A NONLINEAR MODEL FOR LONG-RANGE SEGREGATION

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ABSTRACT. We study a system of fully nonlinear elliptic equations, depending on a small parameter ε, that models long-range segregation of populations. The diffusion is governed by the negative Pucci operator. In the linear case, this system was previously investigated by Caffarelli, the second author, and Quitalo in [6] as a model in population dynamics. We establish the existence of solutions and prove convergence as $\varepsilon \to 0^+$ to a free boundary problem in which populations remain segregated at a positive distance. In addition, we show that the supports of the limiting functions are sets of finite perimeter and satisfy a semi-convexity property.

1. INTRODUCTION

In this paper, we study the following fully nonlinear system of elliptic equations: for $i = 1, \dots, K$,

$$\begin{cases} \mathcal{M}^-(u_i^\varepsilon) = \frac{1}{\varepsilon^2} u_i^\varepsilon \sum_{j \neq i} H_R(u_j^\varepsilon)(x) & \text{in } \Omega, \\ u_i^\varepsilon = f_i & \text{on } (\partial\Omega)_{\leq R}, \end{cases} \tag{1.1}$$

where Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, $\varepsilon > 0$, $0 < R \leq 1$. The boundary neighborhood is defined as

$$(\partial\Omega)_{\leq R} := \{x \in \Omega^c : d(x, \partial\Omega) \leq R\},$$

where $d(x, \partial\Omega) := \inf_{y \in \partial\Omega} |x - y|$ denotes the distance of x from $\partial\Omega$. The boundary data are nonnegative Hölder continuous functions with supports separated by at least distance R (see assumptions (1.6)-(1.8)).

The operator $\mathcal{M}^-$ is the negative extremal Pucci operator, which is uniformly elliptic and fully nonlinear (see Section 2 for its definition). Each equation in the system is coupled to the others through a nonlocal zero-order interaction term $H_R(u_j^\varepsilon)$, which depends on the parameter R. We consider two different cases for H_R:

$$H_R(w)(x) = \fint_{B_R(x)} w^p(y)\, dy, \quad p \geq 1, \tag{1.2}$$

and

$$H_R(w)(x) = \sup_{B_R(x)} w. \tag{1.3}$$

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We prove the existence of a positive solution $(u_1^\varepsilon, \dots, u_K^\varepsilon)$ to the system (1.1), and show that, up to a subsequence, these solutions converge as $\varepsilon \to 0^+$ to a limit configuration $(u_1, \dots, u_K)$. The supports of the limit functions u_i are mutually disjoint and are separated from each other by a distance of at least R. Furthermore, we begin the study of the geometric properties of the boundaries of these supports within Ω, the so-called free boundaries. Many of the proofs presented here are adaptations of arguments developed by Caffarelli, the second author, and Quitalo in [6], where system (1.1) is studied in the case of the Laplace operator.

A deeper analysis of the free boundary regularity, as well as the asymptotic behavior of solutions as $R \to 0^+$, will be addressed in a forthcoming paper. In particular, we will need to introduce a notion of regular point and, in order to accomplish this, we expect to be able to show that the distance between the supports of the limiting functions is exactly R. For $n = 2$ we expect the free boundaries to be Lipschitz curves and to have a finite number of singular points (edges). Motivated by [6], where the analysis of the free boundary was carried out without the use of monotonicity formulas, we also expect to be able to develop the analysis of the free boundaries for the problem considered in this paper without these formulas.

1.1. **Background and motivations.** In population dynamics, Gause-Lotka-Voltera models describe coexistence of species that live in the same territory, diffuse, and compete for limited resources.

One of the simplest forms of such models consists of a system of equations of the form: for $i = 1, \dots, K$ and $\varepsilon > 0$,

$$L_i(u_i^\varepsilon) = \frac{u_i^\varepsilon}{\varepsilon^2} F(u_1^\varepsilon, \dots, u_K^\varepsilon), \tag{1.4}$$

in some domain Ω, where u_i^ε is a positive function representing the density of the i-th species, L_i encodes the diffusion of u_i^ε, and $u_i^\varepsilon F(u_1^\varepsilon, \dots, u_K^\varepsilon)/\varepsilon^2$ models the attrition of the species i due to competition with the others. The interaction functional F is strictly positive whenever the supports of two or more species overlap. The smaller the parameter ε, the stronger the competition among species. In the limit as $\varepsilon \to 0^+$ the high competition forces the species to segregate, meaning $u_i u_j = 0$ for $j \neq i$, in a such a way that $u_i F(u_1, \dots, u_K) = 0$.

A classical example of system (1.4) is given by: for $i = 1, \dots, K$ and $\varepsilon > 0$,

$$\Delta u_i^\varepsilon = \frac{1}{\varepsilon^2} \sum_{j \neq i} u_i^\varepsilon u_j^\varepsilon. \tag{1.5}$$

The existence of positive solutions to (1.5) was initially investigated by Dancer and Du [16, 17] in the case of three species. Convergence to a segregated limit configuration as $\varepsilon \to 0^+$ was later proven by Dancer, Hilhorst, Mimura, and Peletier [18]. More general classes of linear competitive systems, including (1.5) as a special case, have been studied by Conti, Terracini, and Verzini [11, 12, 13]. For related optimal partition problems involving the first eigenvalue of the Laplace operator, see also [3, 9, 10]. The geometric properties of the free boundaries $\partial\{u_i > 0\} \cap \Omega$ have been investigated by Caffarelli, Karakhanyan and Lin [4, 5]. There, it is shown that each free boundary splits into two parts: a regular set, which is a locally analytic surface, and a singular set, which is a closed set of Hausdorff dimension at most $n-2$. Singular points occur where the boundaries of three or more connected components of the supports intersect. In two dimensions, such points correspond to junctions where the supports meet at equal angles. See also [28] for similar results applied to a broader class of systems.

The system (1.5) when the Laplace operator is replaced by the fully nonlinear Pucci operator, has been studied by Quitalo [27], who established the existence of positive solutions and convergence to a limiting segregated configuration. In the case of two populations, Caffarelli, Quitalo, and the second and third authors [7] showed that the limiting problem becomes a two-phase free boundary problem with the associated free boundary condition $\frac{\partial u_1}{\partial \nu_1} = \frac{\partial u_2}{\partial \nu_2}$, where ν_1 and ν_2 denote the interior normal directions to the respective supports. This formulation allowed for the application of the sup-convolutions method, originally developed by Caffarelli in the linear setting, to prove that the regular points form an open subset of the free boundary locally of class $C^{1,\alpha}$. For a comprehensive discussion of the sup-convolution method and the general theory of free boundaries, the reader is referred to the monograph by Caffarelli and Salsa [8]. For two-phase free boundary problems governed by fully nonlinear operators, we refer the reader to [1, 20, 19] and the references therein.

In all the works mentioned above, the interaction between populations is adjacent, meaning that $u_i(x)$ interacts with the other densities evaluated at the same point x. However, there are many processes where the growth of species i is inhibited by populations j occupying an entire neighborhood around x, see for example [15, 26]. As a first step in studying this nonlocal interaction, Caffarelli, the second author, and Quitalo [7] introduced system (1.1) with the Laplace operator and $R = 1$. They proved existence of solutions and convergence to a limiting segregated configuration. Unlike the case of adjacent segregation, here the species segregate at a distance of at least 1 from each other, with the distance exactly equal to 1 when H_1 is defined as in (1.2) with $p = 1$, or as in (1.3). This form of segregation ensures that species no longer interact in the limit. Moreover, under suitable assumptions on the boundary data, it was shown that in dimension 2 the free boundaries are Lipschitz curves, and the number of singular points (edges) is finite. These edges arise at points that are exactly distance 1 from two or more other connected components of the supports. At such points, the edges of the free boundaries all coincide. A free boundary condition was also derived.

One of the main challenges in studying segregation at a distance is that the classical monotonicity formulas, such as the Alt-Caffarelli-Friedman and Almgren monotonicity formulas, cannot be applied.

In the present paper, we study segregation at a distance governed by fully nonlinear diffusion, with long-range interaction at distance R, as modeled by system (1.1). We prove the existence of solutions and their convergence to a limiting segregated configuration, and we begin the analysis of the geometric properties of the resulting free boundaries. One of the main motivations for studying this problem is to gain a better understanding of the adjacent interaction model as $R \to 0^+$, namely the limiting free boundary problem of system (1.5) with the Laplace operator replaced by the fully nonlinear Pucci operator. In this nonlinear setting, many tools available in the linear theory, such as Almgren monotonicity formula and energy methods, are no longer applicable. From this perspective, the limiting free boundary problem of system (1.1) can be viewed as a regularization of the adjacent free boundary problem, since in the former, the free boundaries satisfy an exterior ball condition (see Theorem 1.3). In particular, we aim to gain a better understanding of the singular sets in the adjacent free boundary problem in dimension two, as no results were established on this topic in [7].

1.2. **Main results.** We assume the boundary data satisfy

$$f_i : (\partial\Omega)_{\leq R} \to \mathbb{R}, f_i \geq 0, f_i \neq 0, \quad f_i \text{ is Hölder continuous,} \tag{1.6}$$

and that there is a constant $c > 0$ such that for any $x \in \partial\Omega \cap \operatorname{supp} f_i$:

$$|B_r(x) \cap \operatorname{supp}\ f_i| \geq c|B_r(x)|, \tag{1.7}$$

and

$$d(\operatorname{supp} f_i, \operatorname{supp} f_j) \geq 1 \tag{1.8}$$

for all $i \neq j$.

Our first main result establishes existence of positive solutions to (1.1):

Theorem 1.1 (Existence of Solutions)**.** *Let Ω be a bounded Lipschitz domain of $\mathbb{R}^n$. Assume* (1.6) *holds true. Then for any $\varepsilon > 0$, and $0 < R \leq 1$, there exist positive functions $u_1^\varepsilon, \ldots, u_K^\varepsilon \in C^\alpha(\overline{\Omega}) \cap C^{2,\alpha}_{loc}(\Omega)$, for some $0 < \alpha < 1$, which are solutions of problem* (1.1)*.*

The next result concerns the limiting behavior of solutions as $\varepsilon \to 0^+$. As in the linear case, the supports of the limiting functions remain separated by at least distance R.

Theorem 1.2 (Limit Problem)**.** *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Assume* (1.6), (1.7) *and* (1.8) *hold true. For any $\varepsilon > 0$ and $0 < R \leq 1$, let $(u_1^\varepsilon, \ldots, u_K^\varepsilon)$ be a solution to* (1.1)*. Then there exists a subsequence of $\{(u_1^\varepsilon, \ldots, u_K^\varepsilon)\}_{\varepsilon>0}$ that converges locally uniformly in Ω to a limit function $(u_1, \ldots, u_K)$ as $\varepsilon \to 0^+$. Moreover, the limit function $(u_1, \ldots, u_K)$ has the following properties:*

(1) Each function u_i is locally Lipschitz continuous on Ω.
(2) $\mathcal{M}^-(u_i) = 0$ on $\{u_i > 0\} \cap \Omega$, for any $i = 1, \ldots, K$.
(3) For any $1 \leq i < j \leq K$, the supports of the function u_i and u_j are at distance at least R from each other. That is, u_i vanishes on the set $\{x \in \Omega: d(x, \operatorname{supp}\ u_j) \leq R\}$ for any $i \neq j$.

We now turn to the regularity properties of the sets $\{u_i > 0\} \cap \Omega$ and their corresponding free boundaries $\partial\{u_i > 0\} \cap \Omega$.

Let $(u_1, \ldots, u_K)$ be a subsequential limit of $\{(u_1^\varepsilon, \ldots, u_K^\varepsilon)\}_{\varepsilon>0}$ as in Theorem 1.2. The following two geometric properties hold:

Theorem 1.3 (A Semiconvexity Property of the Free Boundary). *If $x_0 \in \partial\{u_i > 0\} \cap \Omega$ for some $i = 1, \dots, K$, there is an exterior tangent ball of radius R at x_0.*

Theorem 1.4. *For each $i = 1, \dots, K$, the set $\{u_i > 0\} \cap \Omega$ has finite perimeter.*

1.3. **Organization of the paper.** The paper is organized as follows. In Section 2, we present regularity results and comparison principles for fully nonlinear equations that that will be used throughout the paper. In Section 3, we apply the Schauder fixed-point theorem to establish the existence of solutions to system (1.1) (Theorem 1.1). Section 4 is devoted to proving exponential decay properties for the functions u_i^ε, that are fundamental to let $\varepsilon \to 0^+$ and obtain convergence to locally Lipschitz limiting functions u_i (Theorem 1.2). In Section 5, we prove Theorems 1.3 and 1.4. Finally, in the Appendix, we provide a geometric property of the distance function that is used in Section 5.

1.4. **Notations.** In the paper, we will denote by $C > 0$ any constant depending only on the dimension n, the domain Ω and the boundary data f_i.

We let $B_r(x_0)$ denote the ball of radius $r > 0$ centered at $x_0 \in \mathbb{R}^n$.

Given a subset E of $\mathbb{R}^n$, we let ∂E, $\mathcal{H}^{n-1}(E)$, $|E|$, and $P(E)$ denote the topological boundary, $(n-1)$-dimensional Hausdorff measure, Lebesgue measure, and perimeter of the set E, respectively.

The support of a function f is denoted as supp f, and the average of f over the set E as $\fint_E f\,dx := \frac{1}{|E|}\int_E f\,dx$.

For $0 < \alpha < 1$ and $k \in \mathbb{N}$, we denote by $C^{k,\alpha}(E)$ the usual class of functions with bounded $C^{k,\alpha}$ norm over the domain $E \subset \mathbb{R}^n$. For $\alpha = 0$ we simply write $C^k(E)$. For $k = 0$ we simply write $C^\alpha(E)$.

For $p > 1$ and $k \in \mathbb{N}$, we denote by $W^{k,p}(E)$ the usual Sobolev space of functions which are $L^p(E)$ and whose weak derivatives are in $L^p(E)$ up to order k.

2. Preliminaries

In this section, we present some fundamental results for fully nonlinear elliptic equations that will be used throughout this paper. We begin by recalling the definition and some basic properties of Pucci extremal operators. Next, we review some key regularity results and comparison principles for viscosity solutions of equations involving Pucci operators. We then recall the Schauder fixed-point theorem. Finally, we finish with a remark on the scaling property of the operator H_R.

2.1. **The Pucci extremal operators.** Let $0 < \lambda \le \Lambda$ be given positive constants. For any $n \times n$ symmetric real matrix M, the Pucci's extremal operators are defined by

$$\mathcal{M}^-(M) := \lambda \sum_{e_i > 0} e_i + \Lambda \sum_{e_i < 0} e_i$$
$$\mathcal{M}^+(M) := \Lambda \sum_{e_i > 0} e_i + \lambda \sum_{e_i < 0} e_i,$$

where $e_i = e_i(M)$, $1 \le i \le n$, are the eigenvalues of the matrix M.

For a function u of class C^2, we define

$$\mathcal{M}^-(u) := \mathcal{M}^-(D^2u). \tag{2.1}$$

The operators $\mathcal{M}^-$ and $\mathcal{M}^+$ are fully nonlinear and uniformly elliptic. For a more comprehensive discussion on the general theory of fully nonlinear uniformly elliptic operators, we refer the reader to [2]. Moreover, these operators enjoy the following basic properties:

Lemma 2.1. *Let $0 < \lambda \le \Lambda$. For any $n \times n$ symmetric matrices M and N, the following properties hold:*

(1) $\mathcal{M}^-(M) = -\mathcal{M}^+(-M)$.
(2) $\mathcal{M}^\pm(tM) = t\mathcal{M}^\pm(M)$, *if* $t \ge 0$.
(3) $\mathcal{M}^-(M) \le \lambda trM$ *and* $\mathcal{M}^+(M) \ge \Lambda trM$.
(4) $\mathcal{M}^-(M) + \mathcal{M}^-(N) \le \mathcal{M}^-(M+N) \le \mathcal{M}^-(M) + \mathcal{M}^+(N)$.
(5) $\mathcal{M}^+(M) + \mathcal{M}^-(N) \le \mathcal{M}^+(M+N) \le \mathcal{M}^+(M) + \mathcal{M}^+(N)$.

Property (3) of Lemma 2.1 implies that for any C^2 function u, we have

$$\mathcal{M}^-(u) \le \lambda \Delta u, \tag{2.2}$$

which will be used several times later in the paper.

2.2. **Regularity results.** In this subsection, we recall some well-known regularity results for viscosity solutions of equations involving Pucci operators. For the definition of a viscosity solution, we refer the reader to [2, 14]. We begin with the Harnack inequality for Pucci operators:

Theorem 2.2. [2, Theorem 4.3] (Harnack Inequality)**.** *Let $u \in C(B_1(0))$ be non-negative in $B_1(0)$ and satisfy in the viscosity sense*

$$\mathcal{M}^+u \ge -|f| \quad \textit{and} \quad \mathcal{M}^-u \le |f| \quad \textit{in } B_1(0),$$

where f is a bounded continuous function in $B_1(0)$. Then

$$\sup_{B_{\frac{1}{2}}(0)} u \le C(\inf_{B_{\frac{1}{2}}(0)} u + \|f\|_{L^n(B_1(0))}),$$

where $C > 0$ is a universal constant.

Theorem 2.3. [2, Theorem 7.1] (Interior $W^{2,p}$ Regularity of Viscosity Solutions)**.** *Let $u \in C(B_1(0))$ be a viscosity solution of $\mathcal{M}^-(D^2u) = g$ in $B_1(0)$, with $g \in L^p(B_1(0))$, $n < p < \infty$. Then $u \in W^{2,p}(B_{\frac{1}{2}}(0))$ and*

$$\|u\|_{W^{2,p}(B_{\frac{1}{2}})} \le C(\|u\|_{C(B_1(0))} + \|f\|_{L^p(B_1(0))}),$$

where $C > 0$ is a universal constant.

Theorem 2.4. [2, Theorem 6.6] (Interior $C^{2,\alpha}$ Regularity of Viscosity Solutions)**.** *Let $u \in C(B_1(0))$ be a viscosity solution of $\mathcal{M}^-(D^2u) = 0$ in $B_1(0)$. Then $u \in C^{2,\alpha}(B_{\frac{1}{2}})$, and*

$$\|u\|_{C^{2,\alpha}(B_{\frac{1}{2}})} \le C\, \|u\|_{C(B_1)}\,,$$

where $0 < \alpha < 1$ and $C > 0$ are universal constants.

The following result combines interior $C^{2,\alpha}$ regularity with C^{α} regularity up to the boundary. For the interior $C^{2,\alpha}$ regularity for equations with Pucci operators and a C^{α} right-hand side, we refer to [2, Theorem 8.1]. For the C^{α} regularity up to the boundary in Lipschitz domains, see [24, Theorem VII.1] and [27, Proposition 3.10].

Theorem 2.5. *Let Ω be a bounded Lipschitz domain. Assume $g \in C(\Omega)$ and $f \in C^{\beta}(\partial\Omega)$, for some $0 < \beta < 1$. Let $u \in C(\overline{\Omega})$ be the viscosity solution of*

$$\begin{cases} \mathcal{M}^{-}u = g & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases}$$

Then, there exists $0 < \alpha \leq \beta$ such that $u \in C^{\alpha}(\overline{\Omega})$. If in addition $g \in C^{\gamma}(\Omega)$, for some $0 < \gamma < 1$, then there exists $0 < \alpha \leq \gamma$ such that $u \in C^{2,\alpha}_{loc}(\Omega)$.

2.3. **The comparison principle.** We will frequently use the following comparison principle throughout the paper, for which we refer the reader to [2, Theorem 3.6].

Theorem 2.6 (Comparison Principle)**.** *Let Ω be a bounded domain of $\mathbb{R}^n$. Let a, $g \in C(\Omega)$ with $a \geq 0$ in Ω. Assume that u, $v \in C(\overline{\Omega})$ satisfy in the viscosity sense*

$$\mathcal{M}^{-}u \leq a(x)u + g(x) \quad \text{and} \quad \mathcal{M}^{-}v \geq a(x)v + g(x) \quad \text{in } \Omega.$$

If $u \geq v$ on $\partial\Omega$, then $u \geq v$ in Ω.

The following minimum principle is an immediate corollary of the above comparison theorem.

Theorem 2.7 (Minimum Principle)**.** *Let Ω be a bounded domain of $\mathbb{R}^n$. Let $a \in C(\Omega)$ with $a \geq 0$ in Ω. Assume that $u \in C(\overline{\Omega})$ satisfy in the viscosity sense*

$$\mathcal{M}^{-}u \leq a(x)u \quad \text{in } \Omega.$$

If $u \geq 0$ on $\partial\Omega$, then $u \geq 0$ on Ω.

We conclude this subsection by recalling the so-called strong minimum principle ([2, Proposition 4.9]):

Theorem 2.8 (Strong Minimum Principle)**.** *Let Ω be a domain of $\mathbb{R}^n$, and assume $a \in C(\Omega)$. Let $u \in C(\overline{\Omega})$ satisfy in the viscosity sense*

$$\mathcal{M}^{-}u \leq a(x)u \quad \text{in } \Omega.$$

Assume that $u \geq 0$ in Ω and $u(x_0) = 0$ for some $x_0 \in \Omega$, then u vanishes identically in Ω.

A consequence of the above theorems is that if Ω is a bounded domain and u is a continuous function on Ω satisfying $\Delta u \geq 0$ in the viscosity sense, then $\sup_{\Omega} u = \sup_{\partial\Omega} u$. This result will be used several times later in the paper.

2.4. **The Schauder fixed-point theorem.** The following fixed-point theorem will be used to prove the existence of a solution to (1.1).

Theorem 2.9. *([23, Corollary 11.2]) Let B be a nonempty closed convex subset of a real Banach space X, and let $T : B \to B$ be continuous on B such that $T(B)$ is precompact. Then T has a fixed point.*

2.5. **Scaling properties of H_R.** Let H_R be defined as in (1.2) or (1.3). Let us define $w_{R,x}(y) = w(x + R(y - z))$. Then, we have the following scaling property:

$$H_1(w_{R,x})(z) = H_R(w)(x). \tag{2.3}$$

3. Existence of Solutions for the Model (1.1)

Using Theorem 2.9, we can prove Theorem 1.1.

Proof of Theorem 1.1: We adapt the proof in [6, Theorem 4.1] to the nonlinear case with the Pucci operator, see also [27, Theorem 2.2]. Let $X{:=}\ \{u = (u_1, \dots, u_K) : \overline{\Omega} \to \mathbb{R}^K\colon\ u$ is continuous on $\overline{\Omega}\}$, which is a real Banach space under the sup-norm defined by $\|u\|_\infty{:=}\ \max_{1\le j\le K}\|u_j\|_\infty$.

For each $1 \le i \le K$, let ϕ_i to be the (unique) viscosity solution of the problem

$$\begin{cases} \mathcal{M}^-(\phi_i) = 0 & \text{in } \Omega, \\ \phi_i = f_i & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

whose existence is guaranteed by Perron's method ([14, Theorem 4.1]).

Next, consider $B := \{u = (u_1, \dots, u_K) \in X\colon\ 0 \le u_i \le \phi_i$ in Ω, $u_i = f_i$ on $(\partial\Omega)_{\le R}$, $1 \le i \le K\}$. Clearly, B is a nonempty closed convex subset of X. We define a mapping $T^\varepsilon\colon\ B \to B$ by $T^\varepsilon(u_1, \dots, u_K) = (v_1^\varepsilon, \dots, v_K^\varepsilon)$ if and only if $(v_1^\varepsilon, \dots, v_K^\varepsilon)$ is the viscosity solution of the problem

$$\begin{cases} \mathcal{M}^-(v_i^\varepsilon) = \frac{1}{\varepsilon^2} v_i^\varepsilon \sum_{j\ne i} H_R(u_j)(x) & \text{in } \Omega, \\ v_i^\varepsilon = f_i & \text{on } (\partial\Omega)_{\le R}. \end{cases} \tag{3.2}$$

Note that the zero-order term in (3.2) is non-negative, so the comparison principle and consequently Perron's method apply. The proof of the existence of a viscosity solution of (1.1) is completed if we show that the mapping T^ε has a fixed point. To this end, by Theorem 2.9, it suffices to prove the following:

(1) T^ε maps B into B.
(2) T^ε is continuous on B.
(3) $T^\varepsilon(B)$ is precompact.

(1) T^ε maps B into B: Let $(u_1, \dots, u_K) \in B$, and $(v_1^\varepsilon, \dots, v_K^\varepsilon) = T^\varepsilon(u_1, \dots, u_K)$. We need to check that $(v_1^\varepsilon, \dots, v_K^\varepsilon) \in B$. Fix any $1 \le i \le K$. Since $v_i^\varepsilon = f_i \ge 0$ on $\partial\Omega$, by the maximum principle, we have $v_i^\varepsilon \ge 0$ in Ω. In particular, v_i^ε satisfies in the viscosity sense $\mathcal{M}^-(v_i^\varepsilon) \ge 0$ in Ω. Therefore, the comparison principle implies that $v_i^\varepsilon \le \phi_i$ in Ω. This shows that $(v_1^\varepsilon, \dots, v_K^\varepsilon) \in B$.

(2) T^ε is continuous on B: Let $\{(u_{1m}, \dots, u_{Km})\}_{m\in\mathbb{N}} \subset B$ be such that $(u_{1m}, \dots, u_{Km}) \to (u_1, \dots, u_K)$ uniformly in Ω as $m \to \infty$. Denote $(v_1^\varepsilon, \dots, v_K^\varepsilon) = T^\varepsilon(u_1, \dots, u_K)$

and $(v^\varepsilon_{1m},\dots,v^\varepsilon_{Km}) = T^\varepsilon(u_{1m},\dots,u_{Km})$, for all $m\in\mathbb{N}$. We need to check that $(v^\varepsilon_{1m},\dots,v^\varepsilon_{Km}) \to (v^\varepsilon_1,\dots,v^\varepsilon_K)$ uniformly in Ω as $m\to\infty$. To see this, it suffices to show that there is a constant $C_\varepsilon>0$ (which depends on ε, Ω, n and the boundary data), such that, for all $m\in\mathbb{N}$ and for all $1\le i\le K$,

$$\|v^\varepsilon_{im}-v^\varepsilon_i\|_\infty \le C_\varepsilon \max_{1\le j\le K}\|u_{jm}-u_j\|_\infty . \tag{3.3}$$

Fix $r>0$ large enough so that $\Omega\subset B_r(0)$. For any $m\in\mathbb{N}$ we consider the function

$$h_m(x) := D\max_{1\le j\le K}\|u_{jm}-u_j\|_\infty (r^2-|x|^2),$$

where $D>0$ is a constant to be chosen later. Since h_m is smooth, by Lemma 2.1, we see that, for all $1\le i\le K$, the function $h_m+v^\varepsilon_i$ satisfies in the viscosity sense,

$$\begin{aligned}\mathcal{M}^-(h_m+v^\varepsilon_i) &\le \mathcal{M}^+(h_m)+\mathcal{M}^-(v^\varepsilon_i)\\ &= -2n\lambda D\max_{1\le j\le K}\|u_{jm}-u_j\|_\infty + \frac{1}{\varepsilon^2}v^\varepsilon_i\sum_{j\ne i}H_R(u_j)(x)\\ &= -2n\lambda D\max_{1\le j\le K}\|u_{jm}-u_j\|_\infty + \frac{1}{\varepsilon^2}v^\varepsilon_i\sum_{j\ne i}\left(H_R(u_j)(x)-H_R(u_{jm})(x)\right)\\ &+\frac{1}{\varepsilon^2}v^\varepsilon_i\sum_{j\ne i}H_R(u_{jm})(x).\end{aligned} \tag{3.4}$$

We claim that there is a constant $C>0$ such that, for all $x\in\Omega$

$$\sum_{j\ne i}\left(H_R(u_j)(x)-H_R(u_{jm})(x)\right) \le (K-1)C\max_{1\le j\le K}\|u_{jm}-u_j\|_\infty . \tag{3.5}$$

To see this, note that if H_R is given by (1.3), we have, for any $j\ne i$ and $y\in B_R(x)$, $u_j(y)-u_{jm}(y)\le \max_{1\le j\le K}\|u_{jm}-u_j\|_\infty$, so that $u_j(y)\le \max_{1\le j\le K}\|u_{jm}-u_j\|_\infty + u_{jm}(y)$. Taking the supremum over all $y\in B_R(x)$ and then summing over over all $j\ne i$ yields (3.5) with $C=1$.

Now assume that H_R is given by (1.2). We then have, for $j\ne i$,

$$\begin{aligned}H_R(u_j)(x)-H_R(u_{jm})(x) &= ⨍_{B_R(x)}(u_j^p(y)-u_{jm}^p(y))\,dy\\ &\le ⨍_{B_R(x)} p\bar{C}^{p-1}|u_j(y)-u_{jm}(y)|\,dy\\ &\le p\bar{C}^{p-1}\max_{1\le j\le K}\|u_{jm}-u_j\|_\infty ,\end{aligned}$$

where $\bar{C}=\max_{1\le j\le K}\{\sup_{m\in\mathbb{N}}\|u_{jm}\|_\infty,\|u_j\|_\infty\}<\infty$. Taking the summation over $j\ne i$, we obtain (3.5) with $C=p\bar{C}^{p-1}$. From (3.4) and (3.5) we infer that, choosing $D=D_\varepsilon$ such that

$$D\ge \frac{\max_{1\le i\le K}\|\phi_i\|_\infty (K-1)C}{2n\lambda\varepsilon^2},$$

the function $h_m + v_i^\varepsilon$ satisfies in the viscosity sense

$$\mathcal{M}^-(h_m + v_i^\varepsilon) \leq \frac{1}{\varepsilon^2} v_i^\varepsilon \sum_{j\neq i} H_R(u_{jm})(x) \leq \frac{1}{\varepsilon^2}(h_m + v_i^\varepsilon) \sum_{j\neq i} H_R(u_{jm})(x).$$

Since in addition $h_m + v_i^\varepsilon \geq v_i^\varepsilon = f_i = v_{im}^\varepsilon$ on $\partial\Omega$, by the comparison principle we have $h_m + v_i^\varepsilon \geq v_{im}^\varepsilon$ in Ω, which implies that for all $x \in \Omega$,

$$v_{im}^\varepsilon(x) - v_i^\varepsilon(x) \leq h_m(x) \leq r^2 D \max_{1\leq j\leq K} \|u_{jm} - u_j\|_\infty .$$

Similarly, one can prove that $v_i^\varepsilon - v_{im}^\varepsilon \leq r^2 D \max_{1\leq j\leq K} \|u_{jm} - u_j\|_\infty$ in Ω. Estimate (3.3) follows. This concludes the proof of (2).

(3) $T^\varepsilon(B)$ is precompact: Let $\{(u_{1m}, \dots, u_{Km})\}_{m\in\mathbb{N}} \subset B$ be a bounded sequence in B and $(v_{1m}^\varepsilon, \dots, v_{Km}^\varepsilon)= T^\varepsilon(u_{1m}, \dots, u_{Km}) \in B$, for $m \in \mathbb{N}$. By Theorem 2.5, there is $0 < \alpha < 1$ such that $\{(v_{1m}^\varepsilon, \dots, v_{Km}^\varepsilon)\}$ is bounded in $C^\alpha(\overline{\Omega}; \mathbb{R}^K)$. By the compact embedding of $C^\alpha(\overline{\Omega}; \mathbb{R}^K)$ in X, there is a subsequence of $\{(u_{1m}, \dots, u_{Km})\}$ that converges in B. This shows that $T^\varepsilon(B)$ is precompact.

By (1)-(3) and Theorem 2.9, T^ε has a fixed point in B. This completes the proof of the existence of a viscosity solution $(u_1^\varepsilon, \dots, u_K^\varepsilon)$ to (1.1). Moreover, by the strong minimum principle, each function u_i^ε is positive in Ω.

By Theorem 2.5 and (1.6), $u_i^\varepsilon \in C^\alpha(\Omega \cup (\partial\Omega)_{\leq R})$ for all $1 \leq i \leq K$ and some $0 < \alpha < 1$. Let us show that this implies that $H_R(u_i^\varepsilon) \in C^\alpha(\Omega)$ for all $1 \leq i \leq K$. Let x_1, $x_2 \in \Omega$. First, assume that H_R is defined as in (1.2), then

$$\begin{aligned}
|H_R(u_i^\varepsilon)(x_1) - H_R(u_i^\varepsilon)(x_2)| &= \left| \fint_{B_R(0)} [(u_i^\varepsilon)^p(x_1 + y) - (u_i^\varepsilon)^p(x_2 + y)]\, dy \right| \\
&\leq p\|u_i^\varepsilon\|_\infty^{p-1} \fint_{B_R(0)} |u_i^\varepsilon(x_1 + y) - u_i^\varepsilon(x_2 + y)|\, dy \\
&\leq C_\varepsilon |x_1 - x_2|^\alpha,
\end{aligned}$$

as desired.

Next, assume that H_R is defined as in (1.3). Let $z_1 \in \overline{B_R(x_1)}$ be such that $H_R(u_i^\varepsilon)(x_1) = u_i^\varepsilon(z_1)$ and define $z_2 := z_1 + x_2 - x_1$. Note that $|z_2 - x_2| \leq R$, so $z_2 \in \overline{B_R(x_2)}$, and hence

$$H_R(u_i^\varepsilon)(x_1) - H_R(u_i^\varepsilon)(x_2) \leq u_i^\varepsilon(z_1) - u_i^\varepsilon(z_2) \leq C_\varepsilon |z_1 - z_2|^\alpha = C_\varepsilon |x_1 - x_2|^\alpha.$$

This implies that $H_R(u_i^\varepsilon) \in C^\alpha(\Omega)$.

Since the product of functions in $C^\alpha(\Omega)$ belongs to $C^\alpha(\Omega)$, we deduce that the right-hand side of equation (1.1) lies in this space. Applying Theorem 2.5 once more, we conclude that $u_i^\varepsilon \in C^\alpha(\overline{\Omega}) \cap C^{2,\alpha}_{loc}(\Omega)$ for all $1 \leq i \leq K$ and some $0 < \alpha < 1$. □

4. Uniform Estimates and the Limit Problem

This section is dedicated to the proof of Theorem 1.2. We will show that a subsequence of $\{(u_1^\varepsilon, \dots, u_K^\varepsilon)\}_{\varepsilon>0}$ converges uniformly on compact subsets of Ω to a limit function $(u_1, \dots, u_K)$ by establishing a Lipschitz estimate that is uniform in ε. To this end, we will prove that each function u_i^ε exhibits exponential decay in neighborhoods of size R around the regions where the other functions u_j^ε, $j \neq i$, stay away from zero, see Lemmas 4.4 and 4.5 below. This will ensure that the right-hand side of (1.1) go to 0 uniformly as $\varepsilon \to 0^+$ in those regions. We begin by stating two lemmas that are fundamental to establishing this exponential decay.

Lemma 4.1. *([6, Lemma 5.1]) Let ω be a subharmonic function in $B_1(0)$ such that $\omega \le 1$ in $B_1(0)$ and $\omega(0) = m > 0$. Also, let $D_0 \subset \mathbb{R}^n$ be a smooth domain with curvatures bounded by a positive constant C_0. Then there exists a universal constant $\tau_0 = \tau_0(n, C_0) > 0$ such that if $d(D_0, 0) \le \tau_0 m$, then $\sup_{\partial D_0 \cap B_1(0)} \omega \ge \frac{m}{2}$.*

Lemma 4.2. *([6, Lemma 5.2]) Let $\omega \in C(B_r(0))$ be a positive viscosity subsolution of the linear uniformly elliptic equation $a_{ij} D_{ij}\omega = \theta^2 \omega$ in $B_r(0)$. Then there exist two constants $c, C > 0$ such that $\frac{w(0)}{\sup_{B_r(0)} w} \le Ce^{-c\theta r}$.*

We will also use the following result whose proof follows [6, Lemma 5.3].

Lemma 4.3. *Let $0 < r < 1$, and let ω be a function satisfying the following conditions in $B_{2r}(0)$:*

(1) $0 \le \omega \le 1$;
(2) ω is subharmonic;
(3) $\omega(0) = m$.

Then there exists a universal constant $0 < \tau < 1$ such that, if $|\bar{x}| \le 1 + \tau m r/2$, we have for any $x \in B_{\frac{\tau m}{4}}(\bar{x})$,

$$H_1(\omega)(x) \ge \frac{m}{2} \quad \text{if } H_1 \text{ is defined as in (1.3)}, \tag{4.1}$$

and

$$H_1(\omega)(x) \ge C m^{p+n} r^n \quad \text{if } H_1 \text{ is defined as in (1.2)}, \tag{4.2}$$

for some constant $C > 0$ depending on p, τ, and n.

Proof. Let τ be the constant τ_0 given by Lemma 4.1 with $C_0 = 2$. Without loss of generality, we may assume $\tau < 1$. Let $\bar{x}$ be such that $|\bar{x}| \le 1 + \tau m r/2$. We will apply the lemma to the rescaled function $\tilde{\omega}(x) = \omega(rx)$ with $D_0 = B_{\frac{1}{r} - \frac{\tau m}{2}}(\frac{\bar{x}}{r})$. Note that D_0 is at distance less or equal to τm from the origin and its principal curvatures, all equal to $\frac{1}{\frac{1}{r} - \frac{\tau m}{2}}$, are bounded by

$$\frac{1}{\frac{1}{r} - \frac{\tau m}{2}} = \frac{2r}{2 - \tau m r} \le 2.$$

By Lemma 4.1, there exists a point $\bar{z} \in B_r(0) \cap \partial B_{1 - \frac{\tau m r}{2}}(\bar{x})$ such that $\omega(\bar{z}) \ge m/2$. Now, let $x \in B_{\frac{\tau m r}{4}}(\bar{x})$. Then

$$|x - \bar{z}| \le |x - \bar{x}| + |\bar{x} - \bar{z}| \le 1 - \frac{\tau m r}{4}.$$

In particular, $B_{\frac{\tau m r}{8}}(\bar{z}) \subset B_1(x)$. Since $\bar{z} \in B_r(0)$, we also have $B_{\frac{\tau m r}{8}}(\bar{z}) \subset B_{2r}(0)$.

First, consider the case where H_1 is defined as in (1.3). Then,

$$H_1(\omega)(x) = \sup_{B_1(x)} \omega \geq \omega(\bar{z}) \geq \frac{m}{2}.$$

This proves (4.1).

Next, assume H_1 as in (1.2). Since $p \geq 1$, w^p is still subharmonic in $B_{2r}(0)$. Then, by the mean value formula (see [23, Theorem 2.1]) applied in $B_{\frac{\tau m r}{8}}(\bar{z}) \subset B_{2r}(0)$, we obtain

$$\begin{aligned} H_1(\omega)(x) &= \fint_{B_1(x)} \omega^p(y)\,dy \geq \frac{1}{|B_1(x)|}\int_{B_{\frac{\tau m r}{8}}(\bar{z})} \omega^p(y)\,dy = \frac{\tau^n m^n r^n}{8^n}\fint_{B_{\frac{\tau m r}{8}}(\bar{z})} \omega^p(y)\,dy \\ &\geq \frac{\tau^n m^n r^n}{8^n}\omega^p(\bar{z}) \geq \frac{\tau^n m^{n+p} r^n}{2^p 8^n}. \end{aligned}$$

This proves (4.2) and completes the proof of the lemma.

□

Following [6, Lemma 5.3], we now prove the exponential decay of the functions u_j^ε away from the boundary of Ω, and at distances less than R from the support of u_i^ε, $i \neq j$.

Lemma 4.4. *Let $(u_1^\varepsilon, \dots, u_K^\varepsilon)$ be a solution of problem (1.1). For $i = 1, \dots, K$, $0 < r < 1$ and $\sigma > 0$, let*

$$\Gamma_i^{\sigma,r} := \{x \in \Omega \,:\, d(x, \text{supp } f_i) \geq 2Rr,\, u_i^\varepsilon(x) = \sigma\},$$

and

$$m := \frac{\sigma}{\sup_{\partial\Omega} f_i}.$$

Then, for τ defined as in Lemma 4.3, in the sets

$$\Sigma_{i,j}^{\sigma,r} := \left\{x \in \Omega \,:\, d(x, \Gamma_i^{\sigma,r}) \leq R + \frac{\tau m R r}{2},\, d(x, \text{supp } f_j) \geq \frac{\tau m R r}{4}\right\},$$

we have $u_j^\varepsilon \leq C e^{\frac{-c\sigma^\alpha r^\beta R}{\varepsilon}}$ for all $j \neq i$ and some $C, c, \alpha, \beta > 0$ depending on p, τ, n and the ellipticity constants.

Proof. Let $\bar{x} \in \Sigma_{i,j}^{\sigma,r}$. We claim that, for $j \neq i$,

$$\Delta u_j^\varepsilon \geq \frac{C\sigma^{\bar{\alpha}} r^{\bar{\beta}}}{\varepsilon^2} u_j^\varepsilon \quad \text{in } B_{\frac{\tau m R r}{4}}(\bar{x}), \tag{4.3}$$

where C, $\bar{\alpha} > 0$ and $\bar{\beta} \geq 0$ are constants depending on p, τ, n and λ. Assuming (4.3), we can apply Lemma 4.2 to obtain

$$u_j^\varepsilon(\bar{x}) \leq C e^{-\frac{\bar{c}\sigma^{\frac{\bar{\alpha}}{2}} r^{\frac{\bar{\beta}}{2}}}{\varepsilon} \frac{\tau m R r}{4}} = C e^{-\frac{c\sigma^\alpha r^\beta R}{\varepsilon}},$$

where $\alpha = \bar{\alpha}/2 + 1$ and $\beta = \bar{\beta}/2 + 1$. This completes the proof of the lemma, provided that (4.3) holds.

To verify (4.3), note that since $d(\bar{x}, \mathrm{supp}\ f_j) \geq \tau mRr/4$, the ball $B_{\frac{\tau mRr}{4}}(\bar{x})$ does not intersect supp f_j. Therefore, recalling (2.2) and observing that $u_j^\varepsilon = 0$ in $B_{\frac{\tau mRr}{4}}(\bar{x}) \cap \Omega^c$, we have

$$\lambda \Delta u_j^\varepsilon \geq \mathcal{M}^-(u_j^\varepsilon) \geq \frac{1}{\varepsilon^2} u_j^\varepsilon \sum_{k \neq j} H_R(u_k^\varepsilon) \geq \frac{1}{\varepsilon^2} u_j^\varepsilon H_R(u_i^\varepsilon) \quad \text{in } B_{\frac{\tau mRr}{4}}(\bar{x}). \tag{4.4}$$

We now estimate $H_R(u_i^\varepsilon)$ in $B_{\frac{\tau mRr}{4}}(\bar{x})$. Let $\bar{y} \in \Gamma_i^{\sigma,r}$ satisfy $|\bar{x} - \bar{y}| \leq R + \tau mRr/2$. Since $d(\bar{y}, \mathrm{supp}\ f_i) \geq 2Rr$, u_i^ε (extended by zero in $B_{2Rr}(\bar{y}) \cap \Omega^c$) satisfies $\Delta u_i^\varepsilon \geq 0$ in $B_{2Rr}(\bar{y})$. Moreover, since u_i^ε is subharmonic in Ω, it attains its maximum at the boundary of Ω, so that $\frac{u_i^\varepsilon}{\sup_{\partial\Omega} f_i} \leq 1$ in Ω. Define,

$$\omega(y) := \frac{u_i^\varepsilon(\bar{y} + Ry)}{\sup_{\partial\Omega} f_i}.$$

Then, $0 \leq \omega \leq 1$, $\omega(0) = \frac{u_i^\varepsilon(\bar{y})}{\sup_{\partial\Omega} f_i} = m$, $\Delta\omega \geq 0$ in $B_{2r}(0)$. Since $|(\bar{x} - \bar{y})/R| \leq 1 + \tau mr/2$, by Lemma 4.3,

$$H_1(\omega)\,(x) \geq C m^{\overline{\alpha}} r^{\overline{\beta}} \quad \text{for any } x \in B_{\frac{\tau mr}{4}}\left(\frac{\bar{x} - \bar{y}}{R}\right),$$

where $\overline{\alpha} = 1$ and $\overline{\beta} = 0$ if H_1 is defined as in (1.3), $\overline{\alpha} = p + n$ and $\overline{\beta} = n$ if H_1 is defined as in (1.2). Recalling the scaling property (2.3), we obtain

$$H_R(u_i^\varepsilon)(x) \geq C m^{\overline{\alpha}} r^{\overline{\beta}} \quad \text{for any } x \in B_{\frac{\tau mRr}{4}}(\bar{x}).$$

Inequality (4.3) follows. The proof of the lemma is thus completed. □

The following result, whose proof follows [6, Lemma 5.5], states that for any $i \neq j$, the function u_i^ε decays exponentially to 0 in a strip of size R around the support of f_j.

Lemma 4.5. *Let $(u_1^\varepsilon, \ldots, u_K^\varepsilon)$ be a solution of problem (1.1). For $j = 1, \ldots, K$, $\sigma > 0$, and $0 < r < R$, let*

$$\bar{\Gamma}_j^\sigma := \{x \in (\partial\Omega)_{\leq R} \,:\, f_j(x) \geq \sigma\} \subset \Omega^c.$$

Then on the sets $\{x \in \Omega \,:\, d(x, \bar{\Gamma}_j^\sigma) \leq R - r\}$, we have $u_i^\varepsilon \leq Ce^{\frac{-c\sigma^\alpha r^\beta}{\varepsilon}}$ for all $i \neq j$, for some $C, c, \alpha, \beta > 0$ depending on p, n, the ellipticity constants and the modulus of continuity of f_j.

Proof. Let $\bar{x} \in \{x \in \Omega \,:\, d(x, \bar{\Gamma}_j^\sigma) \leq R - r\}$. We claim that for $i \neq j$,

$$\Delta u_i^\varepsilon \geq \frac{C\sigma^{\bar{\alpha}} r^{\bar{\beta}}}{\varepsilon^2} u_i^\varepsilon \quad \text{in } B_{\frac{r}{2}}(\bar{x}), \tag{4.5}$$

where C, $\bar{\alpha} > 0$ and $\bar{\beta} \geq 0$ are constants depending on p, n, λ, and the modulus of continuity of f_j. Assuming (4.5), we can apply Lemma 4.2 to obtain

$$u_i^\varepsilon(\bar{x}) \leq Ce^{-\frac{\bar{c}\sigma^{\frac{\bar{\alpha}}{2}} r^{\frac{\bar{\beta}}{2}}}{\varepsilon}\frac{r}{2}} = Ce^{-\frac{c\sigma^\alpha r^\beta}{\varepsilon}},$$

where $\alpha = \bar{\alpha}/2+1$ and $\beta = \bar{\beta}/2+1$. This completes the proof of the lemma, provided that (4.5) holds.

To verify (4.5), note that by assumption (1.8) the ball $B_{\frac{r}{2}}(\bar{x})$ does not intersect supp f_i. Therefore, recalling (2.2) and observing that $u_i^\varepsilon = 0$ in $B_{\frac{r}{2}}(\bar{x}) \cap \Omega^c$, we have

$$\lambda \Delta u_i^\varepsilon \geq \mathcal{M}^-(u_i^\varepsilon) \geq \frac{1}{\varepsilon^2} u_i^\varepsilon \sum_{k \neq i} H_R(u_k^\varepsilon) \geq \frac{1}{\varepsilon^2} u_i^\varepsilon H_R(u_j^\varepsilon) \quad \text{ in } B_{\frac{r}{2}}(\bar{x}). \tag{4.6}$$

We now estimate $H_R(u_j^\varepsilon)$ in $B_{\frac{r}{2}}(\bar{x})$. Let $\bar{y} \in \bar{\Gamma}_j^\sigma$ satisfy $|\bar{x} - \bar{y}| \leq R - r$, and let $x \in B_{\frac{r}{2}}(\bar{x})$. Then, $|x - \bar{y}| \leq R - r/2$.

First, consider the case where H_R is given by (1.3). Then,

$$H_R(u_j^\varepsilon)(x) = \sup\nolimits_{B_R(x)} u_j^\varepsilon \geq u_j^\varepsilon(\bar{y}) = f_j(\bar{y}) \geq \sigma.$$

Consequently, from (4.6), we obtain (4.5) with $\bar{\alpha} = 1$ and $\bar{\beta} = 0$.

Next, consider the case where H_R is given by (1.2). Let $\gamma > 0$ depend on the modulus of continuity of f_j so that $f_j > \frac{\sigma}{2}$ in $B_{\sigma^\gamma}(\bar{y}) \cap \text{supp } f_j$. Define $r_0 := \min\{\sigma^\gamma, \frac{r}{4}\}$. Then, for $z \in B_{r_0}(\bar{y})$,

$$|x - z| \leq |x - \bar{y}| + |\bar{y} - z| \leq R - \frac{r}{2} + r_0 \leq R - \frac{r}{4}.$$

In particular, $B_{r_0}(\bar{y}) \subset B_R(x)$. Therefore, using assumption (1.7), we get

$$\begin{aligned}
H_R(u_j^\varepsilon)(x) &= \fint_{B_R(x)} (u_j^\varepsilon)^p(z)\, dz \geq \frac{1}{|B_R(x)|} \int_{B_{r_0}(\bar{y}) \cap \text{supp } f_j} (u_j^\varepsilon)^p(z)\, dz \\
&= \frac{1}{|B_R(x)|} \int_{B_{r_0}(\bar{y}) \cap \text{supp } f_j} f_j^p(z)\, dz \geq \frac{1}{|B_R(x)|} \int_{B_{r_0}(\bar{y}) \cap \text{supp } f_j} \left(\frac{\sigma}{2}\right)^p dz \\
&\geq C \sigma^p r_0^n \\
&= C \min\left\{\sigma^{p+\gamma}, \frac{\sigma^p r}{4}\right\} \\
&\geq C \sigma^{p+\gamma} r.
\end{aligned}$$

Consequently, from (4.6), we obtain (4.5) with $\bar{\alpha} = p + \gamma$ and $\bar{\beta} = 1$. This completes the proof of the lemma. $\square$

The following result, a corollary of Lemma 4.4, provides interior Lipschitz estimates for the functions u_i^ε, which are uniform in ε.

Corollary 4.6. *Let $(u_1^\varepsilon, \dots, u_K^\varepsilon)$ be a solution of problem (1.1). Let $\bar{y}$ be a point in Ω such that $u_i^\varepsilon(\bar{y}) = \sigma$, $d(\bar{y}, \text{supp } f_j) \geq R + \tau m R r$ for $j \neq i$, and $d(\bar{y}, \partial\Omega) \geq 2Rr$, where $m = \frac{\sigma}{\sup_{\partial\Omega} f_i}$, $0 < r < 1$, and τ, α, and β, are given in Lemma 4.4. Then, there exists $C > 0$ such that for $0 < \varepsilon \leq \sigma^{2\alpha} r^{2\beta}$,*

$$|\nabla u_i^\varepsilon| \leq \frac{C}{Rr} \quad \text{ in } B_{\frac{\tau m R r}{8}}(\bar{y}) \tag{4.7}$$

and

$$\mathcal{M}^-(u_i^\varepsilon) \to 0 \quad \text{ in } B_{\frac{\tau m R r}{2}}(\bar{y}) \tag{4.8}$$

uniformly as $\varepsilon \to 0^+$.

Proof. First, note that $B_{\frac{\tau m R r}{2}}(\bar{y}) \subset B_{2Rr}(\bar{y}) \subset \Omega$, where the first inclusion follows from the fact that $\tau < 1$ and $m = \frac{\sigma}{sup_{\partial\Omega} f_i} \leq 1$.

We claim that for any $z \in B_{\frac{\tau m R r}{2}}(\bar{y})$, we have

$$u_j^\varepsilon(\bar{x}) \leq C e^{\frac{-c\sigma^\alpha r^\beta R}{\varepsilon}} \tag{4.9}$$

for all $\bar{x} \in B_R(z)$ and all $j \neq i$. To see why (4.9) holds, assume $z \in B_{\frac{\tau m R r}{2}}(\bar{y})$ and $\bar{x} \in B_R(z)$. Then

$$|\bar{x} - \bar{y}| \leq R + \frac{\tau m R r}{2}.$$

Moreover, since $d(\bar{y}, \text{supp } f_j) \geq R + \tau m R r$, we have

$$d(\bar{x}, \text{supp } f_j) \geq \frac{\tau m R r}{2}.$$

Let $\Gamma_i^{\sigma,r}$ and $\Sigma_{i,j}^{\sigma,r}$ be defined as in Lemma 4.4. Note that $\bar{y} \in \Gamma_i^{\sigma,r}$ and $\bar{x} \in \Sigma_{i,j}^{\sigma,r}$. Therefore, by Lemma 4.4, estimate (4.9) follows.

Now, using (4.9), for all $z \in B_{\frac{\tau m R r}{2}}(\bar{y})$ and for $0 < \varepsilon \leq \sigma^{2\alpha} r^{2\beta}$, we obtain

$$0 \leq \mathcal{M}^-(u_i^\varepsilon(z)) \leq u_i^\varepsilon(z) \frac{C e^{\frac{-c\sigma^\alpha r^\beta R}{\varepsilon}}}{\varepsilon^2} \leq \frac{C e^{-c\varepsilon^{-\frac{1}{2}} R}}{\varepsilon^2} \to 0$$

as $\varepsilon \to 0^+$, which proves (4.8).

It therefore remains to establish (4.7).

By (4.8), the function u_i^ε satisfies

$$\mathcal{M}^-(u_i^\varepsilon) = g \quad \text{in } B_{\frac{\tau m R r}{2}}(\bar{y}),$$

with $\|g\|_{L^\infty(B_{\frac{\tau m R r}{2}}(\bar{y}))} \leq C$ for some $C > 0$ independent of ε. Define the rescaled function

$$v_i^\varepsilon(x) := 4 \frac{u_i^\varepsilon\left(\frac{\tau m R r}{4} x + \bar{y}\right)}{\tau m R r}.$$

Note that v_i^ε satisfies

$$\mathcal{M}^-(v_i^\varepsilon) = \overline{g} \quad \text{in } B_2(0),$$

with $\|\overline{g}\|_{L^\infty(B_2(0))} \leq C$ and

$$v_i^\varepsilon(0) = \frac{4\sigma}{\tau m R r} = \frac{4 \sup_{\partial\Omega} f_i}{\tau R r} \leq \frac{C}{Rr}.$$

By the Harnack inequality, Theorem 2.2,

$$v_i^\varepsilon(x) \leq C(n)(v_i^\varepsilon(0) + C) \leq \frac{C}{Rr} \quad \text{for all } x \in B_1(0).$$

By Theorem 2.3 and Sobolev embeddings, we infer that

$$|\nabla v_i^\varepsilon| \leq \frac{C}{Rr} \quad \text{in } B_{\frac{1}{2}}(0).$$

Estimate (4.7) follows. This completes the proof of the corollary. □

Finally, the following result is a corollary of Lemma 4.5.

Corollary 4.7. *Let $(u_1^\varepsilon, \dots, u_K^\varepsilon)$ be a solution of problem* (1.1)*. For $i = 1, \dots, K$, define*

$$B_i := \cup_{j\neq i}\{x \in \Omega \,:\, d(x, \text{supp } f_j) \le R\}.$$

Then $u_i^\varepsilon \to 0$ as $\varepsilon \to 0^+$ in B_i.

Proof. Let $x_0 \in B_i$. Then there exists $j \neq i$ such that $d(x_0, \text{supp } f_j) \le R$. First, assume that $d(x_0, \text{supp } f_j) < R$. Observe that

$$\{x \in \Omega \,:\, d(x, \text{supp}\, f_j) < R\} \subset \bigcup_{r,\sigma>0} \left\{x \in \Omega \,:\, d(x, \bar\Gamma_j^\sigma) \le R - r\right\},$$

where $\bar\Gamma_j^\sigma$ is defined as in Lemma 4.5. Hence, there exist $r, \sigma > 0$ such that

$$x_0 \in \left\{x \in \Omega : d(x, \bar\Gamma_j^\sigma) \le R - r\right\}.$$

By Lemma 4.5, it follows that

$$u_i^\varepsilon(x_0) \le Ce^{\frac{-c\sigma^\alpha r^\beta}{\varepsilon}}$$

for some constants $C, c, \alpha, \beta > 0$. Letting $\varepsilon \to 0^+$ in the above inequality, we obtain

$$u_i^\varepsilon(x_0) \to 0 \quad \text{as } \varepsilon \to 0^+.$$

Next, assume that $d(x_0, \text{supp } f_j) = R$. Consider the set $A_r := \{x \in \Omega : d(x, \text{supp } f_j) \ge R - r\}$. Note that A_r has an exterior tangent ball of radius $R - r$ at every point of its boundary. Moreover, we have just showed that $\sup_{\partial A_r \cap \Omega} u_i^\varepsilon \to 0$ as $\varepsilon \to 0^+$. Since points x in Ω such that $d(x, \text{supp}\, f_j) = R$ are at distance r from ∂A_r, a barrier argument shows that there exist $C, r_0 > 0$ such that $u_i^\varepsilon(x) \le \sup_{\partial A_r\cap\Omega} u_i^\varepsilon + Cr$ for all $x \in B_{r_0}(x_0) \cap \{x \in \Omega \,:\, d(x, \text{supp}\, f_j) = R\}$. Evaluating the latter inequality at $x = x_0$, sending first $\varepsilon \to 0^+$ and then $r \to 0^+$, yields $u_i^\varepsilon(x_0) \to 0$ as $\varepsilon \to 0^+$.

This completes the proof of the corollary. □

Proof of Theorem 1.2: For each for $i = 1 \dots, K$, define

$$\Omega_i := \{x \in \Omega \,:\, d(x, \text{supp } f_j) > R \text{ for all } j \neq i\}.$$

Claim: *There exists a subsequence $\{u_i^{\varepsilon_l}\}_l$ locally uniformly convergent in Ω_i as $l \to \infty$ to a locally Lipschitz continuous function u_i.*

To prove the claim, fix $r \in (0, R)$ and define

$$\Omega_i^r := \{x \in \Omega \,:\, d(x, \partial\Omega) > 2Rr,\, d(x, \text{supp } f_j) \ge R + \tau Rr \text{ for all } j \neq i\},$$

where τ is defined as in Lemma 4.3. Note that $\Omega_i = \cup_{r\in(0,R)}\Omega_i^r$. Let α and β be as in Lemma 4.4. Fix $\theta < \frac{1}{2\alpha}$, and define $\sigma_\varepsilon := \varepsilon^\theta$. Observe that

$$\varepsilon = \sigma_\varepsilon^{\frac{1}{\theta}} = \sigma_\varepsilon^{2\alpha}\sigma_\varepsilon^{\frac{1}{\theta}-2\alpha} = \sigma_\varepsilon^{2\alpha}\varepsilon^{\theta(\frac{1}{\theta}-2\alpha)}$$

and note that $\frac{1}{\theta} - 2\alpha > 0$. Therefore, we can choose $\varepsilon_0 = \varepsilon_0(r)$ such that $\varepsilon \le \sigma_\varepsilon^{2\alpha} r^{2\beta}$ whenever $0 < \varepsilon < \varepsilon_0$. Now define

$$v_i^\varepsilon := (u_i^\varepsilon - \sigma_\varepsilon)_+.$$

Then, the functions v_i^ε are uniformly Lipschitz continuous on Ω_i^r. Indeed, if $u_i^\varepsilon(x) \leq \sigma_\varepsilon$, then clearly $v_i^\varepsilon = 0$. Next, let $x \in \Omega_i^r$ be such that $u_i^\varepsilon(x) > \sigma_\varepsilon = \varepsilon^\theta$. Define $\sigma := u_i^\varepsilon(x)$ and $m := \frac{\sigma}{\sup_{\partial\Omega} f_i} \leq 1$. Then, for all $j \neq i$, $d(x, \operatorname{supp} f_j) \geq R + \tau R r \geq R + \tau m R r$. In addition, $d(x, \partial\Omega) > 2Rr$ and for $0 < \varepsilon < \varepsilon_0$ we have $\varepsilon \leq \sigma_\varepsilon^{2\alpha} r^{2\beta} \leq \sigma^{2\alpha} r^{2\beta}$. Thus, we can apply Corollary 4.6 to obtain

$$|\nabla v_i^\varepsilon(x)| = |\nabla u_i^\varepsilon(x)| \leq \frac{C}{rR}.$$

Therefore, by the Arzelà-Ascoli Theorem we may extract a subsequence $\{v_i^{\varepsilon_l}\}_l$ uniformly convergent to a Lipschitz continuous function u_i in Ω_i^r as $l \to \infty$. Since $|u_i^\varepsilon - v_i^\varepsilon| \leq \varepsilon^\theta$, the same subsequence $\{u_i^{\varepsilon_l}\}_l$ converges uniformly to u_i in Ω_i^r. Taking a sequence $r_k \to 0^+$ as $k \to \infty$ and using a diagonal argument, we can find a subsequence of $\{u_i^\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ converging locally uniformly to a locally Lipschitz function u_i in Ω_i. This concludes the proof of the claim.

Now let $B_i := \Omega \setminus \Omega_i$. Then, by Corollary 4.7 $u_i^{\varepsilon_l} \to 0$ in B_i as $l \to \infty$.

Combining this with the previous claim proves convergence of the subsequence $\{u_i^{\varepsilon_l}\}_l$ to a function u_i which is locally Lipschitz in Ω_i and in the interior of B_i.

To conclude the proof of part (1) of the theorem, it remains to show that the function u_i is locally Lipschitz on $\partial\Omega_i \cap \Omega = \partial B_i \cap \Omega$.

First, note that the second part of the theorem follows immediately from the proof of the claim and Corollary 4.6.

Now, let $x_0 \in \partial\Omega_i \cap \Omega$, then $u_i(x_0) = 0$. If $x_0 \notin \partial\{u_i > 0\}$ then in a neighborhood of x_0 $u_i = 0$ and of course u_i is Lipschitz in that neighborhood. Suppose instead that $x_0 \in \partial\{u_i > 0\}$. By the definition of Ω_i, there exists an exterior ball of radius R at every point of $\partial\Omega_i \cap \Omega$, and we have $\mathcal{M}^-(u_i) = 0$ on $\{u_i > 0\}$. Then, a standard barrier argument shows that that there exists $r_0, C > 0$ such that $0 \leq u_i(x) = u_i(x) - u_i(x_0) \leq C|x - x_0|$ for all $x \in B_{r_0}(x_0)$. This establishes the local Lipschitz continuity of u_i near x_0, completing the proof of part (1) of the theorem.

We now prove part (3) of Theorem 1.2. Let $x_0 \in \Omega \cup (\partial\Omega)_{\leq R}$ be such that $u_i(x_0) > 0$. Let us show that if $y_0 \in \Omega$ is such that $|x_0 - y_0| \leq R$ then $u_j(x_0) = 0$ for all $j \neq i$.

If $x_0 \in (\partial\Omega)_{\leq R}$ then this follows from Corollary 4.7.

Assume now that $x_0 \in \Omega$. Let $0 < r < 1$ be such that $d(y_0, \partial\Omega) \geq 2Rr$. Let $\sigma_l := u_i^{\varepsilon_l}(x_0)$, then $\sigma_l \geq u_i(x_0)/2 > 0$ for l sufficiently large. By Lemma 4.4, there exist $C, c, \alpha, \beta > 0$ such that $u_j^{\varepsilon_l}(y_0) \leq C e^{\frac{-c\sigma_l^\alpha r^\beta R}{\varepsilon}}$ for all $j \neq i$. Letting l go to infinity we obtain $u_j(y_0) = 0$.

This completes the proof of (3) and of the theorem. □

5. A Semiconvexity Property of the Free Boundary

We have shown in the previous section that there is a subsequential limit $(u_1, \ldots, u_K)$ of $\{(u_1^\varepsilon, \ldots, u_K^\varepsilon)\}_{\varepsilon>0}$. In this section, we study the geometry of the sets

$$S(u_i) := \{x \in \Omega : u_i > 0\}, \quad i = 1, \ldots, K,$$

and the corresponding free boundaries $\partial S(u_i)\cap\Omega$. We will show that each set $S(u_i)$ has finite perimeter (Theorem 1.4), and that its free boundary satisfies a semi-convexity property (Theorem 1.3).

The proof of the following result is based on the strong minimum principle, Theorem 2.8.

Lemma 5.1. *Let $F(u_i) := \{x \in \mathbb{R}^n : d(x, S(u_i)) \geq R\}$, and set $N(u_i) := \{x \in \Omega : d(x, F(u_i)) > R\}$. Then $\partial S(u_i) \subset \partial N(u_i)$.*

Proof. To show that $\partial S(u_i) \subset \partial N(u_i)$, it suffices to prove that every connected component of $S(u_i)$ is also a connected component of $N(u_i)$.

First, observe that $S(u_i) \subset N(u_i)$. Indeed if $p \in S(u_i)$, and since $S(u_i)$ is open, we have that $d(p, F(u_i)) > R$. Thus, $p \in N(u_i)$.

Now, for all $\sigma > 0$, consider the sets

$$S_\sigma(u_i) := \{x \in \Omega \,:\, u_i > \sigma\},$$

$$F_\sigma(u_i) := \{x \in \mathbb{R}^n \,:\, d(x, S_\sigma(u_i)) \geq R\}$$

and

$$N_\sigma(u_i) := \{x \in \Omega \,:\, d(x, F_\sigma(u_i)) > R\}.$$

We claim that

$$(F_\sigma(u_i))^c = \bigcup_{x\in S_\sigma(u_i)} B_R(x) = \bigcup_{x\in N_\sigma(u_i)} B_R(x). \tag{5.1}$$

To check the first equality of (5.1), note that if $p \in \cup_{x\in S_\sigma(u_i)} B_R(x)$, then $p \in B_R(x)$ for some $x \in S_\sigma(u_i)$. It follows that $d(p, S_\sigma(u_i)) < R$, so $p \in (F_\sigma(u_i))^c$. Conversely, if $p \in (F_\sigma(u_i))^c$, then $d(p, S_\sigma(u_i)) < R$, so $p \in B_R(x)$ for some $x \in S_\sigma(u_i)$. This proves that $(F_\sigma(u_i))^c = \cup_{x\in S_\sigma(u_i)} B_R(x)$. To prove the second equality, note that if $p \in \cup_{x\in N_\sigma(u_i)} B_R(x)$, then $p \in B_R(x)$ for some $x \in N_\sigma(u_i)$. Since $|x - p| < R$, by the definition of $N_\sigma(u_i)$ we must have that $p \in (F_\sigma(u_i))^c$. On the other hand, since $S_\sigma(u_i) \subset N_\sigma(u_i)$, it follows that $\cup_{x\in S_\sigma(u_i)} B_R(x) \subset \cup_{x\in N_\sigma(u_i)} B_R(x)$. We have shown that $\cup_{x\in N_\sigma(u_i)} B_R(x) = \cup_{x\in S_\sigma(u_i)} B_R(x)$. Consequently, (5.1) holds.

Next, we claim that for all $\sigma > 0$,

$$\mathcal{M}^-(u_i) = 0 \text{ in } N_\sigma(u_i) \tag{5.2}$$

provided $N_\sigma(u_i)$ is nonempty. Let $\{u_i^{\varepsilon_l}\}_l$ be a subsequence converging locally uniformly to u_i in Ω. Fix $x \in S_\sigma(u_i)$, then $\sigma_l := u_i^{\varepsilon_l}(x) \geq \sigma$ for l sufficiently large. Moreover, by Theorem 1.2-(3), we know that $d(x, \operatorname{supp} f_j) > R$, for all $j \neq i$. Therefore, there exists $r > 0$ such that $x \in \Gamma_i^{\sigma_l, r}$ and $B_R(x) \subset \Sigma_{i,j}^{\sigma_l, r}$ where $\Gamma_i^{\sigma_l, r}$ and $\Sigma_{i,j}^{\sigma_l, r}$ are defined as in Lemma 4.4. By Lemma 4.4, it follows that

$$u_j^{\varepsilon_l} \leq Ce^{\frac{-c\sigma_l^\alpha r^\beta R}{\varepsilon_l}} \leq Ce^{\frac{-c\sigma^\alpha r^\beta R}{\varepsilon_l}} \quad \text{in } B_R(x).$$

Thus, for all $j \neq i$,

$$u_j^{\varepsilon_l} \leq Ce^{\frac{-c\sigma^\alpha r^\beta R}{\varepsilon_l}} \quad \text{in} \bigcup_{x\in S_\sigma(u_i)} B_R(x).$$

Recalling (5.1), we conclude that for all $j \neq i$,

$$\frac{H_R(u_j^{\varepsilon_l})}{\varepsilon_l^2} \to 0 \quad \text{in } N_\sigma(u_i),$$

which implies the desired result (5.2).

Finally, we prove that every connected component of $S(u_i)$ is also a connected component of $N(u_i)$. To see this, let A be a connected component of $S(u_i)$. Since $S(u_i) \subset N(u_i)$, there exists B connected component of $N(u_i)$ such that $A \subset B$.

For $\sigma > 0$, define

$$A_\sigma := A \cap S_\sigma(u_i), \quad B_\sigma := B \cap N_\sigma(u_i).$$

By (5.2) $\mathcal{M}^-(u_i) = 0$ in B_σ. Moreover, since $A_\sigma \subset B_\sigma$, we know that $u_i \not\equiv 0$ in B_σ. Then, by the strong minimum principle, Theorem 2.8, it follows that $u_i > 0$ in B_σ, that is $B_\sigma \subset S(u_i)$. Since this holds for every $\sigma > 0$, we conclude that $B \subset S(u_i)$. But we already had that $A \subset B$, and A was a connected component of $S(u_i)$. Since B is also connected, we must have $A = B$.

This completes the proof of the lemma. □

Proof of Theorem 1.3: For $i = 1, \ldots, K$, let $S(u_i)$ and $N(u_i)$ be defined as in Lemma 5.1. By definition of $N(u_i)$, for every $x \in \partial N(u_i) \cap \Omega$ there exists an exterior ball of radius R tangent at x. Theorem 1.3 then immediately follows from Lemma 5.1. □

The following theorem is shown in the Appendix.

Theorem 5.2. *Let E be a compact subset of $\mathbb{R}^n$, and let $E_t := \{x \in \mathbb{R}^n : d(x, E) < t\}$, $t > 0$. Then E_t has finite perimeter.*

Proof of Theorem 1.4: Let $F(u_i)$ and $N(u_i)$ be defined as in Lemma 5.1. Note that

$$N(u_i) = \{x \in \mathbb{R}^n \,:\, d(x, \partial F(u_i)) > R\} \cap \Omega$$

and $\partial F(u_i)$ is a compact set. By Theorem 5.2, $N(u_i)$ has finite perimeter. Theorem 1.4 then immediately follows from Lemma 5.1. □.

6. Appendix: a proof of Theorem 5.2

Theorem 5.2 can be proven either using PDE techniques, as in [6], or using techniques from geometric measure theory, as in [25]. Here we explain the proof of Theorem 5.2 given in [25], which is based on a covering argument.

For a set $E \subset \mathbb{R}^n$, we denote by d_E the distance function from E, given by

$$d_E(x) := \inf_{y \in E} |x - y| \quad \text{for all } x \in \mathbb{R}^n.$$

For $t > 0$, we define

$$E_t := \{x \in \mathbb{R}^n \,:\, d_E(x) < t\}, \tag{6.1}$$

and

$$U_t := \{x \in \mathbb{R}^n : 0 < d_E(x) < t\} = \bigcup_{x_0 \in \partial E} B_t(x_0) \setminus (\partial E \cup E). \tag{6.2}$$

Note that E_t is open and $\partial E_t = \{x \in \mathbb{R}^n : d_E(x) = t\} = d_E^{-1}(\{t\})$.

We also recall, that

$$P(E_t) \leq \mathcal{H}^{n-1}(\partial E_t) = \lim_{\delta \to 0^+} \mathcal{H}_\delta^{n-1}(\partial E_t),$$

where

$$\mathcal{H}_\delta^{n-1}(\partial E_t) := \inf \left\{ \sum_{i=1}^{\infty} \omega_{n-1} r_i^{n-1} \ : \ \partial E_t \subset \bigcup_{i=1}^{\infty} A_i,\ r_i = \frac{1}{2} \sup_{x,y \in A_i} |x-y| \leq \delta \right\}$$

and ω_{n-1} denotes the volume of the unit ball in $\mathbb{R}^{n-1}$, see for instance [22].

Let $x_0, x \in \mathbb{R}^n$ and $\phi \in [0, \frac{\pi}{2}]$. We define the open "sector" with center at x_0 and radius $t := |x - x_0|$ by

$$A_\phi(x_0, x) := \{y \in \mathbb{R}^n : 0 < |x_0 - y| < |x_0 - x|, (y - x_0) \cdot (x - x_0) > |x_0 - x||x_0 - y| \cos\phi\}.$$

For all $0 \leq r_1 \leq r_2 \leq 1$, we further define

$$A_\phi(x_0, x; r_1, r_2) := \{y \in A_\phi(x_0, x) : r_1|x_0 - x| < |x_0 - y| < r_2|x_0 - x|\}$$

We will need the following properties of sectors proven in [25].

(1) For a fixed $t > 0$, there exists $0 < \delta_0 < t$ and a dimensional constant $C > 0$ such that

$$\delta^{n-1}\omega_{n-1} \leq C \frac{|A_{\phi(\delta)}(x_0, x)|}{t} \tag{6.3}$$

for all $\delta \in (0, \delta_0)$ and arbitrary $x_0, x \in \mathbb{R}^n$ with $|x_0 - x| = t$. Here $\phi(\delta) := \arccos(1 - \frac{\delta^2}{2t^2})$.

(2) Let $x_0, x \in \mathbb{R}^n, \phi \in [0, \frac{\pi}{2}]$ and $z \in \overline{A_\phi(x_0, x)}$. Denote by $t := |x_0 - x|$ and $|x_0 - z| = rt$ with some $r \in [0, 1]$. Then $t(1 - r) \leq |x - z| \leq t(1 - r + \phi r) \leq t(1 - r + \phi)$.

(3) Fix a $t > 0$. Then there exist $0 < \delta_0 < t$, $0 < a < 1$ and $0 < r_1 < r_2 < 1$ such that

$$\overline{A_{a\phi(\delta)}(x_0, x; r_1, r_2)} \cap \overline{A_{a\phi(\delta)}(y_0, y; r_1, r_2)} = \emptyset \tag{6.4}$$

for all $\delta \in (0, \delta_0)$ and $x_0, y_0, x, y \in \mathbb{R}^n$ with $t = |x - x_0| = |y - y_0|$, whenever $\overline{B_\delta(x)} \cap \overline{B_\delta(y)} = \emptyset$ and $t \leq \min(|x_0 - y|, |y_0 - x|)$. The constants a, r_1, r_2 are explicit and do not depend on any other values, and here $\phi(\delta) := \arccos(1 - \frac{\delta^2}{2t^2})$.

Theorem 5.2 is a consequence of the following result.

Theorem 6.1. *([25], Theorem 2) Let $E \subset \mathbb{R}^n$ be a compact set, and let E_t and U_t be defined as is* (6.1) *and* (6.2)*, respectively. Then there exists a dimensional constant $C > 0$ such that*

$$P(E_t) \leq \mathcal{H}^{n-1}(\partial E_t) \leq C \frac{|U_t|}{t} \tag{6.5}$$

for all $t > 0$.

Proof. We prove only the second inequality since the first inequality is well known. Fix $t > 0$. Let δ_0 be as in property (3) above and assume $\delta \in (0, \delta_0)$.

Clearly, $\partial E_t \subset \bigcup_{x\in\partial E_t} \overline{B_\delta(x)}$. By Vitali's Covering Theorem (see for instance [22, Theorem 1.24]), there exists a countable subset $J \subset \partial E_t$ such that

$$\partial E_t \subset \bigcup_{x\in J} \overline{B_{5\delta}(x)}, \tag{6.6}$$

and the family $\{B_\delta(x)\}_{x\in J}$ is pairwise disjoint.

Since E is compact, for each $x \in J$ there exists $x_0 \in \partial E$ such that $|x - x_0| = t$. Let us define $A_x := \overline{A_{a\phi(\delta)}(x_0, x; r_1, r_2)}$, with a, r_1 and r_2 as in property (3). Then, for all $z \in A_x$, we have

$$d_E(z) \le |z - x_0| \le tr_2 < t. \tag{6.7}$$

Moreover, since the family $\{B_\delta(x)\}_{x\in J}$ is pairwise disjoint, by property (3) we have that A_x and A_y are disjoint for all $x, y \in J$, $x \ne y$.

We now show that,

$$A_x \cap E = \emptyset \tag{6.8}$$

provided $\delta > 0$ is small enough. To see this, pick any $z \in A_x$, then $|x_0 - z| = rt$ for some $r \in [r_1, r_2]$. By property (2), we have that

$$|x - z| \le t(1 - r + a\phi(\delta)) < t(1 - r + \phi(\delta)) \le t(1 - r_1 + \phi(\delta)) \le t,$$

if δ is so small that $\phi(\delta) \le r_1$. Since $B_t(x) \subset E^c$, this implies that $z \notin E$.

By (6.7) and (6.8) it follows that each A_x is contained in U_t and

$$\sum_{x\in J} |(A_{\phi(\delta)}(x_0, x))| \le C' \sum_{x\in J} |A_x| = C' \Big| \bigcup_{x\in J} A_x \Big| \le C'|U_t|, \tag{6.9}$$

for some suitable constant $C' > 0$, as long as $\delta > 0$ is small enough. Now combine (6.6), (6.9) and property (1) above to obtain

$$\mathcal{H}^{n-1}_{5\delta}(\partial E_t) \le \sum_{x\in J} (5\delta)^{n-1}\omega_{n-1} \le 5^{n-1}\frac{C''}{t} \sum_{x\in J} |A_{\phi(\delta)}(x_0, x)| \le 5^{n-1} C' C'' \frac{|U_t|}{t},$$

for some $C'' > 0$. Letting $\delta \to 0^+$, we obtain the desired result.

□

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