there exists a sequence of C^∞ sets $E_{\epsilon_k} \subset\subset E$ converging in measure to E and such that

$$\lim_{\epsilon_k \to 0} \mathrm{Per}(E_{\epsilon_k}, \mathbb{R}^n) = \mathrm{Per}(E, \mathbb{R}^n).$$

Proof. The proof of this lemma is an improvement of Theorem 3.42 in [2] under the extra condition (59). In fact, we consider the standard mollified functions $u_\epsilon = \varphi_E * \rho_\epsilon$ and $v_\epsilon = \varphi_F * \rho_\epsilon$, where spt $\rho \subset B_1$, $\rho \equiv 1$ on $B(x, \frac{1}{2})$, and $\rho_\epsilon = \frac{1}{\epsilon^n}\rho(\frac{\cdot}{\epsilon})$. We note that $u_\epsilon + v_\epsilon = 1$. If $x \in \partial F$ and $\epsilon < \tilde{r}$, we obtain from (59)

$$\begin{aligned} v_\epsilon(x) &= \frac{1}{\epsilon^n}\int_{B(x,\epsilon)\cap F} \rho\left(\frac{x-y}{\epsilon}\right) dy \\ &\geq \frac{1}{\epsilon^n}\int_{B(x,\frac{\epsilon}{2})\cap F} \rho\left(\frac{x-y}{\epsilon}\right) dy \\ &= \frac{1}{\epsilon^n}\left|B\left(x, \frac{\epsilon}{2}\right) \cap F\right| \\ &\geq C\epsilon^{-n}\epsilon^n = C. \end{aligned}$$

Therefore, we can choose t close enough to 1 so that

$$\{v_\epsilon < 1 - t\} = \{u_\epsilon > t\} \subset\subset E. \tag{60}$$

Using an exercise problem in [2, p. 39] we have that, for almost every $t \in (0, 1)$,

$$\lim_{\epsilon \to 0} \mathrm{Per}(\{u_\epsilon > t\}, \mathbb{R}^n) = \mathrm{Per}(E, \mathbb{R}^n). \tag{61}$$

Hence, we choose t such that (60) and (61) holds, and we define $E_{\epsilon_k} \equiv \{u_{\epsilon_k} > t\}$. We can now conclude as in [2]. ∎

Acknowledgments. I am greatly indebted to Luis A. Caffarelli for introducing me to this subject. I am also very thankful to Lawrence C. Evans, Rafael de la Llave, Ovidiu Savin, and the referees for many useful comments. The proof of Lemma A.7 was communicated by Luigi Ambrosio.

REFERENCES

[1] O. ALVAREZ, *Homogenization of Hamilton-Jacobi equations in perforated sets*, J. Differential Equations, 159 (1999), pp. 543–577.

[2] L. AMBROSIO, N. FUSCO, AND D. PALLARA, *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2000.

[3] F. AUER, *Minimale hyperflächen in riemannsche n-torus*, Albert-Ludwigs Univ. thesis, Freiburg, Germany, 1997.

[4] F. AUER AND V. BANGERT, *Minimising currents and the stable norm in codimension one*, C. R. Acad. Sci. Paris Sér. I Math., 333 (2001), pp. 1095–1100.

[5] V. Bangert, *Minimal geodesics*, Ergodic Theory Dynam. Systems, 10 (1990), pp. 263–286.

[6] V. Bangert, *Minimal foliations and laminations*, in Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994), Birkhäuser, Basel, 1995, pp. 453–464.

[7] M. Bardi, M. G. Crandall, L. C. Evans, H. M. Soner, and P. E. Souganidis, *Viscosity Solutions and Applications*, Lecture Notes in Math. 1660, I. Capuzzo Dolcetta and P. L. Lions, eds., Springer-Verlag, Berlin, 1997.

[8] M. Bardi and I. Capuzzo-Dolcetta, *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations*, Birkhäuser Boston, Boston, MA, 1997.

[9] G. Barles and P. E. Souganidis, *On the large time behavior of solutions of Hamilton–Jacobi equations*, SIAM J. Math. Anal., 31 (2000), pp. 925–939.

[10] D. Burago, *Periodic metrics*, in Seminar on Dynamical Systems (St. Petersburg, 1991), Progr. Nonlinear Differential Equations Appl. 12, Birkhäuser, Basel, 1994, pp. 90–95.

[11] D. Burago, S. Ivanov, and B. Kleiner, *On the structure of the stable norm of periodic metrics*, Math. Res. Lett., 4 (1997), pp. 791–808.

[12] L. A. Caffarelli and A. Córdoba, *An elementary regularity theory of minimal surfaces*, Differential Integral Equations, 6 (1993), pp. 1–13.

[13] L. A. Caffarelli and A. Córdoba, *Correction: "An elementary regularity theory of minimal surfaces"* [Differential Integral Equations, 6 (1993), pp. 1–13], Differential Integral Equations, 8 (1995), p. 223.

[14] L. A. Caffarelli and R. de la Llave, *Planelike minimizers in periodic media*, Comm. Pure Appl. Math., 54 (2001), pp. 1403–1441.

[15] I. Capuzzo-Dolcetta and H. Ishii, *On the rate of convergence in homogenization of Hamilton-Jacobi equations*, Indiana Univ. Math. J., 50 (2001), pp. 1113–1129.

[16] M. C. Concordel, *Periodic homogenization of Hamilton-Jacobi equations: Additive eigenvalues and variational formula*, Indiana Univ. Math. J., 45 (1996), pp. 1095–1117.

[17] M. C. Concordel, *Periodic homogenisation of Hamilton-Jacobi equations.* II. *Eikonal equations*, Proc. Roy. Soc. Edinburgh Sect. A, 127 (1997), pp. 665–689.

[18] E. De Giorgi, *Frontiere orientate di misura minima*, Seminario di Matematica della Scuola Normale Superiore di Pisa, 1960-61, Editrice Tecnico Scientifica, Pisa, 1961.

[19] L. C. Evans and D. Gomes, *Effective Hamiltonians and averaging for Hamiltonian dynamics.* I, Arch. Ration. Mech. Anal., 157 (2001), pp. 1–33.

[20] L. C. Evans and D. Gomes, *Effective Hamiltonians and averaging for Hamiltonian dynamics.* II, Arch. Ration. Mech. Anal., 161 (2002), pp. 271–305.

[21] L. C. Evans, *The perturbed test function method for viscosity solutions of nonlinear PDE*, Proc. Roy. Soc. Edinburgh Sect. A, 111 (1989), pp. 359–375.

[22] L. C. Evans, *Periodic homogenisation of certain fully nonlinear partial differential equations*, Proc. Roy. Soc. Edinburgh Sect. A, 120 (1992), pp. 245–265.

[23] L. C. Evans, *Partial differential equations*, Graduate Stud. Math. 19, AMS, Providence, RI, 1998.

[24] L. C. Evans, *Effective Hamiltonians and quantum states*, in Séminaire: Équations aux Dérivées Partielles, 2000–2001, Sémin. Équ. Dériv. Partielles, Exp. No. XXII, École Polytech., Palaiseau, 2001.

[25] L. C. Evans and R. F. Gariepy, *Measure Theory and Fine Properties of Functions*, CRC Press, Boca Raton, FL, 1992.

[26] H. Federer, *Geometric Measure Theory*, Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer-Verlag, New York, 1969.

[27] E. Giusti, *Minimal Surfaces and Functions of Bounded Variation*, Birkhäuser Verlag, Basel, 1984.

[28] M. Goldberg, *Stable norms—examples and remarks*, in General inequalities, 7 (Oberwolfach, 1995), Internat. Ser. Numer. Math. 123, Birkhäuser, Basel, 1997, pp. 61–64.

[29] M. Grüter, *Optimal regularity for codimension one minimal surfaces with a free boundary*, Manuscripta Math., 58 (1987), pp. 295–343.

[30] K. Horie and H. Ishii, *Homogenization of Hamilton-Jacobi equations on domains with small scale periodic structure*, Indiana Univ. Math. J., 47 (1998), pp. 1011–1058.

[31] Z. Lin, L. Xiao-Biao, M. Torres, and Z. Hongkai, *Theoretical and numerical analysis of the weighted minimal surface problem*, Methods Appl. Anal., 10 (2003), pp. 199–214.

[32] P. L. Lions, G. Papanicolaou, and S. R. S. Varadhan, *Homogenization of Hamilton-Jacobi Equations*, manuscript.

[33] D. Massart, *Stable norms of surfaces: Local structure of the unit ball of rational directions*, Geom. Funct. Anal., 7 (1997), pp. 996–1010.

[34] L. Simon, *Lectures on Geometric Measure Theory*, Australian National University, Centre for Mathematical Analysis, Canberra, 1983.